Computer Modelling & New Technologies, 2004, Volume 8, No.2, 40-47
Transport and Telecommunication Institute, Lomonosov Str.1, Riga, LV-1019, Latvia

ESTIMATING OF PARAMETERS OF FATIGUE CURVE OF COMPOSITE MATERIAL

A.YU. PARAMONOVA, M.A. KLEINHOF, YU.M. PARAMONOV

Transport and Telecommunication Institute, Riga Technical University, Riga, Latvia

1. Introduction

Every year the use of composite material in aircraft structure increases. To provide reliability of flight we should study the fatigue phenomenon of this material. One of the main quantitative characteristics of this phenomenon is fatigue curve. There are many offers for its description. Short review of this problem is given in papers [1,2]. We will not repeat it because this paper is in some way a development of [1]. This paper is devoted mainly to processing of dataset of fatigue tests of 125 carbon-fibre laminate specimens. The purpose is to get estimates of parameters of fatigue damage accumulation model, based on the Markov chain theory [1]. It was shown, that, although the model is too simple and does not provide numerical coincidence with experimental fatigue test data, nevertheless it allows to get fatigue curve equation by the use of static strength distribution parameters and some additional parameters, which have some 'physical' interpretation. In this paper we consider inverse problem: by the use of this model, which can be considered now as nonlinear regression analysis model, we'll try to get estimates of local static strength distribution parameters. But it is not the end in itself. The likelihood of these estimates and the likelihood of "theoretical" and experimental fatigue curves can be considered as proof of likelihood of the studied model.

The model discussed in [1] has as 6 parameters all together. For experimental data processing 5 of them is used. In this paper approximately for the same level of precision of fatigue curve description we have used only 4 parameters. It is significant decreasing of the difficulty of statistical analysis. The method of approximate estimation of unknown parameters is offered. Numerical example is given.
Overview. The reminder of the discussed model ideas is given in Section 2. Method for model parameters estimation is discussed in Section 3. Numerical example is given in Section 4.

2. Cumulative damage model based on the Markov chains theory

We consider the process of fatigue damage accumulation as Markov chain with the following matrix of transition probabilities:

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{23} & \cdots & p_{1r} & p_{1(r+1)} \\ 0 & p_{22} & p_{23} & p_{24} & \cdots & p_{2r} & p_{2(r+1)} \\ 0 & 0 & p_{33} & p_{34} & \cdots & p_{3r} & p_{3(r+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & p_{rr} & p_{r(r+1)} \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix},$$

corresponding to Markov Chain (MC) with one (r+1)th absorbing state and *r* non-recurrent states.
Cumulative distribution function (CDF) of time to absorption for this process
$F_T(t) = p_{1\ r+1}(t)$, $t = 1,2,3,\ldots$, where $p_{1\ r+1}(t)$ is the element of first row and (r+1)th column of matrix $P(t) = P^t$. It can be defined also in this way

$$F_T(t) = aP^t b, \qquad (1)$$

where $a = (100...0)$ is the row vector, $b = (00...01)'$ is the column vector. It is worth to notice, that product $P^t b$ gives a column vector of cumulative times distribution functions for absorption, which components correspond to different start states of MC : $(F_T^{(1)}(t), F_T^{(2)}(t),..., F_T^{(r)}(t))'$. In general case it can be used to get CDF, when the probability distribution on start states of MC, $\pi = (\pi_1, \pi_2,..., \pi_r, \pi_{(r+1)})$, is known $F_T(t) = \pi P^t b = \pi\ (F_T^{(1)}(t), F_T^{(2)}(t),..., F_T^{(r)}(t))'$. This possibility should be considered as some reserve, which can be used to take into account some specific features of specific composite structure, induced by some specific technology. Just now we do not use this possibility, because we deliberately try to decrease the number of parameters of the considered model. Moreover, in this paper we consider S*imple Markov Chain Model of Fatigue Life* (SMCMFL) of composite material. It is a model, for which only transitions to the nearest previous states are allowed and

$$P = \begin{bmatrix} q_1 & p_1 & 0 & & & \cdots & 0 \\ 0 & q_2 & p_2 & 0 & & \cdots & 0 \\ 0 & 0 & q_3 & p_3 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & & & \cdots & 0 & q_r & p_r \\ 0 & & & \cdots & 0 & 0 & 1 \end{bmatrix},$$

