STOCHASTIC MODEL OF AN ADAPTIVE SAMUEL-MARSHALL TYPE SINGLE COMPONENT MARKET

Vineta Minkevica and Karlis Sadurskis
Institute of Information Technology
Riga Technical University
1 Kalku street, Riga, LV-1658, Latvia
E-mail: vineta@itl.rtu.lv;
skarlis@latnet.lv

ABSTRACT

A mathematical model of an adaptive Samuel-Marshall type single component market described by quasi-linear functional differential equations with dependent on phase coordinates and frequently switched an ergodic Markov process is presented. The proposed method is based on an averaging procedure with respect to time along the critical solutions of the generative average linear equation and with respect to the invariant measure of the Markov process. It is proved that exponential stability of the resulting deterministic equation is sufficient for exponential $p-$stability of the initial random system for all positive numbers p and for sufficiently fast switching.

INTRODUCTION

Let us consider the $n-$dimensional functional differential equation in a quasi-linear form with a small parameter $\varepsilon \in [0,1]$

$$\frac{du^{\varepsilon}(t)}{dt} = \int_{-h}^{0} \{ dG(\theta, y(t/\varepsilon))\} u^{\varepsilon}(t+\theta) + \varepsilon F(t, u_t^{\varepsilon}, y(t), \varepsilon), \qquad (1)$$

where $\{y(t), t \geq 0\}$ is a homogenous ergodic Feller type Markov process (Dynkin 1965) on the probability space $(\Omega, \mathsf{F}, \mathbf{P})$ with values in the compact phase space $\mathbf{Y}$, with infinitesimal operator Q, transition probability $P(t, y, dz)$ and unique invariant measure $\mu(dy)$ satisfying the condition of exponential ergodicity, that is, there exist positive constants M and δ such that $\| P(t, y, .) - \mu \| \leq M \exp\{-\delta t\}$ for any $t \geq 0$; ε is a small positive parameter; u_t^{ε} is a part of solution defined by the equality $u_t^{\varepsilon} = \{ u^{\varepsilon}(t+\theta), \theta \in [-h, 0]\}$ with some positive number h; $G(\theta, y)$ is a matrix consisting of bounded variation on θ functions; the perturbing term $F(t\varphi, y, \varepsilon)$ is a continuous mapping of the product space $\mathbf{R}_+ \times \mathbf{C}_n([-h, 0]) \times \mathbf{Y} \times [0,1)$ to the space $\mathbf{R}^n$, satisfying $F(t, 0, y, \varepsilon) \equiv 0$ and the Lipschitz condition on the second argument for any $y \in \mathbf{Y}$, $\varepsilon \in [0,1)$, $t \in \mathbf{R}_+$. Under these conditions the random equation (1) with initial problem $u^{\varepsilon}(s+\theta) = \varphi(\theta)$, $-h \leq \theta \leq 0$ has (Hale and Sjord 1993) a unique solution $u^{\varepsilon} = \{u^{\varepsilon}(t), t \geq 0\}$ for any continuous function φ. This solution is a continuous stochastic process with probability one.

Averaging the linear part of the equation (1) according to the invariant measure of Markov process

$$\overline{G}(\theta) = \int_{\mathbf{Y}} G(\theta, y)\mu(dy),$$

One can define a generative equation for the equation (1):

$$\frac{d\overline{x}(t)}{dt} = \int_{-h}^{0} \{ d\overline{G}(\theta)\} \overline{x}(t+\theta). \qquad (2)$$

It is well known (Hale and Sjord 1993) that equation (2) defines in the space $\mathbf{C}_n$ a strong continuous semigroup $T(t)$ with infinitesimal operator given for sufficiently smooth function φ by

$$(\mathbf{A}\varphi)(\theta) = \begin{cases} \dfrac{d\varphi(\theta)}{d\theta}, & -h \leq \theta < 0, \\ g(\varphi), & \theta = 0. \end{cases}$$

The spectrum $\sigma(\mathbf{A})$ of this operator is given by $\sigma(\mathbf{A}) = \{ z : \det\{U(z)\} = 0\}$ where

$$U(z) = Iz - \int_{-h}^{0} e^{z\theta} d\overline{G}(\theta).$$

If $\sigma(\mathbf{A}) \cap \{z : \mathrm{Re}\, z > 0\} = \varnothing$ and $\sigma_0 = \sigma(\mathbf{A}) \cap \{z : \mathrm{Re}\, z = 0\} \neq \varnothing$ the generative equation is on the border of stability and some preliminary preparation is necessary in order to obtain the resulting averaged equation.

