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THE ESTIMATION OF PARAMETERS OF FATIGUE CURVE OF COMPOSITE MATERIAL

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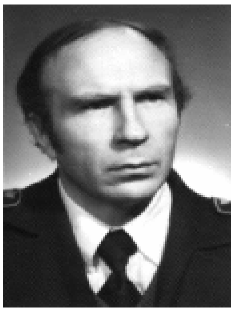

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Abstract. Maximum likelihood estimation of parameters of mathematical model of fatigue damage accumulation based on laminate fatigue life data processing is considered. The model, which is founded on the use of Markov chain theory, allows us to see connection between static strength distribution parameters and S-N fatigue curve. It was shown already that the model is too simple and does not provide numerical coincidence with experimental fatigue test data, nevertheless it could be used as nonlinear regression model of S-N fatigue curve. Simple method of approximate estimation of model parameters is offered. Numerical example is given. Some parameters of this fatigue curve model can be considered as local static strength distribution function parameters. Using this model, we can predict fatigue curve changes as consequence of static strength parameter changes.

Keywords: strength, fatigue life, composite.

Introduction

Use of composite material in aircraft structure increases every year. To provide reliability of flight we should study the fatigue phenomenon of this material. One of the main quantitative characteristics of this phenomenon is fatigue curve. There are many offers for its description. Wide discussion on this subject is presented in [5]. Short review of this problem is given in paper [1]. We will not repeat it because this paper is in some way a development of [1]. This paper is devoted mainly to processing dataset of fatigue tests of 125 carbon-fiber laminate specimens. The purpose is to get estimates of parameters of fatigue damage accumulation model, based on the Markov chain theory [1]. It was shown that, although the model is too simple and does not provide numerical coincidence with experimental fatigue test data, it allows to give fatigue curve equation by the use of static strength distribution parameters and some additional parameters, which have some 'physical' interpretation. In this paper we consider inverse problem: by the use of this model, which can be considered now as nonlinear regression analysis model, we'll try to get estimates of local static strength distribution parameters. But it is not the end in itself. The likelihood of these estimates and the likelihood of "theoretical" and experimental fatigue curves can be considered as a proof of likelihood of the studied model.

The model discussed herein has 6 parameters all together. For experimental data processing 5 of them are used [1]. In this paper approximately for the same level of precision of fatigue curve description we have used only 4 parameters. It is significant decreasing of the difficulty of statistical analysis. The method of approximate estimation of unknown parameters is offered. Numerical example is given.

Overview. The reminder of the discussed model ideas is given in Section 2. Method for model parameters estimation is discussed in Section 3. Numerical example is given in Section 4.

1. Cumulative damage model based on the Markov chains theory

We consider the process of fatigue damage accumulation as Markov chain with following matrix of transition probabilities

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{23} & \cdots & p_{1r} & p_{1(r+1)} \\ 0 & p_{22} & p_{23} & p_{24} & \cdots & p_{2r} & p_{2(r+1)} \\ 0 & 0 & p_{33} & p_{34} & \cdots & p_{3r} & p_{3(r+1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \cdots & p_{rr} & p_{r(r+1)} \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix},$$

corresponding to a MC (Markov Chain) with one (r+1)th absorbing state and r nonrecurrent states.

Cumulative distribution function (cdf) of time to absorption for this process

$$F_T(t) = p_{1\ r+1}(t),\ t = 1,2,3,\ldots,$$

where $p_{1\ r+1}(t)$ is the element of first row and (r+1)th column of matrix $P(t) = P^t$.

It can be defined also in this way

$$F_T(t) = aP^t b, \qquad (1)$$

where $a = (100\ldots0)$ is the row vector, $b = (00\ldots01)'$ is the column vector. It is worth to notice, that product $P^t b$ gives a column vector of cumulative distribution functions of times to absorption, components of which correspond to different start states of MC: $(F_T^{(1)}(t), F_T^{(2)}(t),\ldots,F_T^{(r)}(t))'$. In general case it can be used to get cdf when the probability distribution on start states of MC, $p = (p_1, p_2,\ldots,p_r,p_{(r+1)})$, is known

