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ANALYSIS OF FIBER STRENGTH DEPENDENCE ON LENGTH USING AN EXTENDED WEAKEST-LINK DISTRIBUTION FAMILY

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An extended family of the weakest-link models based on the assumption of a two-stage failure process of a fiber specimen was developed in [1, 2]. A generalization of this family is presented in this paper. As in [1, 2] we consider the specimen as a chain of *n* elements (links). The fracture4 process is modelled as follows: in the first stage initiation of defects (before loading or during loading), and in the second stage a specimen fracture takes place. As distinct from our previous publications, the strength of items without defects is taken into account and two types of the influence of defect number on the specimen strength are considered. The comparison of the models and the choice of the best one are made using cross validation method. The offered models sometimes describe more adequately the experimentally observed fiber strength scatter and the strength dependence on fiber length than the traditional models do.

Keywords: *distribution function, composite, static strength*

1. Introduction

The significant dependence of static strength of a composite on the scatter of static strength of its components can be illustrated by the following example. Let us consider three parallel items with 10 N, 15 N and 30 N strength and identical stiffness. It may seem surprising that they will fail at the applied load of 30 N, as if the strength of every item is equal to 10 N. Why?

The reason is that under 30 N load at first the weakest item will fail because its strength is equal to 10 N. At the uniform distribution of total loads, its load is equal to 10 N also. Now the load acting on each "surviving" item is equal to 15 N. So the second item, the strength of which is equal to the same value of 15 N, will fail. Now the load for the last strongest item will be equal to 30 N. It will fail also because its strength is just equal to this load. This process ("domino phenomenon") is shown on Fig.1. The same phenomenon takes place if element strengths are proportional to the terms of harmonic series: 1, ½, 1/3, …, 1/n, see Fig. 2.

So we see that the composite strength dependence on the strength scatter of its constituents can be very significant.