$q_i=1-p_i$, $i=1,...,r$. The main characteristics of this type MC are well known. A time to failure (time to absorption) $T=X_1+X_2+...+X_r$, where X_i (time the process spends in *i*-th state), $i = 1,...,r$, are independent random variables. A random variable X_i has geometric distribution with a probability mass function (PMF) $P(X_i = n) = (1-p_i)^{n-1} p_i$. Expectation value $E(X_i) = 1/p_i$ and variance $V(X_i) = (1-p_i)/p_i^2$. So probability generating function for random variable T (which can be used to obtain PMF of T) $G_T(z) = \sum_{i=0}^{\infty} p_T(i) \cdot z^i = \prod_{i=1}^{r} \frac{z\, p_i}{1 - z(1-p_i)}$.

The PMF can be calculated also by the use of formula $p_T(t) = F_T(t) - F_T(t-1)$. If we assume, that one step in MC corresponds to k_M cycles in fatigue test, then for calculation of expectation value and variance of T we should use formulae:

$$E(T) = k_M \sum_{i=1}^{r} 1/p_i\,, \tag{2}$$

$$V(T) = k_M{}^2 \sum_{i=1}^{r} (1-p_i)/p_i^2\ . \tag{3}$$

But the main and most difficult problem is to connect these probabilities, p_i, i=1,...,r, with parameter of composite material component strength distribution and applied stress level in such a way that we'll can get the fatigue curve equation. Our offer is to assume that in one step of Markov process (1 cycle or may be 1000 cycles) only one parallel structural item (for example strand) can be failed. If we have *(R-i)* still alive parallel structural items and the same cdf $F(s)$ for every item, then the fracture probability of at least one item is equal $p_i = 1-(1-F(s_i))^{R-i}$, where R is initial number of items, i is the number of items, which are failed already, s_i is the corresponding stress applied uniformly to all *(R-i)* items. We suppose also that $s_i = \frac{SR - S_f i}{R-i} = \frac{S(1 - S_f\, i/SR))}{1 - i/R}$, where S is initial stress (force) in every item (at the start of the test), S_f is stress (force), which already failed item can carry yet (because at least at the beginning of damage accumulation the rupture of fibres can be in different cross sections).

Let us consider the case when cdf *F(s)* has location and scale parameters:

$F(s)=F_0((g(s)-\theta_0)/\theta_1)$, (4)

where g(s) is some known function, $F_0(.)$ is some known CDF. For example, later on we'll use normal CDF and $g(s)=\log(s)$. Now the considered model has 6 parameters: θ_0, θ_1, *r, R,* k_M, S_f. They have the following interpretation:

θ_0, θ_1 are parameters of CDF of strength of composite item (strand or fiber); for example, if $g(s)=s$ (normal distribution of strength) then θ_0 is expectation value and θ_1 is standard deviation of item strength;

R is the number of composite items in critical volume, failure of which corresponds to total failure of specimen;

r is a critical number of failed elements inside of this critical volume, corresponding to failure of this volume; the ratio *r/R* defines the part of the cross section area, the destruction of which we consider as failure of specimen; the value *r* defines mainly the variance and coefficient of variation of fatigue life;

k_M is number of cycles corresponding to one step in MC;

S_f is residual strength of failed item (it depends on the orientation and number of layers, the characteristics of matrix,...).

So now on we'll use $F_T(t;S,\eta)$ as specific notation of CDF of random variable T instead of more general notation, $F_T(t)$.

3. Estimation of model parameters

Formulae (1), (2) can be used in both direction: for calculation of mean and p-quantile fatigue curves, if parameters are known, or for nonlinear regression analysis for model parameters estimation, if fatigue life dataset is known. Mean and p-quantile fatigue curves are defined by formulae

$E(T(S_j))=k_M\sum_{i=1}^{r}1/p_i(S_j,\eta)$, $t_p(S_j)=F_T^{-1}(p;S_j,\eta)$, where $E(T(S_j))$, $t_p(S_j)$ are mean value and p -quantile of fatigue life for stress S_j.

The parameters of the model can be estimated by the use of Method of Moments (MM), Least Square Method (LSM) and Maximum Likelihood Method (MLM), which is more preferable. For the profound investigation of this model can be recommended nonlinear regression procedure of SAS system. But in any case to find 6 unknown parameters is a very difficult problem. So we limit ourselves to only approximate solution of this problem. First of all we put: $k_M=1$, $S_f=0$, then we'll get approximate estimation of the remaining parameters: r,R, θ_0 and θ_1, and, finally, for fixed approximate estimates of parameters r,R we can find estimates of θ_0 and θ_1 by the use of MLM. Approximate estimate of parameter r can be found, if we assume, that approximately $p_1=p_2=...=p_r=p$.