$$F_T(t) = pP^t b = p\ \ (F_T^{(1)}(t), F_T^{(2)}(t),\ldots,F_T^{(r)}(t))'.$$

This possibility should be considered as some reserve, which can be used to take into account some specific features of specific composite structure, induced by some specific technology. Just now we do not use this possibility, because we deliberately try to decrease the number of parameters of considered model. Moreover, in this paper we consider S*imple Markov Chain Model of Fatigue Life* (SMCMFL) of composite material, it is the model, for which only transitions to the nearest 'seniors' states are allowed and

$$P = \begin{bmatrix} q_1 & p_1 & 0 & & & \ldots & 0 \\ 0 & q_2 & p_2 & 0 & & \ldots & 0 \\ 0 & 0 & q_3 & p_3 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & & & \ldots & 0 & q_r & p_r \\ 0 & & & \ldots & 0 & 0 & 1 \end{bmatrix},\ q_i = 1 - p_i,\ i = 1,\ldots,r.$$

The main characteristics of this type MC are well known. Time to failure (time to absorption) $T = X_1 + X_2 + \ldots + X_r$, where X_i (time the process spends in ith state), $i = 1,\ldots,r$, are independent random variables. Random variable X_i has geometric distribution with pmf (probability mass function)

$$P(X_i = n) = (1 - p_i)^{n-1} p_i.$$

Expectation value

$$E(X_i) = 1/p_i$$

and variance

$$V(X_i) = (1 - p_i)/p_i^2.$$

So probability generating function for random variable T (which can be used to obtain pmf of T)

$$G_T(z) = \sum_{i=0}^{\infty} p_T(i) \cdot z^i = \prod_{i=1}^{\infty} \frac{z\, p_i}{1 - z(1 - p_i)}.$$

The pmf can be calculated also by the use of formula

$$p_T(t) = F_T(t) - F_T(t-1).$$

If we assume, that one step in MC corresponds to k_M cycles in fatigue test, then for calculation of expectation value and variance of T we should use formulae

$$E(T) = k_M \sum_{i=1}^{r} 1/p_i \, , \quad (2)$$

$$V(T) = k_M{}^2 \sum_{i=1}^{r} (1 - p_i)/p_i^2 \, . \quad (3)$$

But the main and most difficult problem is to connect these probabilities, p_i, i=1,...,r, with parameter of composite material component strength distribution and applied stress level in such a way that we'll can get the fatigue curve equation. Our offer is to assume that in one step of Markov process (1 cycle or may be 1000 cycles) only one parallel structural item (for example strand) can be failed. If we have ***(R-i)*** still alive parallel structural items and the same cdf $F(s)$ for every item, then the fracture probability of at least one item is equal

$$p_i = 1 - (1 - F(s_i))^{R-i} \, ,$$

where R is initial number of items, i is the number of items, which are failed already, s_i is the corresponding stress applied uniformly to all ***(R-i)*** items. We suppose also that

$$s_i = \frac{SR - S_f i}{R - i} = \frac{S(1 - S_f \, i/SR))}{1 - i/R} \, ,$$

where S is initial stress (force) in every item (at the start of the test), S_f is stress (force) which already failed item can carry yet (because at least at the beginning of damage accumulation the rupture of fibers can be in different cross sections).

Let us consider the case when cdf $F(s)$ has location and scale parameters:

$$F(s)=F_0((g(s)-q_0)/q_1), \quad (4)$$

where g(.) is some known function, $F_0(.)$ is some known cdf.

For example, later on we'll use normal cdf and $g(s) = \log(s)$. Now the considered model has 6 parameters: q_0, q_1, r, R, k_M, S_f. They have following interpretation:

q_0, q_1 are parameters of cdf of strength of composite item (strand or fiber); for example, if $g(s)=s$ (normal distribution of strength) then q_0 is expectation value and q_1 is standard deviation of item strength;

R is the number of composite items in critical volume, failure of which corresponds to the total failure of specimen;

r is a critical number of failed elements inside of this critical volume, corresponding to failure of this volume; the ratio r/R defines the part of the cross section area, the destruction of which we consider as failure of specimen; the value r defines mainly the variance and coefficient of variation of fatigue life;

k_M is number of cycles corresponding to one step in MC;

S_f is residual strength of failed item (it depends on the orientation and number of layers, the characteristics of matrix,...).

So now on we'll use $F_T(t; S, h)$ as specific notation of cdf of random variable T instead of more general notation, $F_T(t)$.