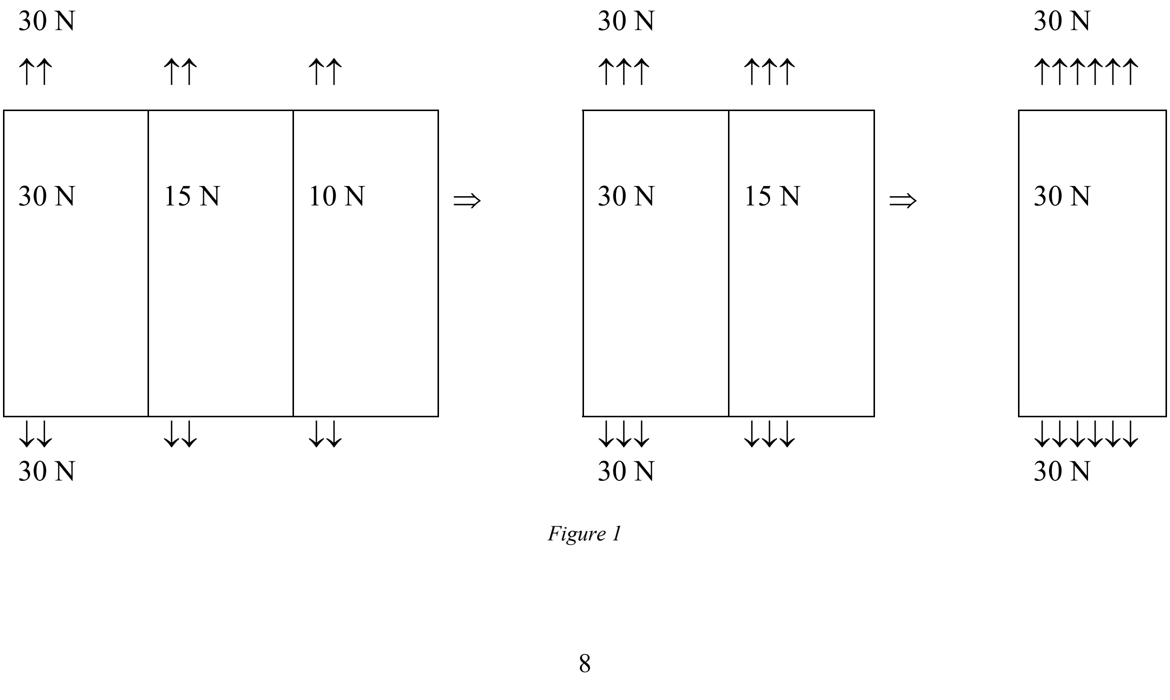

Figure 1

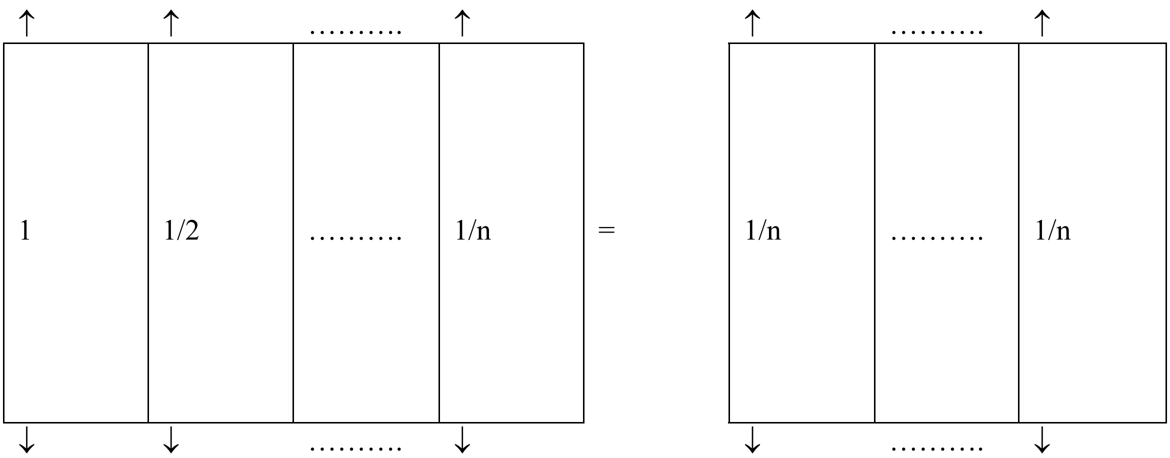

Figure 2

Power-Weibull (PW) model of distribution [3, 4, 5]

$$F(s) = 1 - \exp(-(L/l_1)^{\gamma}(s/\beta_1)^{\alpha}), \tag{1}$$

which has been intensively studied in literature, while providing a good empirical fit to the strength data of specimens with different length, L, lacks the theoretical appeal of the weakest-link models (It should be noted that here parameter β_1 corresponds to $L = l_1$, β_1 changes if l_1 changes.). We derive a new weakest-link model family (WLMF) based on the assumption of a two-stage failure process. For modelling purposes we consider a specimen (fiber) as a chain of n elements (links) of length l_1. First, the process develops along the specimen and defects appear in K elements. Here K is integer random variable, $0 \le K \le n$. Two types of the second stage will be considered in this paper. First type: in every element (containing defects or intact) the development of fracture process takes place and the strength of the weakest item (link) defines the strength of the specimen. Second type: development of fracture process takes place only in one, critical element. Then only the probability that the second stage will take place depends on the number of elements but the strength distribution of this element (the process of accumulation of elementary damages in crosswise direction up to specimen failure) does not depend on this number.

We consider two different versions of the first stage also. First version: defects appear before the loading and their number does not depend on the subsequent loading. Second version: defects appear during loading (instantly or gradually) and their number depends on the load.

2. General Description of the Model Family

2.1. The Fracture Process Takes Place in Every Element

2.1.1. Models of instant fracture

Let K, $0 \le K \le n$ be the number of elements in which defects appear. Let $Y_1, Y_2, ..., Y_K$ be independent random variables which are the strengths of these elements with the same cumulative distribution function (cdf) $F_Y(x)$; $Z_1, Z_2, ..., Z_{n-K}$, $F_Z(x)$ are the same for the elements without defects. It seems reasonable to assume that the random strength of the specimen is the strength of the weakest item

