USING INTERVAL-VALUED ENTROPY FOR RISK ASSESSMENT

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In this paper we propose to use entropy as a measure of risk. We describe the framework of our approach towards risk assessment, which can be said to be information theoretical in some sense. We present different criteria as different systems and different values of criteria as different states of these systems. Probabilities of a system being in particular states are interval-valued. In our paper we present the problem of entropy generalization to the interval-valued case and we show some of the possible solutions, viz., ways of generalization.

1 Introduction

Most of the decisions that we have to make in the real life situations are made under high degree of uncertainty, as available information is frequently imprecise and/or incomplete. In [1, 2] we present an approach to decision analysis based on fuzzy granules, which enables one to use imprecise and/or incomplete information to describe alternatives and to make decisions using such imperfect information. A good source of information on the notion of granularity adopted in our approach can be found in [3].

In [4, 5] we consider the possibility of risk evaluation, which is based on the framework for decision analysis developed in [1]. In the next chapter we consider the general structure of our approach towards risk analysis and later on we proceed to examination of interval-valued entropy.

2 Framework for Risk Analysis

Usually risk is related to uncertainty of future events. In other words, risk is related to lack of knowledge about future events. So it would be natural to define risk as the *amount* of lacking information. Thus, in this approach we adapt the idea that risk reflects how much we do not know about the future.

Entropy is the basic notion in the information theory field. Informally we can define entropy as the measure of uncertainty of a system that at a given moment can be in one of the states, where set of all the possible states is defined and the probability that system is in some state is known.

We can define entropy formally as follows. Let us assume that there are n different states that a system can be in:

$$s_1, s_2, \dots, s_n.$$

With the formula (1) we will denote that the probability that system S is in state s_i is p_i:

$$P(S \sim s_i) = p_i, \quad i = 1 \,..\, n. \tag{1}$$

If values of all p_i in (1) are known, then entropy of a system can be calculated according to the following formula:

$$H = -\sum_{i=1}^{n} p_i \log p_i \,. \tag{2}$$

For further information on entropy and information theory you can refer to [6].

As was mentioned before, in our approach we use entropy as a measure of risk. Entropy is a notion from information theory and it is a measure of "uncertainty" of a system. Entropy can be interpreted as the amount of lacking information about a system, which shows how much we do not know about a system. In our approach we use this notion of uncertainty as one of the risk factors, which can be used as one of the criteria to rank alternatives.

Thus, it can be said that our approach is information theoretical, as we present each alternative as a set of different systems, each corresponding to a particular criterion.

Figure 1 presents our approach towards risk analysis. First we have to define what criteria are of interest for us. In an example presented in Figure 1 it can be seen that each alternative is evaluated using L criteria. After that for each alternative we construct fuzzy granules that describe each criterion and evaluate the probabilities that a particular criterion will take some value. The probabilities are interval-valued. A particular value of some criterion corresponds to a separate state in a system. For example, in Figure 1 the first system corresponds to $criterion_1$, which can take n different values with corresponding probabilities. After having evaluated probabilities of values for all the criteria we can calculate entropy for each system. In order to get evaluation of overall entropy, we can sum entropies calculated for separate systems. The overall entropy value obtained can be considered as evaluation of uncertainty for a particular alternative. We should repeat this procedure for each alternative.