2. Estimation of parameters of model

Formulae (1), (2) can be used in both direction: for calculation of mean and p-quantile fatigue curves, if parameters are known, or for nonlinear regression analysis for model parameters estimation, if fatigue life dataset is known. Mean and p-quantile fatigue curves are defined by formulae

$$E(T(S_j)) = k_M \sum_{i=1}^{r} 1/p_i(S_j, h) \, ,$$

$$t_p(S_j) = F_T^{-1}(p; S_j, h) \, .$$

where $E(T(S_j))$, $t_p(S_j)$ are mean value and p-quantile of fatigue life for stress S_j.

The parameters of the model can be estimated by the use of Method of Moments (MM), Least Square Method (LSM) and Maximum Likelihood Method (MLM), which is more preferable. For the profound investigation of this model can be recommended nonlinear regression procedure of SAS system. But in any case finding 6 unknown parameters is a very difficult problem. So we limit ourselves to only approximate solution of this problem. First of all we put: $k_M = 1$, $S_f = 0$, then we'll get approximate estimation of remain parameters: r, R, q_0 and q_1, and, finally, for fixed approximate estimates of parameters r, R we can find estimates of q_0 and q_1 by the use of MLM.

Approximate estimate of parameter r can be found, if we assume, that approximately

$$p_1 = p_2 = ... = p_r = p \, ,$$

then

$$E(T) \cong \frac{r}{p} \, ; \; V(T) \cong \frac{r}{p^2} \, ;$$

coefficient of variation

$$C_V = \sqrt{V(T)} / E(T) \cong 1/\sqrt{r} \, .$$

And approximate estimate of parameter r is defined by formula

$$\hat{r} \cong]1/(\hat{C}_V)^2[+1,$$

where] x [is the nearest integer towards minus infinity.

Value $E(T) \cong \frac{r}{p}$ is very large (10^5-10^7!!!), r is small enough (see section 4) so the value of p is very small and $F(s)$ is very small too. All this gives us idea to make following enough serious assumption (not too bad final result is the only justification of it!):

$$p_i = 1-(1-F(s_i))^{R-i} \cong p \cong (R-r)F_0((g(S)-q_0)/q_1)$$

for all i=1,2,…,r.

Then we have the following approximate formula

$$E(T(S)) = \frac{D_f}{F_0((g(S)-q_0)/q_1)}, \qquad (5)$$

where $D_f = r/(R-r)$.

Then at fixed D_f we can get following linear regression model

$$y_i = F_0^{-1}(D_f / E(T(S_i))) = -q_0/q_1 + (1/q_1)g(S_i) = b_0 + b_1 x_i,$$
$$i = 1,2,...,n. \qquad (6)$$

Parameters b_0 and b_1 of this model can be estimated by the use of some statistical program of linear regression analysis at every fixed value of parameter D_f. And it is not too serious problem to find only one nonlinear parameter D_f. Then we have following estimates for q_0 and q_1:

$$\hat{q}_1 = 1/\hat{b}_1, \ \hat{q}_0 = -\hat{b}_0/\hat{b}_1.$$

Estimate of parameter *R* can be made after estimation of ratio $r = r/R$. Remind, that this ratio defines the part of the cross section area, the destruction of which we consider as total failure of specimens. In the Daniels's model of static strength this value corresponds to the value of $F(x^*)$, where $x*$ is such, that

$$x^*(1-F(x^*)) = \max_x x(1-F(x)) \ [4].$$

We can estimate this value, using estimates of q_0 and q_1. So we have

$$\hat{r} = F(x^*),$$
$$\hat{R} =]1/((C_V)^2\hat{r})[+1 \qquad (7)$$

Now we have approximate estimates of all four parameters q_0 and q_1, *r* and *R*. At the fixed estimates of *r* and *R* the more precise estimates of q_0 and q_1 can be found by the use of MLM. For probability mass function now we have following formula

$$p_T(t;S,h) = F_T(t;S,h) - F_T(t-1;S,h).$$

But for calculation of cdf we should get P^t. It needs too much time. So we try to find some approximation for cdf. By comparison of normal and lognormal approximation it appears, that lognormal approximation is more appropriate:

$$F_T(t;S,h) \cong \Phi\left(\frac{\log(t)-q_{0LT}}{q_{1LT}}\right),$$

where q_{0LT}, q_{1LT} are such, that we have the same expectation value and standard deviation