$$X = \min(Y_1, ..., Y_K, Z_1, ..., Z_{n-K}), \tag{2}$$

with the corresponding cdf

$$F(x) = 1 - (1 - F_{Z1,n}(x)) \sum_{k=0}^{n} p_k \delta^k , \tag{3}$$

where

$$\delta(x) = (1 - F_Y(x))/(1 - F_Z(x)), \tag{4}$$

$$F_{Z_{1,n}}(x) = 1 - (1 - F_Z(x))^n. \tag{5}$$

Several different assumptions can be made here. Let us consider first the case of defects appearing before loading. It can be assumed that the probability of defect in one item, p, is some constant (and it is a parameter of the model). Then the corresponding binomial probability mass function (pmf) is

$$p_k = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}. \tag{6}$$

If $\lambda = np$ is large enough we can use (as an approximation) Poisson pmf:

$$p_k = \exp(-\lambda)\lambda^k / k!. \tag{7}$$

In this case the equation (3) (approximately) can be written in the following way:

$$F(x) = 1 - (1 - F_{Z_{1,n}}(x))\exp(-\lambda(1 - \delta(x))). \tag{8}$$

If initiation of the defects depends on the applied load then it can be assumed that $p = F_0(x)$, where $F_0(x)$ is the cdf of defect initiation stress.

2.1.2. Models of gradual accumulation of defects

We consider the process of accumulation of defects as an inhomogeneous finite Markov's chain (MC) with finite state space $I = \{i_1, i_2, ..., i_{n+1}, i_{n+2}\}$. MC is in state i_k if there are $(k-1)$ defects, $k = 1, ..., n + 1$. State i_{n+2} is an absorbing state corresponding to the fracture of specimen. Usually we suppose that the Markov's chain starts in state i_1 but in general case the initial distribution is represented by a row vector π given by $\pi = (\pi_1, \pi_2, ..., \pi_{n+1,} \pi_{n+2})$. We further assume that the loading (i.e. the process of nominal stress increase in the specimen cross section) is described by an ascending (up to infinity) sequence $\{x_1, x_2, ..., x_t, ...\}$ and the process of MC state change is described by the transition probabilities matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{23} & \cdots & p_{1(n+1)} & p_{1(n+2)} \\ 0 & p_{22} & p_{23} & p_{24} & \cdots & p_{2(n+1)} & p_{2(n+2)} \\ 0 & 0 & p_{33} & p_{34} & \cdots & p_{3(n+1)} & p_{3(n+2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & p_{(n+1)(n+1)} & p_{(n+1)(n+2)} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

which at the tth-step is a function of x_t, $t = 1, 2, ...$ Let the sequence $\{x_t\}$ be fixed, then P is a function of t. Let us note that if $n = \infty$ then the subscript $(n + 2)$ is not a number but only a symbol, corresponding to the absorbing state i_{n+2}.

In the new model the number of defects and the strength of specimens are random functions of time,

$K(t)$, and

$$X(t) = \min(Y_1, Y_2, ..., Y_{K(t)}, Z_1, Z_2, ..., Z_{n-K(t)}) \tag{9}$$

correspondingly. The specimen fracture occurs when the strength of the specimen becomes equal to or less than the current load (stress). Ultimate strength

$$X = x_{T^*}, \tag{10}$$

where

$$T^* = \max(t : X(t) > x_t) . \tag{11}$$

Cdf of *X* is defined by equation

$$F(x_m) = \pi(\prod_{j=1}^{m} P(j))u , \tag{12}$$

where $P(j)$ is the transition matrix for step number *j,* column vector $u = (0,...,0,1)'$ where only the last component is equal to 1 but all the others are equal to 0.

2.1.3. Specifying models. The specimen strength without defect is very large

For the purpose of specification of the models, the general description of which was given in the previous section, we additionally have to specify the cdf $F_Y(x)$, $F_0(x)$ (for models with defect number dependence on load), $F_Z(x)$, and, additionally, for Markov models, a prior distribution, π, which, of course, in general case can differ from binomial or Poisson distribution. For Markov models we need to specify also the matrix *P* as a function of current stress, x_t.

In this paper we assume that $F_Y(x)$ and $F_0(x)$ are the smallest extreme value (sev) distributions. For the case when location parameter $\theta_0 = 0$ and scale parameter $\theta_1 = 1$ it is assumed that

$$F_Y(x) = 1 - \exp(-\exp(x)) , \tag{13}$$

$$F_0(x) = F_Y(x - \delta_0) , \tag{14}$$

where $x = \log(s)$, *s* is the strength (expressed in MPa). If $\delta_0 > 0$ then at the same probability of events the stress required for new defect initiation is larger than the stress required for the failure of an element with defect.