We will refer to the spectral subspace of the operator $\mathbf{A}$ corresponding to σ_0 as the critical subspace and to the solutions of (2) lying in the critical subspace as the critical solutions. One has to note that the selection of the first linear term in the right hand part of equation (1) can be done somewhat arbitrarily. One can add any arbitrary linear continuous mapping $\varepsilon g_1(\varphi)$ to the linear part to (1) and subtract it from the second term. Using this arbitrariness and because the set σ_0 consists of the finite number of points (Hale and Sjord 1993) $\sigma_0 = \{z_j, j = 1, 2, ..., m\}$ it may be assumed that the selection of the terms in the right hand part of (1) has been done in such manner so that $(\det U(z_j))' \neq 0, \quad j = 1, 2, ..., m$.

One can apply the projective operator P_0 not only on any continuous vector-function $\psi(\theta)$, but also on any

Proceedings 12th International Conference ASMTA 2005
Khalid Al-Begain, Gunter Bolch, Miklos Telek © ECMS, 2005
ISBN 1-84233-112-4 (Set) / ISBN 1-84233-113-2 (CD)

vector- or matrix-valued measurable function. Primarily one needs to rewrite equation (1) in the operator form (Hale and Sjord 1993; Tsarkov 1989)

$$\frac{du_t^\varepsilon}{dt} = \mathbf{A}u_t^\varepsilon + \varepsilon \mathbf{I}F(t, u_t^\varepsilon, y(t), \varepsilon), \qquad (3)$$

where the matrix-valued function $\{\mathbf{I}(\theta), -h \le \theta \le 0\}$ is defined by the equality

$$\mathbf{I}(\theta) = \begin{cases} 0, & \text{if } -h \le \theta < 0, \\ I, & \text{if } \theta = 0 \end{cases}$$

and I is the $n \times n$ identity matrix. Thereafter one must define the spectral projective operator P_0 corresponding to $\sigma_0 \subset \sigma(\mathbf{A})$. We will use its integral representation for that (Kato 1966) in the form

$$(P_0\psi)(\theta) = \frac{1}{2\pi i}\int_{\mathbf{B}} ((Iz - \mathbf{A})^{-1}\psi)(\theta)dz \qquad (4)$$

where $\mathbf{B} = \bigcup_{j=1}^{m} \{z : |z - z_j| = \delta\}$ with sufficiently small $\delta > 0$. It can bee easily noticed that both projective operators P_0 and $I - P_0$ are bounded. Inserting the above matrix-valued function $\mathbf{I}$ into the integral representation (4) one can define the $n \times n$ matrix-function

$$\Gamma(\theta) = \frac{1}{2\pi i}\int_{\mathbf{B}} ((Iz - \mathbf{A})^{-1}\mathbf{I})(\theta)dz = \sum_{j=1}^{m} \operatorname{res}\{U^{-1}(z)e^{z\theta}\}\Big|_{z=z_j} \qquad (5)$$

Let us denote the critical subspace as $\mathbf{X}_0 = P_0\mathbf{C}_n$, the matrix of a basis in this subspace (consisting of m columns and n rows) as $V(\theta)$, the stricture of the operator $\mathbf{A}$ on $\mathbf{X}_0$ as $\mathbf{A}_0$ and let A_0 be the matrix of this stricture, defined by the equation $\mathbf{A}_0V(\theta) = V(\theta)A_0$. Furthermore, one can define the $m \times m$ matrix $\hat{\Psi}$ by the identity $\Gamma(\theta) = V(\theta)\hat{\Psi}$. Let us use the above notations along with the notation $\mathbf{V} = \{V(\theta), -h \le \theta \le 0\}$ and assume the existence of the $m-$dimensional vector function $\breve{F}(x)$ of the argument $x \in \mathbf{R}^m$ defined by

$$\breve{F}(x) = \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathbf{Y}} e^{-tA_0}\hat{\Psi}F(t, \mathbf{V}e^{tA_0}x, y, 0)\mu(dy)dt \cdot$$

Thus we define the averaged (not random) differential equation

$$\frac{d\breve{x}}{dt} = \breve{F}(x) \cdot \qquad (6)$$

STABILITY THEOREMS

By analogy with the corresponding definition of (Khasminsky 1980) we will say that the trivial solution of the random equation (1) is exponentially $p-$stable in the large for all sufficiently small positive ε if there exist positive constants ε_0, a_1 and positive number $a_2(\varepsilon)$ such that

$$\mathbf{E}_{y,\varphi}^{(s)}\left\{\left|u^\varepsilon(t+s)\right|^p\right\} \le a_1 e^{-a_2(\varepsilon)t}\|\varphi\|^p$$

for any $s, t \ge 0$, $y \in \mathbf{Y}$ $\varphi \in \mathbf{C}_n$ and for any $\varepsilon \in (0, \varepsilon_0)$. Here and further throughout this paper the upper and lower indices of expectation denote the conditions $y(s) = y$, $u_s^\varepsilon = \varphi$.