$$q_{0LT} = \log(E(T)) - (\log(C_V^2+1))/2,$$

$$q_{1LT} = (\log(C_V^2+1))^{1/2}.$$

Now the maximum likelihood function in logarithm scale

$$l(h) = \ln(L(h)),$$

where $L(h) = \prod_{i=1}^{n} f_i^{A_i}(1-F_i)^{1-A_i}$, f_i, F_i are probability density function and cumulative distribution function of random variable T (for fixed h and S) ; A_i is equal to 1, if fatigue test is finished by the failure of specimens, and , A_i is equal to 0, if the time of test is limited (right censored observation).

Until now we have considered mainly uniformly load-shearing system of isolated parallel items loaded by tension. But we are supposed to apply this model to the more complex structure, for which fracture of longitudinal items means the failure of specimen as a whole.

3. Numerical example

For numerical example we consider the problem to fit the experimental data of fatigue test of laminate panel. These data was kindly given to the authors by professor W.Q. Meeker, who studied them and gave the following description of these data: "the data come from 125 specimens in four-point out-of-plane bending tests of carbon eight-harness-satin/ epoxy laminate [5]. Fiber fracture and final specimen fracture occurred simultaneously. Thus, fatigue life is defined to be the number of cycles until specimen fracture. The dataset includes 10 right censored observations (known as "runouts" in the fatigue literature)". We have considered already extreme values of fatigue lives for 5 stress levels, which we have got from the Figure 1 in 5 paper [1, 2, 3]. But now we have original information, the same on the base of which the fatigue curve of this figure was made. And, as it was told already, this time we decrease the numbers of model parameters in order to increase the stability of others parameters. We put k=1, S_f = 0. Then four main steps were made for estimation of parameters q_0, q_1, *r* and *R*.

First step. We can make additional assumption, that static strength of items has lognormal distribution: $F_0(.)$ is cdf of standard normal distribution, $g(s) = \log(s)$. By the use of regression analysis (and by sequence of calculation for different D_f) it was found approximate parameter estimates: $\hat{q}_0$=7.6906, $\hat{q}_1$=0.3541 and $\hat{D}_f$ = 0.0229 [4].

Second step. Estimation of *r*. For this purposes the calculation of coefficient of variation C_V =0.5839 for some middle stress, at which there was not censuring, was calculated (for S=340 MPa) . So $\hat{r}$ =3.

Third step. We have got estimate $\hat{R} = 15$, because at obtained approximate estimates $\hat{q}_0$, $\hat{q}_1$, the value of $\hat{r} = F(x^*)$ appears to be equal to 0.2072 (later on for final MLM the same estimate appears to be equal to 0.2022 and estimate $\hat{R} = 15$ did not change). Remind, that x^* is such, that:

$$x^*(1 - F(x^*)) = \max_x x(1 - F(x)) \text{).}$$

Fourth step. The estimation of parameters q_0 and q_1 by the use of MLM appears to be very difficult problem: there are several local maximum of this function as we can see in Fig 1. The projections of this function on the plains q_0=Const and q_1=Const (it is functions: $\max_{q_0} l(h)$ and $\max_{q_1} l(h)$) are shown in Fig 2, 3. The counter lines of equal levels are shown in Fig 4. After detail analysis it was found: $\hat{q}_0$ = 7.6461, $\hat{q}_1$ = 0.34471.