For $F_Z(x)$ we consider two assumptions in this paper. First, sev distribution can be assumed again:

$$F_Z(x) = F_Y(x - \delta_Z) . \tag{15}$$

Again we can say that if $\delta_Z > 0$ then $F_Z(x) < F_Y(x)$.

But the simplest is the assumption that

$$F_Z(x) = \begin{cases} 0, x < C, \\ 1, x \geq C, \end{cases} \tag{16}$$

where C is a very large constant.

Then instead of (2) we have

$$X = \begin{cases} \min(Y_1, ...Y_K), \ K > 0, \\ C, \ \ K = 0. \end{cases} \tag{17}$$

The equation (3) can be written in the form

$$F(x) = \begin{cases} 1 - \sum_{k=0}^{n} p_k \delta^k , \ x < C, \\ 1, \ x \geq C \end{cases} \tag{18}$$

where $\delta = 1 - F_Y(x)$. But equation (8) now has the following form

$$F(x) = \begin{cases} 1 - \exp(-\lambda F_Y(x)), \ x < C, \\ 1, \ x \geq C. \end{cases} \quad (19)$$

In [1, 2] was shown that the cdf

$$F(x) = \sum_{k=0}^{\infty} p_k \{1 - (1 - F_Y(x))^{k+1}\} \quad (20\text{ a})$$

or

$$F(x) = 1 - (1 - F_Y(x)) \exp(-\lambda F_Y(x)), \quad (20\text{ b})$$

where p_k is defined by (7) , $\lambda = np$, $p = F_Y(x)$, $F_Y(x)$ is sev cdf, provides a good empirical fit to the strength data of specimens with different length, *L*. Equation (20 b) can be considered as modification of (8): $F_Y(x)$ is used here instead of $F_{Z_{1,n}}(x)$. But now it is not only an approximation of the "binomial" model. Now we can consider the specimen as continuous one and define λ by equation

$$\lambda = \lambda_1 (L / l_1),$$

where *L* is the specimen length, λ_1 is the intensity of defects (the defect number per length l_1). Then function $F_Y(x)$ can be regarded as an element-length-independent cdf of strength distribution in the cross section with defect, where the number of defective cross sections has the corresponding Poisson distribution.

For Markov models we should specify the matrix *P*. In the case when parameter *C* is very large (the theoretical strength is much higher than the real strength) the probability that in some element the defect appears at the stress x_t under the condition that it has not appeared at the stress x_{t-1} is

$$b(t) = (F_0(x_t) - F_0(x_{(t-1)})) / (1 - F_0(x_{(t-1)})).$$

Consider the case of *s* defects present. The probability that *r* new defects appear, $0 \leq r \leq k = n - s$, and the total number of defects is equal to $m = s + r$

$$\widetilde{p}_{sm}(t) = (b(t))^r (1 - b(t))^{k-r} k! / r!(k - r)!$$

Conditional probability of element fracture at the nominal stress x_t

$$q(t) = (F_Y(x_t) - F_Y(x_{(t-1)})) / (1 - F_Y(x_{(t-1)})).$$

Corresponding probability that none of the elements fail when there are defects in *m* elements is

$$u_m(t) = (1 - q(t))^m.$$

The probability of coincidence of these events, which we consider as independent, is the probability of transition from state $i = s + 1$ to state $j = i + r$

$$p_{ij}(t) = \widetilde{p}_{(i-1)(j-1)}(t) u_{j-1}(t),$$

where $i \leq j \leq (n+1)$.

Conditional fracture probability at state *i*

$$p_{i(n+2)}(t) = 1 - \sum_{j=i}^{n+1} p_{ij}(t).$$

Of course, $p_{ij}(t) = 0$, if $j < i$, and $p_{(n+2)(n+2)}(t) = 1$.