Then $E(T)\cong\frac{r}{p}$; $V(T)\cong\frac{r}{p^2}$; a coefficient of variation $C_V=\sqrt{V(T)}/E(T)\cong 1/\sqrt{r}$.

And approximate estimate of parameter r is defined by formula $\hat{r}\cong]1/(\hat{C}_V)^2[+1$, where] x [is the nearest integer towards minus infinity.

The value $E(T)\cong\frac{r}{p}$ is very large (10^5-10^7!!!), r is small enough (see section 4), so the value of p is very small and $F(s)$ is very small too. All this gives us an idea to make the following serious enough assumption (not too bad final result is the only justification of it!):

$p_i = 1-(1-F(s_i))^{R-i} \cong p \cong (R-r)F_0((g(S)-\theta_0)/\theta_1)$ for all i=1,2,…,r.

Then, we have the following approximate formula

$$E(T(S)) = \frac{D_f}{F_0((g(S)-\theta_0)/\theta_1)}, \tag{5}$$

where $D_f = r/(R-r)$.

Then at the fixed D_f we can get the following linear regression model

$$y_i = F_0^{-1}(D_f / E(T(S_i))) = -\theta_0/\theta_1 + (1/\theta_1)g(S_i) = \beta_0 + \beta_1 x_i,\ i = 1,2,...,n. \tag{6}$$

Parameters β_0 and β_1 of this model can be estimated by the use of some statistical program of linear regression analysis at every fixed value of parameter D_f. And it is not too serious problem to find only one nonlinear parameter D_f. Then we have the following estimates for θ_0 and θ_1: $\hat{\theta}_1 = 1/\hat{\beta}_1$, $\hat{\theta}_0 = -\hat{\beta}_0/\hat{\beta}_1$.

Estimate of parameter *R* can be made after estimation of ratio $\rho = r/R$. Remind, that this ratio defines the part of the cross section area, the destruction of which we consider as total failure of specimens. In the Daniels's model of static strength [4] this value corresponds to the value of $F(x^*)$, where x^* is such, that $x^*(1-F(x^*)) = \max_x x(1-F(x))$.

We can estimate this value, using estimates of θ_0 and θ_1. So we have $\hat{\rho} = F(x^*)$,

$$\hat{R} =]1/((C_V)^2\hat{\rho})[+1. \tag{7}$$

Now we have approximate estimates of all four parameters θ_0 and θ_1, *r* and *R*. At the fixed estimates of *r* and *R* the more precise estimates of θ_0 and θ_1 can be found by the use of MLM. For the probability mass function now we have following formula $p_T(t;S,\eta) = F_T(t;S,\eta) - F_T(t-1;S,\eta)$. But for calculation of CDF we should get P^t. It needs too much time. So we try to find some approximation for CDF. By comparison of normal and lognormal approximation it appears, that lognormal approximation is more appropriate:

$F_T(t;S,\eta) \cong \Phi(\frac{\log(t)-\theta_{0LT}}{\theta_{1LT}})$, where θ_{0LT}, θ_{1LT} are such, that we have the same expectation value and standard deviation $\theta_{0LT} = \log(E(T)) - (\log(C_V^2+1))/2$, $\theta_{1LT} = (\log(C_V^2+1))^{1/2}$.

Now the maximum likelihood function in logarithm scale $l(\eta) = \ln(L(\eta))$, where $L(\eta) = \prod_{i=1}^{n} f_i^{A_i}(1-F_i)^{1-A_i}$, f_i, F_i are the probability density function and cumulative distribution function of random variable T (for fixed η and S); A_i is equal to 1, if a fatigue test is finished by the failure of specimens, and , A_i is equal to 0, if the time of test is limited (right censored observation).

Until now we considered mainly uniformly load-shearing system of isolated parallel items loaded by tension. But we suppose to apply this model to the more complex structure, for which fracture of longitudinal items means the failure of specimen as a whole.