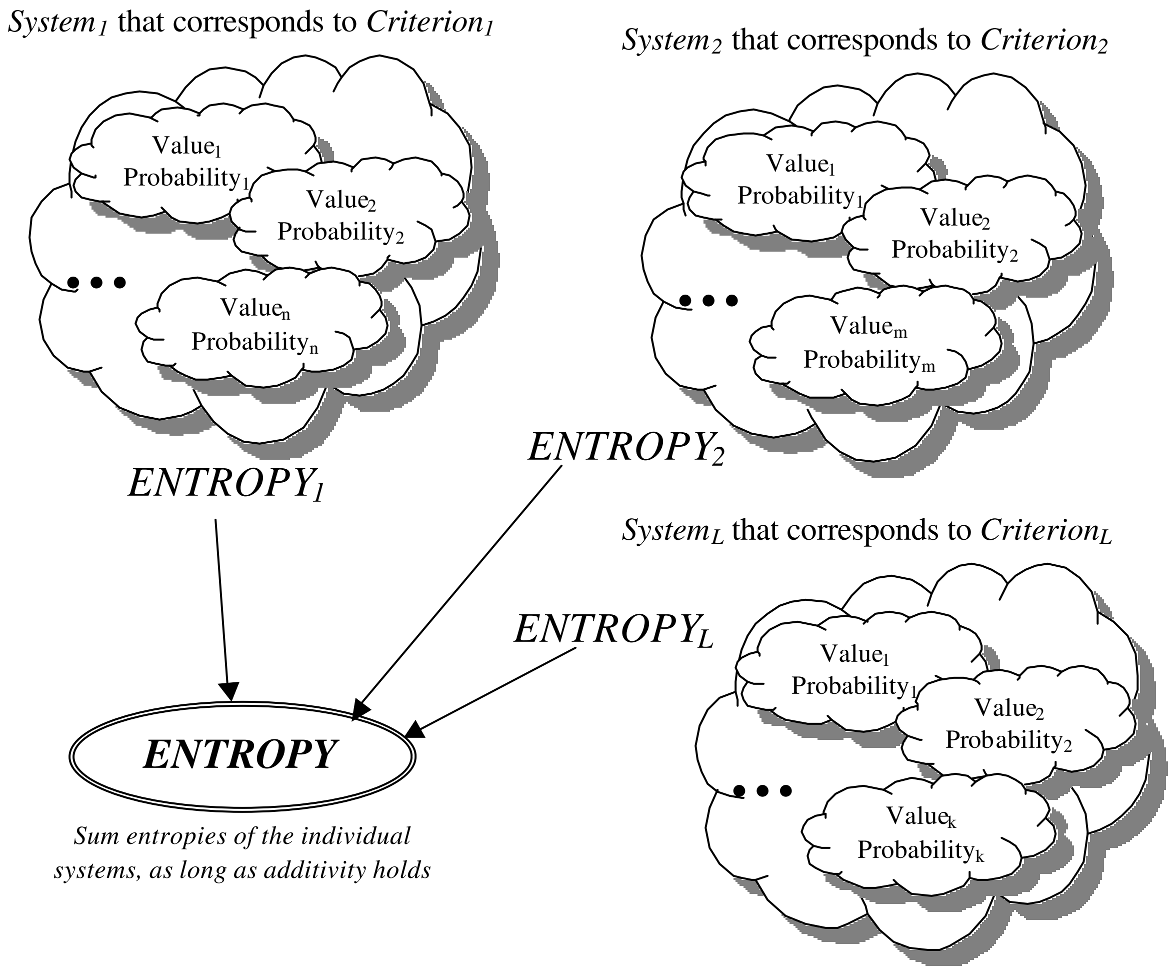

Figure 1. Entropy-based risk evaluation

It should be noted, that in order to calculate the overall entropy by summing up individual entropies, additivity must hold for the generalized definition of entropy. We will discuss this topic when we present generalized definition of entropy.

3 Interval-Valued Entropy

Let us consider how entropy can be calculated if we are dealing with interval-valued probability values. Obviously, the entropy itself will be interval.

In this section we show how a generalised definition of entropy can be obtained, which is suitable for interval-valued probabilities. First we present a general problem statement, which formally describes the problem of finding interval-valued entropy. Then we proceed to possible generalizations of entropy and to the study of additivity feature.

3.1 General Problem Statement

As before, we assume that a system can be in n states, but the probability that the system is in i-th state is interval and is equal to $[p_i^{min}, p_i^{max}]$. In case when probabilities are single-valued rather than interval-valued, it is required that probabilities sum to 1, i.e.

$$\sum_i p_i = 1. \quad (3)$$

If probabilities are interval-valued, then (3) can be rewritten as (4):

$$\sum_i p_i^{\min} \le 1 \le \sum_i p_i^{\max}. \quad (4)$$

It is easy to show that (3) is a special case for (4) when $p_i^{min} = p_i^{max}$ for each i.

General problem statement for the calculation of the interval-valued entropy can be put down as the following optimisation problem.