2.2. The Fracture Process Takes Place Only in One Element

2.2.1. The models of instantaneous failure

In the previous models it is assumed that defects are uniformly distributed along the specimen length. But it is plausible that such uniformity is retained only at the initial stage of loading. More precisely, upon formation of the weakest link in a chain, the development of failure proceeds only in this link, and the specimen length is of no importance any more. The simplest variant of such a model corresponds to the assumption that the law of strength distribution in the element where this process proceeds (in the cross section where the critical defect is formed) is independent of specimen length, which determines only the probability of formation of an element with defect. The mathematical formulation of this hypothesis is as follows

$$X = \begin{cases} Y, K > 0, \\ Z, K = 0. \end{cases} \tag{21}$$

Here, Y and Z are random variables, which are the strength of element where the failure process proceeds with or without defect, correspondingly.

In this case

$$F(x) = \{1-(1-F_0(x))^n\}F_Y(x) + (1-F_0(x))^n F_Z(x)\,. \tag{22}$$

2.2.2. Model of successive formation of at least one defect

The corresponding Markov's chain has only three states. The first state corresponds to the absence of defective elements, the second one means the presence of at least one defective element, and the third, absorbing one, means failure of the specimen. The corresponding probabilities at a *t*th step are determined by the formulae

$$p_{11}(t) = [1-b(t)]^n,\quad p_{12}(t) = (1-p_{11}(t))(1-q(t)),\quad p_{13}(t) = (1-p_{11}(t))q(t),$$

$$p_{21}(t) = 0,\quad p_{22}(t) = 1-q(t),\quad p_{23}(t) = p_{32}(t) = 0,\quad p_{33}(t) = 1\,.$$

Specification of the cdf and of elements of the matrix *P* can be made in the same manner as in section 2.1.3.

3. The Processing of Test Data

The maximum likelihood method can be used for parameter estimation but it is excessively labor-consuming. The estimates of parameters θ_0 and θ_1 (at fixed other parameters) can be found easily using regression analysis of order statistics. Our purpose here is only the investigation of the possibility of using the considered models for prediction of fiber strength distribution changes when the fiber length is varied and the comparison of the models has been done as well. So we have limited ourselves by the use of regression analysis.

Let x_{ij} be *j*th order statistic, $j = 1,2,...,n_i$, n_i is the number of specimens with $L = L_i$, $i = 1,2,...,k_L$, k_L is number of different L_i, $E(X_{ij})$ is the expected value of random order statistic X_{ij}, $E(\overset{0}{X}_{ij})$ is the same but for $\theta_0 = 0$ and $\theta_1 = 1$.

Then we have the following linear regression model

$$E(X_{ij}) = \theta_0 + \theta_1\, E(\overset{0}{X}_{ij}), \tag{23}$$

where $E(\overset{0}{X}_{ij})$ is a function of L_i, n_i and *j*.

This equation can be used for estimation of θ_0 and θ_1 if all the other parameters are fixed.

We compare the above-mentioned models with the PW model (see equation (1)) and LW model (it is the original Weibull model: PW model with $\gamma = 1$). If *S* is random strength of specimen with cdf (1) then for $X = \log(S)$,

$$F_X(x) = 1 - \exp(-\exp((x - \theta_0)/\theta_1), \quad (24)$$

where

$\theta_0 = \log(\beta_1) - (\gamma/\alpha)\log(L/l_1)$, $\theta_1 = 1/\alpha$.

So for PW model we have equation with three unknown parameters $\theta_{00} = \log(\beta_1)$, $\theta_{01} = -\gamma/\alpha$ and θ_1,

$$E(X_{ij}) = \theta_{00} + \theta_{01} \log(L_i/l_1) + \theta_1 E(\overset{0}{X}_{ij}). \quad (25)$$

For LW model we have an equation with two unknown parameters θ_0 and θ_1

$$E(X_{ij}) = \theta_0 + \theta_1(-\log(L_i/l_1) + E(\overset{0}{X}_{ij})). \quad (26)$$

In (25) and (26) the value of $E(\overset{0}{X}_{ij})$ is the expected value of *j*th order statistic for sample from sev distribution with sample size n_i .

It is assumed that roughly $E(\overset{0}{X}_{ij}) = \overset{0}{F}{}^{-1}(\hat{F}(x_{ij}))$, where $\hat{F}(x_{ij}) = (j - 0.3)/(k_L + 0.4)$ is an estimate of $F(x_{ij})$.