4. Numerical example

For numerical example we consider the problem to fit the experimental data of fatigue test of laminate panel. These data was kindly given to the authors by Professor W.Q. Meeker, who studied them in paper [5] and gives the following description of these data: "the data come from 125 specimens in four-point out-of-plane bending tests of carbon eight-harness-satin/ epoxy laminate. Fiber fracture and final specimen fracture occurred simultaneously. Thus, a fatigue life is defined to be the number of cycles until specimen fracture. The dataset includes 10 right censored observations (known as "run outs" in the fatigue literature)". In [1,2,3] we have considered already extreme values of fatigue lives for 5 stress levels, which we have got from the Figure 1 of paper [4]. But now we have original information, the same on the base of which the fatigue curve of this figure was made. And, as it was told already, this time we decrease the numbers of model parameters in order to increase the stability of others parameters. We put k=1, S_f= 0. Then four main steps were made for estimation of parameters θ_0, θ_1, r and R.

First step. We can make additional assumption, that static strength of items has lognormal distribution [2]: $F_0(.)$ is CDF of standard normal distribution, $g(s) = \log(s)$. By the use of regression analysis (and by sequence of calculation for different D_f) it was found approximate parameter estimates: $\hat{\theta}_0$=7.6906, $\hat{\theta}_1$=0.3541 and $\hat{D}_f$ = 0.0229.

Second step. Estimation of r. For this purposes the calculation of coefficient of variation C_V =0.5839 for some middle stress, at which there was not censuring, was calculated (for S=340 MPa). So $\hat{r}$ =3.

Third step. We have got estimate $\hat{R} = 15$, because at obtained approximate estimates $\hat{\theta}_0$, $\hat{\theta}_1$, the value of $\hat{\rho} = F(x^*)$ appears to be equal to 0.2072 (later on for final MLM the same estimate appears to be equal to 0.2022 and estimate $\hat{R} = 15$ did not change). Remind, that x^* is such, that: $x^*(1 - F(x^*)) = \max_x x(1 - F(x))$).

Fourth step. The estimation of parameters θ_0 and θ_1 by the use of MLM appears to be very difficult problem. The general view of this function is as we can see in Fig. 1. The counter lines of equal levels are shown in Fig. 2. After detail analysis (for the fixed already $\hat{r}$ =3, $\hat{R} = 15$) it was found: $\hat{\theta}_0$ = 7.6461, $\hat{\theta}_1$ = 0.34471.