Let $[p_1^{min}, p_1^{max}], [p_2^{min}, p_2^{max}], \ldots, [p_n^{min}, p_n^{max}]$ be interval-valued probabilities, then lower and upper boundary of entropy can be calculated as:

$$H^{\min} = \min\left(-\sum_{i=1}^{n} p_i \log p_i\right) \text{ and } H^{\max} = \max\left(-\sum_{i=1}^{n} p_i \log p_i\right),$$

subject to:

$$p_i^{\min} \le p_i \le p_i^{\max}, \qquad i = \overline{1,n}.$$

$$\sum_{i=1}^{n} p_i = 1$$

Unfortunately, if the problem is presented in this way, it is unfeasible computationally. In the following chapter we present several rather heuristic approaches, which give approximate solutions to the aforementioned problem.

3.2 Heuristic Interval-Valued Entropy

There are several possibilities for generalizing interval-valued entropy, based on different heuristic approaches. First, we can generalize formulae used for single-valued entropy to the case of interval-valued entropy. Second, we can look at the intervals themselves and try to reason about the entropy bounds based on this information. Third, we can use techniques such as genetic algorithms to find upper and lower bounds of the entropy. Let us consider these approaches in more detail.

3.2.1 Formulae Generalization to the Case of Interval-Valued Probabilities

First let us consider the possibility of single-valued entropy formula generalization to the case of interval-valued entropy.

We cannot use formula (2) directly, as the function $h=-p \log p$ is not monotonic. However, if we define p_i^{avg} as (5), then it can be shown that (6) holds.

$$p_i^{avg} = \frac{p_i^{\min} + p_i^{\max}}{2}, \quad (5)$$

$$\forall i : p_i^{\min} \le p_i^{\max} \Rightarrow -p_i^{avg} \log p_i^{\min} \ge -p_i^{avg} \log p_i^{\max}. \quad (6)$$

It should be noted that states with lower probability values are more informative. Thus, we can expect that in order to calculate the upper boundary of entropy $H^{\max}$ we should use the lower probability bounds p_i^{min}. We can find the upper and the lower boundaries of entropy for a system as follows:

$$H^{\max} = -\sum_{i=1}^{n} p_i^{avg} \log p_i^{\min} \quad (7)$$

$$H^{\min} = -\sum_{i=1}^{n} p_i^{avg} \log p_i^{\max} \quad (8)$$

From (6) it follows that $H^{\min} \le H^{\max}$. Moreover, as was mentioned above, entropy for a system with interval-valued probability is interval-valued as well and is equal to (9).

$$H = [H^{\min}, H^{\max}]. \quad (9)$$

The obtained definition of interval-valued entropy is generalisation of the 'traditional' entropy of a system with single-valued probabilities. Now let us examine whether the additivity feature holds for the generalized version of entropy defined in (7), (8).

If additivity holds, it means that if we have two independent systems, say, X and Y, then entropy of a system that is obtained by joining systems X and Y is equal to the sum of individual entropies for X and Y. In other words, if additivity holds, then (10) holds:

$$\mathrm{H(X, Y) = H(X) + X(Y)}. \qquad (10)$$

If we define entropy as (7) and (8), then it can be shown that if we have two systems X and Y with states accordingly $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$, moreover, $P(X{\sim}x_i)=p_i$ and $P(Y{\sim}y_j)=r_j$, then

$$H^{\min}(X,Y) = -\left(\sum_{i=0}^{n} p_i^{q.avg} \log p_i^{\max} + \sum_{j=0}^{m} r_j^{q.avg} \log r_j^{\max}\right) \approx H^{\min}(X) + H^{\min}(Y),$$

and $$H^{\max}(X,Y) = -\left(\sum_{i=0}^{n} p_i^{q.avg} \log p_i^{\min} + \sum_{j=0}^{m} r_j^{q.avg} \log r_j^{\min}\right) \approx H^{\max}(X) + H^{\max}(Y),$$

where $$p_i^{q.avg} = \frac{\left(p_i^{\min}\sum_{j=0}^{m} r_j^{\min} + p_i^{\max}\sum_{j=0}^{m} r_j^{\max}\right)}{2} \text{ and } r_j^{q.avg} = \frac{\left(r_j^{\min}\sum_{i=0}^{n} p_i^{\min} + r_j^{\max}\sum_{i=0}^{n} p_i^{\max}\right)}{2}. \qquad (11)$$