For comparison of different models, the glass fiber dataset described in [1,2] is used (four samples with specimen lengths (L_1, L_2, L_3, L_4)=(10, 20, 40,80 mm), sample sizes (n_1, n_2, n_3, n_4)=(78,74,50,60)). For parameter estimation a version of the cross validation method is applied. At the fixed nonlinear parameters ($l_1, \ldots$) for the linear regression (LR) estimation of parameters θ_0 and θ_1 we use only the dataset corresponding to *L*= 10 mm and *L*=20 mm. We calculate also two additional statistics

$$Q_1 = (\sum_{i=1}^{k_L} (\bar{x}_i - \hat{x}_i)^2 / \sum_{i=1}^{k_L} (\bar{x}_i - \bar{x})^2)^{1/2}, \quad (27)$$

where $\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$; $\hat{x}_i = \sum_{i=1}^{n_i} \hat{x}_{ij}/n_i$; $\hat{x}_{ij} = \hat{\theta}_0 + \hat{\theta}_1 E(\overset{o}{X}_{i.j})$; $\hat{\theta}_0$ and $\hat{\theta}_1$ LR estimates of θ_0 and θ_1, $\bar{x} = \sum_{i=1}^{k_L} \bar{x}_i / k_L$,

and

$$\overline{R}_{LR} = (1 - R^2)^{1/2}, \quad (28)$$

where R^2 is standard statistic of LR analysis (the coefficient of determination).

As nonlinear parameter estimates, the values of the parameters which correspond to the minimum of statistics OSPPt (Order Statistics Probability Plot Test) are taken. OSPPt is the measure of the error of order statistics prediction for sample with L_4 = 80 mm:

$$\text{OSPPt} = (\sum_{j=1}^{n4} (x_{4j} - \hat{x}_{4j})^2 / \sum_{j=1}^{n4} (x_{4j} - \bar{x}_4)^2)^{1/2} . \quad (29)$$

For the convenience of the following references let us list the full number of specifications and assumptions which define the specific model in the considered family and make specific notations of the corresponding assumptions.

We have to specify the conditions under which the initiation of defects takes place. By symbol '**T**' we denote the assumption that the process of initiation of defects is a function of **technology** only, but symbol '**L**' is used if this initiation depends on **load**.

A **prior distribution** of defects needs to be specified for the models in which Markov's chains theory is used. In general case we denote a prior distribution by π but we use symbols '**B**' or '**P**' if binomial or Poisson distribution is used.

If we consider the **instantaneous fracture** of specimen we use symbol '**B**' for binomial distribution of defect number, *K*, symbol '**P**' for Poisson distribution and symbol **'Pm'** for 'truncated' Poisson distribution.

(Remark. We use the words 'truncated in m (discrete) distribution' if instead of discrete rv *X* we consider the rv

$$Xm = \begin{cases} X, \text{ if } X < m+1, \\ m+1, \text{if } X > m. \end{cases}$$

The use of it can be convenient for calculation of the cdf of steps to absorption using formulae of finite Markov's chains theory).

We use simbols '**MB**', '**MBm**' (for truncated binomial distribution), '**MP**' and '**MPm**' (for truncated Poisson distribution) if the Markov's chain is used for description of defect initiation process (Note. Formulae for transition probability matrix in this section are given only for MB case).

F_Z , F_Y and F_0 have to be specified:

- the cdf of strength of elements without defects, F_Z ;
- the cdf of strength of elements with defects, F_Y ;
- the cdf of defect initiation stress F_0 (if the process of defect initiation is assumed to be a function of load). In general case we use symbols F_Z , F_Y and F_0 correspondingly but they should be specified by specific equations or by specific definitions. In this paper (Fig. 3-9) we use symbol **S** if cdf is defined by equation (13), symbol **St** if cdf is defined by equation (14); symbol **Zt**, if cdf is defined by equation (15); symbol **C**, if cdf is defined by equation (16).

If the Markov's chain is used then the sequence of loads (stresses) $\{x_t\}$ should be specified also, but in this paper, as a rule, $\{x_t\}$ is a sequence of numbers uniformly distributed in some interval, which can be seen on Figures with $f(x)$ and $F(x)$ (see Fig. 3, …).

We consider six models in total, but already preliminary investigation shows that the first two (T.B.Zt.S and T.P.C.S) are not appropriate for fiber strength distribution description although it seems that both are very natural. These models correspond to assumptions that there is binomial or Poisson distribution of technological defects which can appear during preparation of specimens. We show this by presenting some examples of calculations.