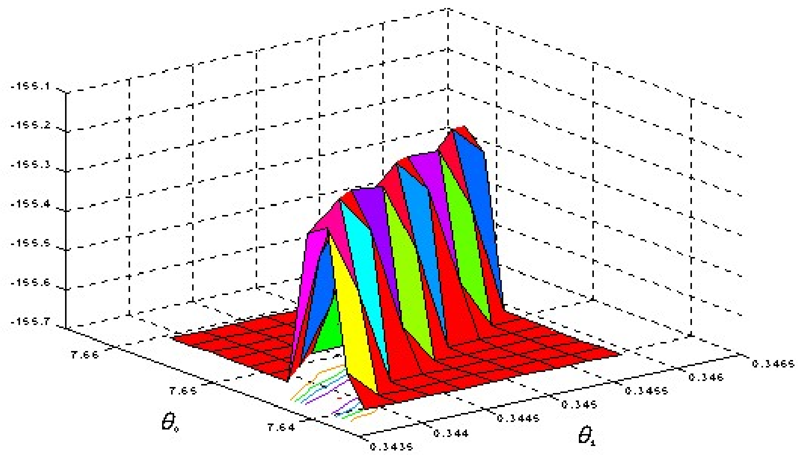

Figure 1. Likelihood function

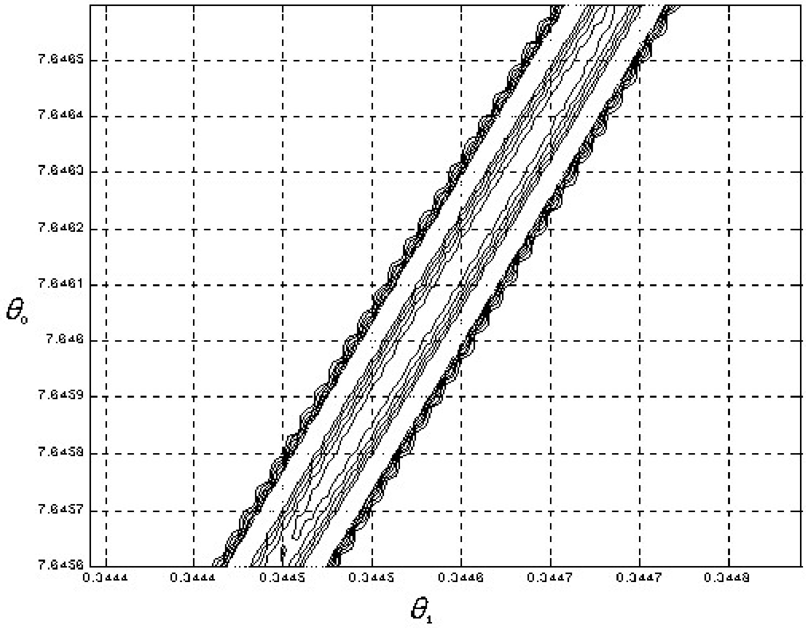

Figure 2. Contour lines of equal levels of likelihood function.
It was found: $\hat{\theta}_0$ = 7.6461 , $\hat{\theta}_1$ =0.34471.

Now we can check the validation of lognormal approximation of $F_T(t;S,\eta)$. We consider two hypotheses

$F_T(t;S,\eta) \cong \Phi(\frac{\log(t)-\theta_{0LT}}{\theta_{1LT}})$ and $F_T(t;S,\eta) \cong \Phi(\frac{t-\theta_{0T}}{\theta_{1T}})$. In the first case we should get straight line

$$\log(t) = \theta_{0LT} + \theta_{1LT}\Phi^{-1}(F_T(t;S,\eta)), \qquad (6)$$

In the second case

$$t = \theta_{0T} + \theta_{1T}\Phi^{-1}(F_T(t,S,\eta)). \qquad (7)$$

For this purpose we calculate the CDF and PDF for stress level S = 290.1 MPa. The result of calculation is shown in Fig. 3. We see, that formula (6) gives nearly straight line, so the lognormal approximation of $F_T(t;S,\eta)$ is more appropriate than the normal approximation.

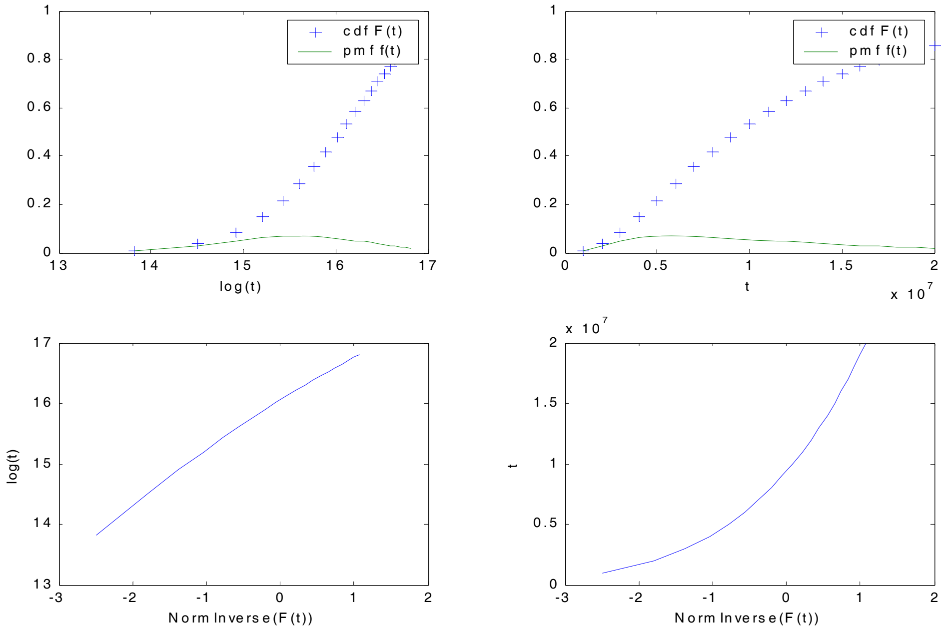

Figure 3. Cumulative distribution (×) and probability mass functions (-) in the upper part .
Functions $\log(t) = \theta_{0LT} + \theta_{1LT}\Phi^{-1}(F_T(t;S,\eta))$ and $t = \theta_{0T} + \theta_{1T}\Phi^{-1}(F_T(t,S,\eta))$ in the lower part

Finally we have got the fatigue curve. The experimental data (×) and results of calculations of the expectation values of extreme order statistics (o) (minimum and maximum) are shown in Figure 4.