Theorem 1.

If the trivial solution of (2) is asymptotically stable the trivial solution of the equation (1) is exponentially $p-$stable for any positive p and sufficiently small positive ε.

Theorem 2.

Let the function $F(t, \mathbf{V}x, y, \varepsilon)$ be uniformly continuous at zero as a function of ε; has uniformly bounded continuous $x-$derivative $DF(t, \mathbf{V}x, y, 0)$; belongs to the domain D(Q) of the operator Q; has continuous bounded $t-$derivative $\frac{\partial}{\partial t}F(t, \mathbf{V}x, y, 0)$; has the above defined average $\breve{F}(x)$ along the solutions of the generative equation and there exists constant b such that

$$\sup_{y,T,s}\left|\int_s^{s+T}\int_{\mathbf{Y}} e^{-tA_0}\hat{\Psi}F(t, \mathbf{V}e^{tA_0}x, y, 0)\mu(dy)dt - T\breve{F}(x)\right| \le b|x|$$

for any $x \in \mathbf{R}^m$. Then if the trivial solution of the averaged equation (6) is globally exponentially stable, then the trivial solution of the random equation (1) is exponentially $p-$stable in the large for all sufficiently small, positive ε.

The proof is based on a projection in the critical subspace of the equation (3), transition to the slow time $\tau = \varepsilon t$ and applying the second Lyapunov method with a specially constructed Lyapunov functional as it has been done in (Katafygiotis and Tsarkov 1999) for equations with small Markov perturbations.

THE MODEL AND RESULTS

Discussing a mathematical model of an adaptive Samuel-Marshall type single component market let us consider the equation

$$\frac{dp(t)}{dt} = D(p(t)) - S(p(t-\tau))$$

where $D(p)$ and $S(p)$ denotes dependence of demand and supply on price p. Producer, having at the disposal resources to react on the increase of price, heightens supply immediately. At the opposite case the reaction is delayed by time τ, necessary for production or transportation of goods. Let us give the reasonable interpretation of time delay $\tau = y(t)$ as a Markov process with two states – zero and one. Suppose $y(t)$ to be ergodic homogenous Markov process in space $\mathbf{Y}$ with transition probabilities $P(t, y, dz)$ and invariant measure $\mu \sim \{\pi, 1-\pi\}$. Let us denote $u(t) = p(t) - \overline{p}$ the price deviance from the equilibrium price $\overline{p}$. Let a and b indicate elasticity of demand and supply correspondingly. Using the ratio

$$c = \frac{a}{b} = \frac{D'(\overline{p})}{S'(\overline{p})}$$

and noting that both the supply and demand are non-linear functions we may have the linearized model near the equilibrium price in the form

$$\frac{du(t)}{dt} = b(cu(t) - u(t - y(t))) + \varepsilon F(u_t, y(t)) \cdot \quad (7)$$

By means of $G(\theta, y) = b(c\mathbf{1}_0(\theta) - \mathbf{1}_y(\theta))$, where $\mathbf{1}_\tau(\theta) = \begin{cases} 0, & \theta \in [-1, 0], \theta \notin \tau \\ 1, & \theta = \tau \end{cases}$, one can rewrite the model in the form

$$\frac{du(t)}{dt} = \int_{-1}^{0} u(t+\theta)dG(\theta, y(t)) + \varepsilon F(u_t, y(t)) \cdot \quad (8)$$

To obtain the form useful for the averaging procedure we introduce operators:

$$(\mathbf{A}(y)\varphi)(\theta) = \begin{cases} \frac{d}{d\theta}\varphi(\theta), & -1 \le \theta < 0, \\ \int_{-1}^{0}\varphi(\theta)dG(\theta, y), & \theta = 0. \end{cases}$$

That acquires the equation in the space of continuous functions $\mathbf{C}$:

$$\frac{d}{dt}u_t^\varepsilon = \mathbf{A}(y(t/\varepsilon))u_t^\varepsilon + \varepsilon \mathbf{I}F(t, u_t^\varepsilon, y(t)) \quad (9)$$