The model T.B.Zt.S corresponds to assumption that during production of fiber specimens in every element (with length l_1) of specimen one defect can appear with probability *p*. (Here and later on we presume that the ratio L_i/l_1 is integer and it is equal to the number of elements in specimen with length L_i (for every i = 1, …, 4)). The results of calculations of $f(x)$, $F(x)$ for θ_0 = 0 and θ_1 = 1, estimates of order statistics, $\hat{x}_{4j}$, as function of x_{ij}, estimates of mean, $\hat{x}_i$, as function of L_i (using LR estimates of θ_0 and θ_1) are shown on Fig. 3, which corresponds to l_1 = 10 mm (it is the length of the shortest specimen), p = 0.5, δ_Z = 7, θ_0 = 7.5326, θ_1 = 0.0562. We see that although the estimates of mean, $\hat{x}_i$, are acceptable, the estimates of $\hat{x}_{4j}$ are less than satisfactory.

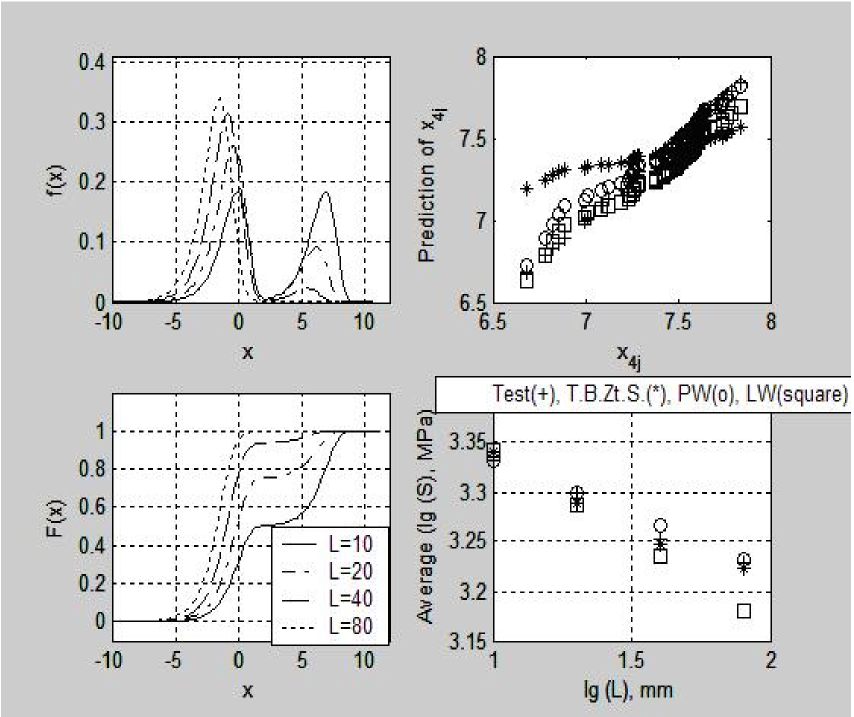

Figure 3

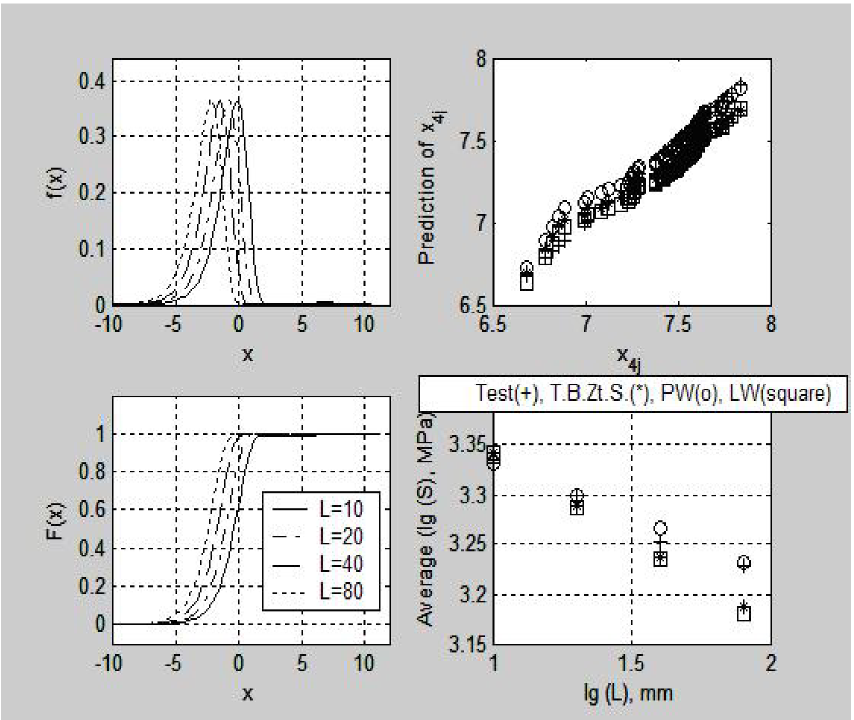

Figure 4

But if better agreement of the estimates $\hat{x}_{4j}$ is achieved (using parameters $p = 0.99$ (for $l_1 = 10$ mm), $\delta_Z = 7$, $\theta_0 = 7.7944$, $\theta_1 = 0.1660$), then the estimates $\hat{x}_i$ deteriorate considerably (see Fig. 4).