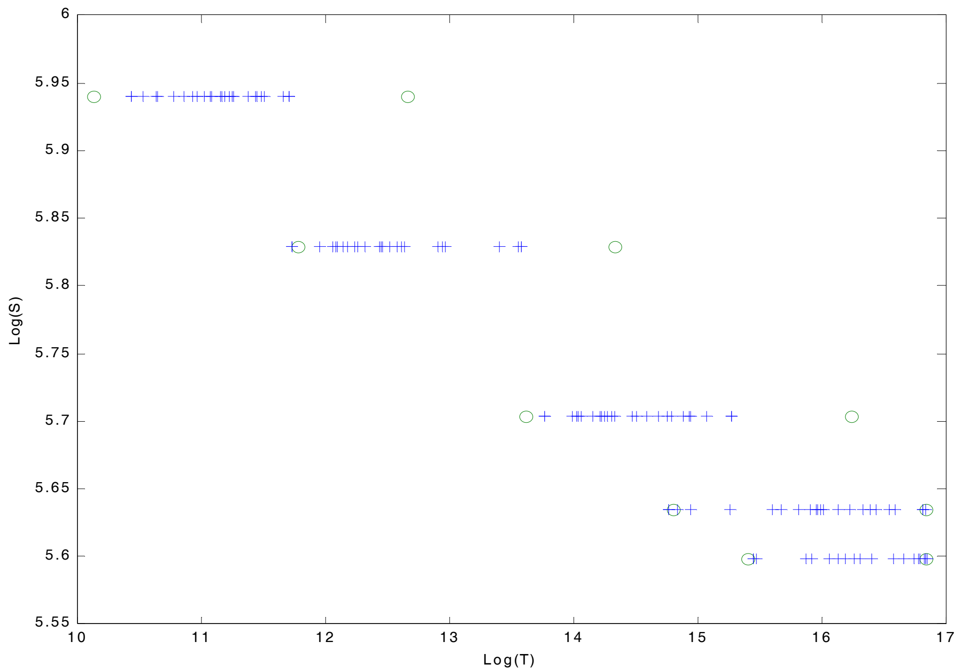

Figure 4. The experimental data (×) and results of calculations of the expectation values of extreme (minimum and maximum) orders statistics (o)

Conclusions

- S*imple Markov Chain Model of Fatigue Life* (SMCMFL) of composite material can be used as nonlinear regression model for fatigue curve approximation. Processing of dataset of fatigue test of carbon-fibre laminate specimens we have got estimates of parameters, which can be interpreted as *equivalent local static strength distribution parameter estimates.* We think, that, probably, there is some discrepancy between these estimations ($\hat{\theta}_0$ = 7.6461, $\hat{\theta}_1$ =0.34471) and real parameters of static strength distribution of carbon fibres, which are unknown for us. For example, in [4] the following estimations of carbon fibre static strength distribution are given: $\theta_0 = 7.198$, $\theta_1 = 0.467$. This discrepancy can be explained by the difference of original material, difference of load types (bending instead of tension), difference of "effective" length of fibres and so on. If we take into account all these circumstances, it seems that it is not too bad result for estimation of static strength distribution function parameters on the base of fatigue data.
- This discrepancy can be used for description of specific features of the specific structure.
- Really we do not need to estimate the static strength distribution parameters on the base of fatigue data. We consider the likelihood of these estimates only as a proof, that considered SMCMFL of composite material has right to exist and it can be used, for example, for forecasting of fatigue curve changes, when there are some changes of real static strength distribution parameters.

References

[1] Парамонов Ю.М., Клейнхоф М.А., Парамонова А.Ю. (2002) Вероятностная модель усталостной долговечности композита для аппроксимации кривой усталости, *Механика композитных материалов,* **38**, № 6, 741-750

[2] Paramonova A.Yu, Kleinhof M.A., Paramonov Yu.M. (2002) Markov chains theory use for fatigue curve of composite material approximation, *Aviation #6*, Tehnika, Vilnius 103-108

[3] Paramonova A.Yu, Kleinhof M.A., Paramonov Yu.M. (2002)The use of Markov chains theory for approximation fatigue curve of composite material. *Proceedings of Third International Conference on MATHEMATICAL METHODS IN RELIABILITY. Methodology and Practice*. June 17-20, 2002, Trondheim, Norway, NTNU , 509-512

[4] Pascual F.G. and Meeker W.Q. (1999) Estimating Fatigue Curves With the Random Fatigue-Limit Model, *Technometrics,* 41, 277-302

Received on the 19th of December 2004