As can be seen, (11) does not differ from (5) much. In (11) summation factors appear. If we are dealing with single-valued probabilities then it is obvious that these sums are equal to *1* and the additivity holds. If the probabilities are interval-valued then from (4) it follows that formulae (12) hold.

$$0 \le \sum_{j=0}^{m} r_j^{\min} \le 1, \qquad \sum_{j=0}^{m} r_j^{\max} \ge 1, \qquad 0 \le \sum_{i=0}^{n} p_i^{\min} \le 1, \qquad \sum_{i=0}^{n} p_i^{\max} \ge 1. \qquad (12)$$

Summation factors (12) can be considered as a sort of scaling factors, where the first may have reducible influence and the second may have augmenting influence, so we may expect that two of these factors compensate each other. Hence, entropy for the joined system *should not* differ much from the sum of individual entropies of the systems considered. Thus, it can be concluded that *quasi-additivity* holds, as the summation factors compensate each other to some extent.

3.2.2 Looking at the Intervals

We can determine the approximate bounds of the entropy by looking at the probability values. The probabilities are interval-valued, so by looking at the interval we can determine what is the maximum possible and the minimum possible contribution of a given probability to the value of entropy. Figure 2 summarizes this approach.

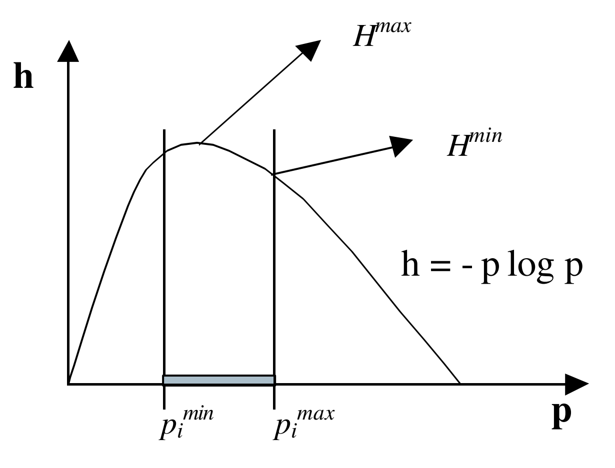

Figure 2. Find boundaries of entropy by examining intervals of probabilities

Minimum and maximum values from each interval contribute to H^{min} and H^{max}. After considering each probability value we obtain evaluation of lower and upper boundaries of the entropy sought.

It can be shown that *quasi-additivity* holds for this heuristic approach as well.

Let us use notation from the chapter 3.2.1, let $P_{ij}^{\min} = p_i^{\min} r_j^{\min}$ and $P_{ij}^{\max} = p_i^{\max} r_j^{\max}$. It can be shown that:

$$H^{\min}(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} \min_{P_{ij}=P_{ij}^{\min}..P_{ij}^{\max}} (-P_{ij}\log P_{ij}) \approx H^{\min}(X) + H^{\min}(Y), \quad (13)$$

$$H^{\max}(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} \max_{P_{ij}=P_{ij}^{\min}..P_{ij}^{\max}} (-P_{ij}\log P_{ij}) \approx H^{\max}(X) + H^{\max}(Y). \quad (14)$$

If probabilities are single-valued then (13) and (14) yield the same result as (2), but the precision decreases as intervals become wider and more uncertain. However, this result does not contradict our intuition: if the system is supplied with uncertain information, then a precise answer obtained from this system would be misleading.

3.2.3 Genetic Algorithms

We can consider using genetic algorithms to solve the problem stated in chapter 3.1. However, this approach requires careful study. We should obtain some sort of evaluations of the confidence level of this approach before we apply it. Moreover, it would be rather difficult to show formally whether additivity holds for genetic-algorithm-based assessments. However, intuitively it seems that some sort of *quasi-additivity* will hold in this approach as well, as additivity holds for (2) and in chapters 3.2.1. through 3.2.3. we present different ways for the generalization of the entropy definition given by (2).

4 Conclusion

In this paper we presented an entropy-based approach towards risk analysis. The idea of using entropy as a measurement of risk is not new. However, in our approach we deal with interval-valued probabilities. Thus, the entropy definition should be generalized. We present the formal problem statement, which is an optimisation problem. Computationally it is hard to solve the presented problem directly, so we propose several heuristic approaches towards entropy generalisation, which are approximate solutions to the presented formal problem statement.

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