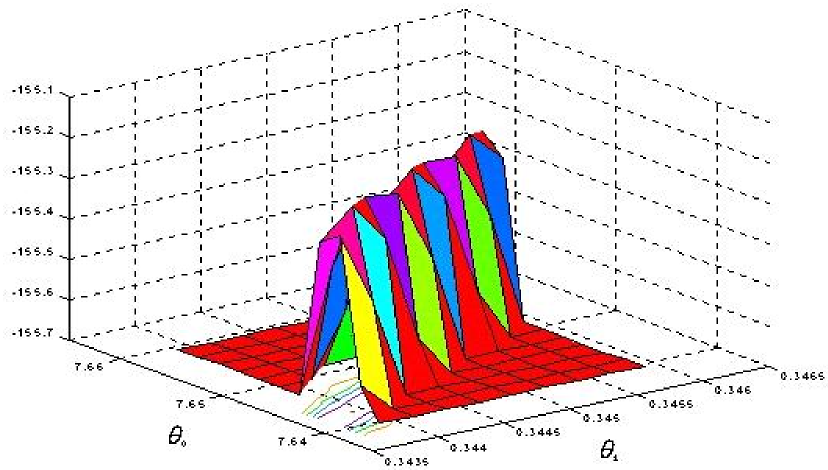

Fig 1. Likelihood function

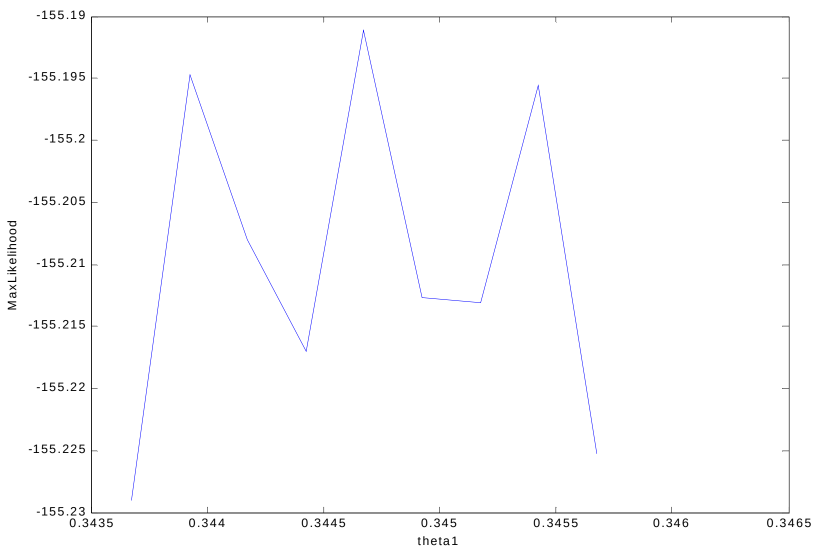

Fig 2. Projection of likelihood function on the plain q_0=Const: $\max_{q_0} l(h)$

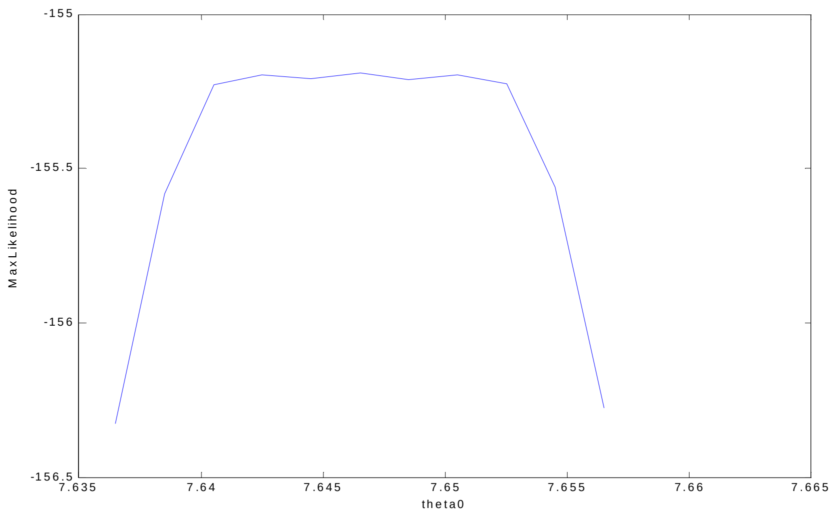

Fig 3. Projection of likelihood function on the plain q_1=Const: $\max_{q_1} l(h)$