Integration by the invariant measure μ provides

$$\frac{d}{dt}\bar{u}_t = \bar{\mathbf{A}}\bar{u}_t \quad (10)$$

where

$$\bar{\mathbf{A}}\bar{u}_t = \int_{\mathbf{Y}} (\mathbf{A}(y)\bar{u}_t)(\theta)\mu(dy) = \begin{cases} \int_{-1}^{0}\bar{u}(t+\theta)d\bar{G}(\theta), & \theta = 0, \\ \frac{d}{d\theta}\bar{u}(t+\theta), & -1 \le \theta < 0 \end{cases}$$

$$\bar{G}(\theta) = \int_{\mathbf{Y}} G(\theta, y)\mu(dy) \cdot$$

Stability conditions depending on the spectrum of operator $\bar{\mathbf{A}}$

$$\sigma(\bar{\mathbf{A}}) = \{\det U(z) = 0\},$$

$$U(z) = I z - \int_{-1}^{0} e^{z\theta} d\bar{G}(\theta),$$

follow:

$$\bar{G}(\theta) = b(c - \pi)\mathbf{1}_0(\theta) - b(1-\pi)\mathbf{1}_{-1}(\theta),$$

$$\frac{d}{dt}\bar{u}(t) = b(c-\pi)\bar{u}(t) - b(1-\pi)\bar{u}(t-1),$$

$$\sigma(\bar{\mathbf{A}}) = \{z : z = b(c - \pi - (1-\pi)e^{-z})\}.$$

As $-1 < c < 0$, it is necessary to have $\pi < \frac{1-c}{2}$ and $b < \frac{\arccos((c-\pi)/(1-\pi))}{\sqrt{(1-c)(1+c-2\pi)}}$ to provide $\sigma(\bar{\mathbf{A}}) \subset \{\operatorname{Re} z < 0\}$.

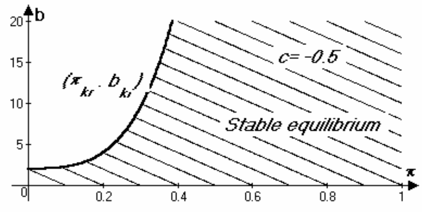

Figure1. *Stability region for the market model.*

In case of $\sigma(\bar{\mathbf{A}}) \cap \{\operatorname{Re} z > 0\} = \varnothing$ and $\sigma(\bar{\mathbf{A}}) \cap \{\operatorname{Re} z = 0\} = \{z_1, z_2, ..., z_m\} \ne \varnothing$, i.e. the spectrum of operator $\bar{\mathbf{A}}$ contains the set $\sigma_0 = \{z_1, z_2, ..., z_m\}$ of m imaginary points one may use the results of *Theorem 2.*

Let us specify the demand and supply functions for the model to provide the case if the system is on the border of stability:

$$D(p) = ap + \varepsilon\gamma p^3 + \alpha$$
$$S(p) = bp + \beta$$
$$\pi = \pi_{kr} - \varepsilon\gamma$$

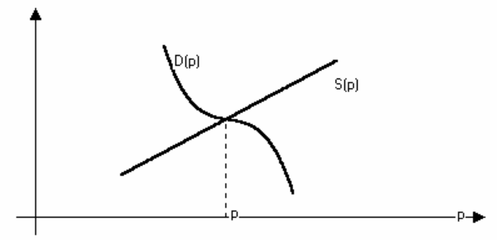

Figure 2. *Specified demand and supply functions for obtaining business cycle.*

Then operator $\bar{\mathbf{A}}$ has a pair of imaginary eigenvalues $\sigma_0 = \{\pm i\nu\}$ with $\nu = \sqrt{(1-c)(1+c-2\pi_{kr}}$. Analysis of the averaged system allows establishing an existence of the stable phase trajectories business cycle on the (D, S) plane for the system (7).

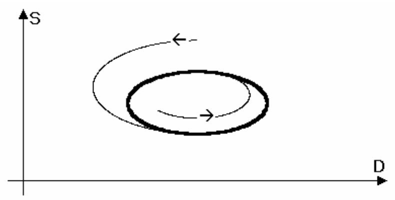

Figure 3. *Business cycle on the demand-supply plane.*

REFERENCES

Dynkin E.B. 1965. “Markov Processes“. Springer-Verlag, Berlin.

Hale J., Sjord M. 1993. “Introduction to Functional Differential Equations”. Springer-Verlag, New York, Hong Kong.

Katafygiotis L., Tsarkov Ye. 1999. “Averaging and Stability of Functional Differential Equations with Markov Parameters”. In: Journal of Applied Mathematics and Stochastic Analysis, 12:1, pp. 1-15.

Kato T. 1966. “Perturbations Theory of Linear Operators”. Springer-Verlag, Berlin, Heidelberg

Khasminsky R. 1980. “Stochastic Stability of Differential Equations” Sijthoff and Noordhoff, Alphen aan den Rijn.

Tsarkov Ye. 1989. “Random Perturbation of Functional Differential Equation”. Zinatne, Riga.