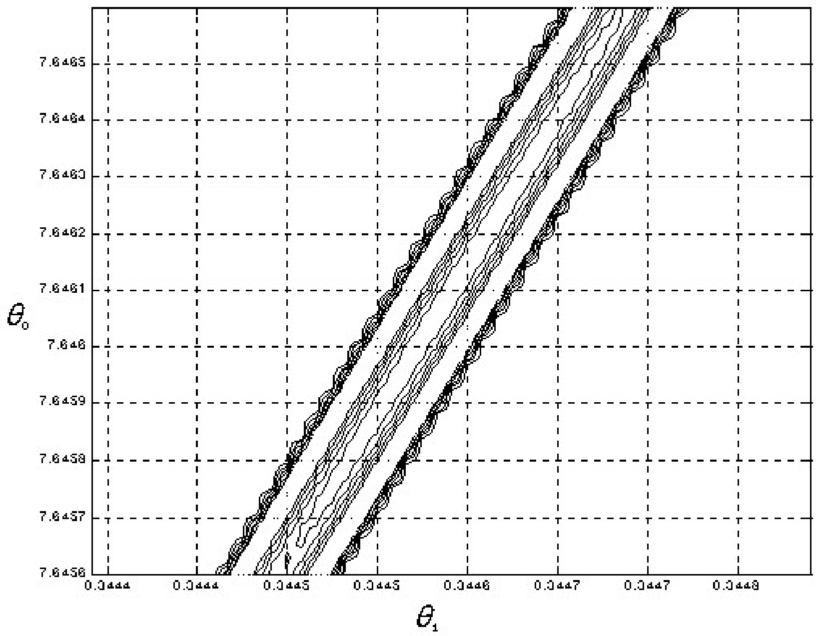

Fig 4. Contour lines of equal levels of likelihood function

It was found: $\hat{q}_0$ = 7.6461, $\hat{q}_1$ =0.34471.

Now we can check the validation of lognormal approximation of $F_T(t;S,h)$. We consider two hypotheses

$$F_T(t;S,h) \cong \Phi\left(\frac{\log(t) - q_{0LT}}{q_{1LT}}\right)$$

and

$$F_T(t;S,h) \cong \Phi\left(\frac{t - q_{0T}}{q_{1T}}\right).$$

In first case we should get straight line

$$\log(t) = q_{0LT} + q_{1LT}\Phi^{-1}(F_T(t;S,h)), \quad (6)$$

In second case

$$t = q_{0T} + q_{1T}\Phi^{-1}(F_T(t,S,h)). \quad (7)$$

For this purpose we calculate the cdf and pdf for stress level S = 290.1 MPa.

The result of calculation is shown in Fig 5. We see that formula (6) gives nearly straight line, so the lognormal approximation of $F_T(t;S,h)$ is more appropriate than normal approximation.

Finally we have got the fatigue curve. The experimental data (×) and results of calculations of the expectation values of extreme order statistics (o) (minimum and maximum) are shown in Figure 6.

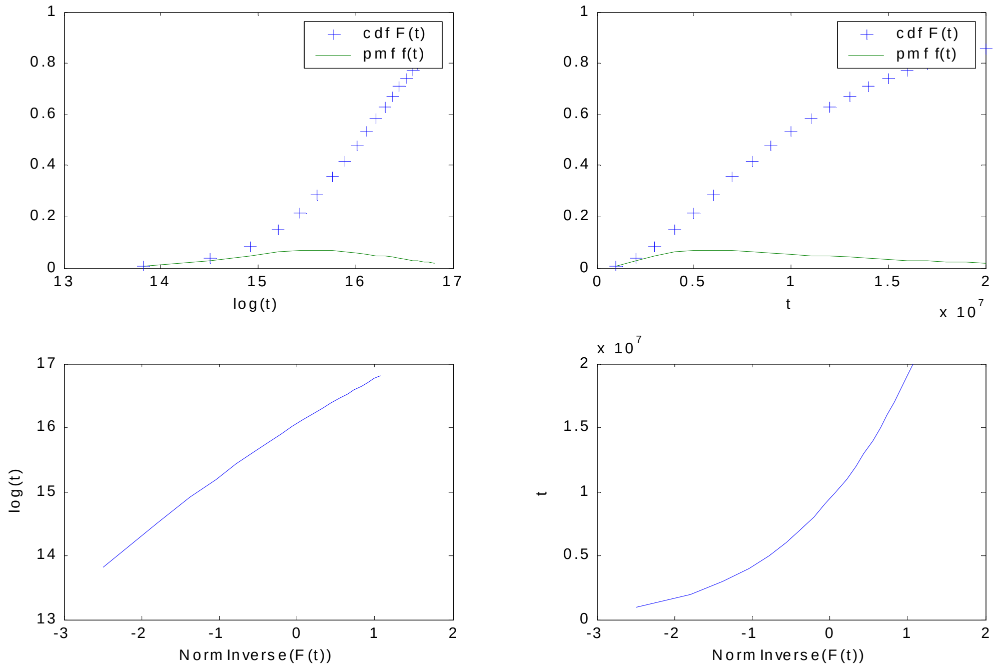

Fig 5. Cumulative distribution (×) and probability mass functions (-) in the upper part. Functions $\log(t)=q_{0LT}+q_{1LT}\Phi^{-1}(F_T(t;S,h))$ and $t=q_{0T}+q_{1T}\Phi^{-1}(F_T(t,S,h))$ in the lower part

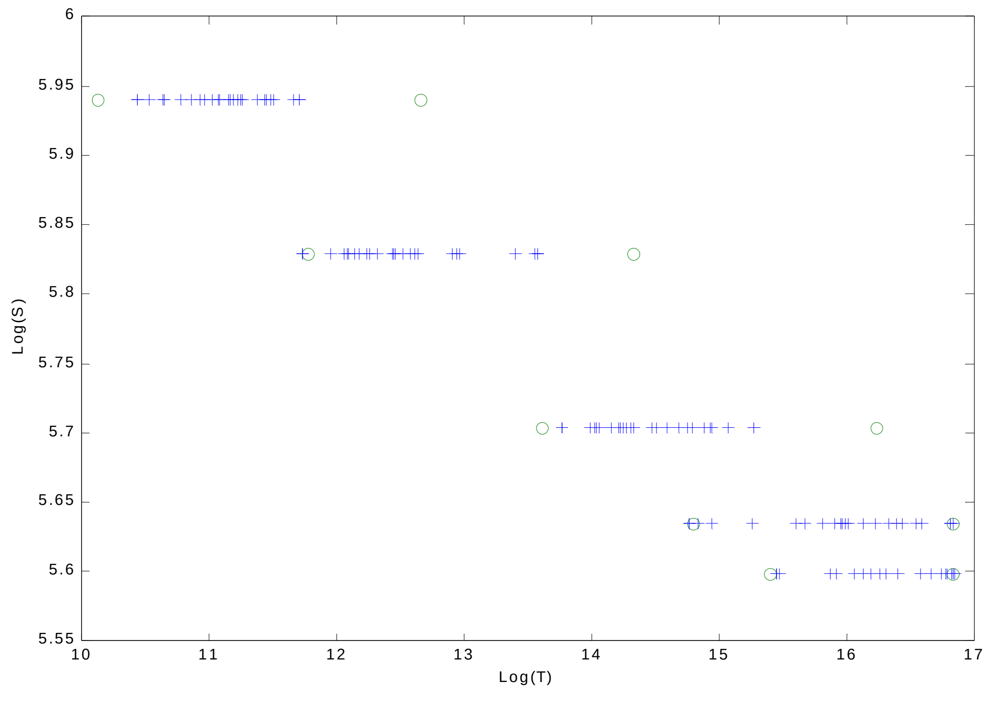

Fig 6. The experimental data (×) and results of calculations of the expectation values of extreme (minimum and maximum) orders statistics (o)

Conclusions

1. S*imple Markov Chain Model of Fatigue Life* (SMCMFL) of composite material can be used as nonlinear regression model for fatigue curve approximation. Processing dataset of fatigue test of carbon-fiber laminate specimens we have got estimates of parameters, which can be interpreted as ***equivalent local*** *static strength distribution parameter estimates.*

We think that probably there is some discrepancy between these estimations ($\hat{q}_0$ = 7.6461, $\hat{q}_1$ =0.34471) and real parameters of static strength distribution of carbon fibers, which are unknown for us. For example, in [4] the following estimations of carbon fiber static strength distribution are given: $q_0=7.198$, $q_1=0.467$. This discrepancy can be explained by the difference of original material, difference of load types (bending instead of tension), difference of "effective" length of fibers and so on. If we take into account all these circumstances, it seems that it is not too bad result for estimation of **static strength distribution function parameters** on the base of **fatigue data**.

2. This discrepancy can be used for description of specific features of specific structure.

3. Really **we do not need** to estimate the static strength distribution parameters on the base of fatigue data. We considered the **likelihood of these estimates** only as a proof, that considered SMCMFL of composite material **has right to exist** and it can be used, for example, for forecasting fatigue curve changes, when there are some changes of real static strength distribution parameters.

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