In accordance with the model T.P.C.S it is assumed that the number of defects in specimen with length L_i has Poisson distribution with parameter $\lambda_1 L_i / l_1$. Cdf $F(x)$ is defined by (19). Results of calculation for $\lambda_1 = 0.65$ (for $l_1 = 10$ mm), $\theta_0 = 7.5107$, $\theta_1 = 0.0389$ and $C = 10$ are shown on Fig. 5.

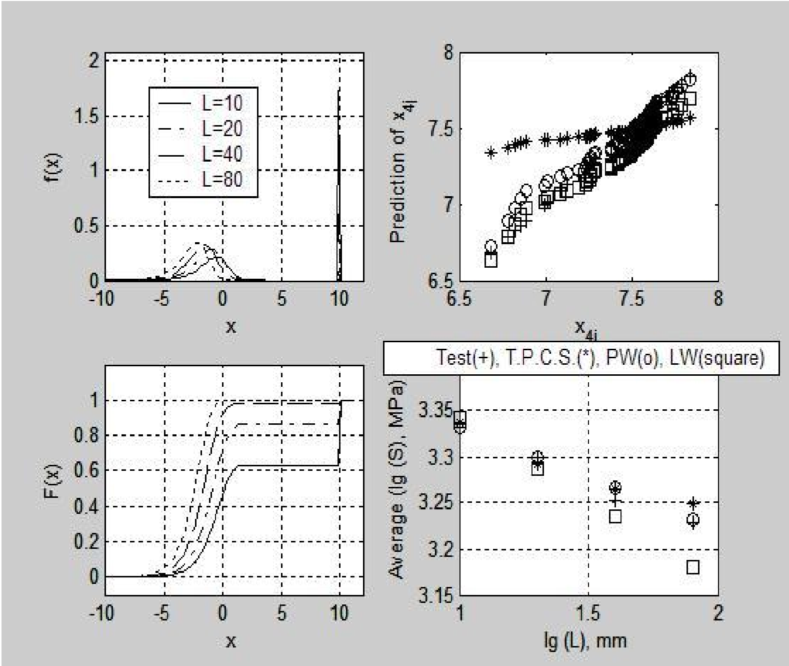

Figure 5

Again we see reasonable estimation of mean , $\hat{x}_i$, but the estimates $\hat{x}_{4j}$ are less than satisfactory. And just as in the previous model, we can improve $\hat{x}_{4j}$, but then the estimates of mean, $\hat{x}_i$, deteriorate.

More detailed search of parameter estimates was made for four models.

For the Model Lmod.P.C.S.S. (see equation (20)), which in [2] was denoted by p-sev-sev), the following parameter estimates were determined (for $C=\infty$): $\hat{\lambda}$ = 1.1 (for l_1 = 1 mm), $\hat{\theta}_0$ = 8.1406, $\hat{\theta}_1$ = 0.2743. These estimates correspond to the minimum of $\overline{R}_{LR}$. Estimates $\hat{x}_{4j}$ and $\hat{x}_i$ are shown on Fig. 6. Although for this model the values of $\overline{R}_{LR}$, Q_1 are better than for PW and LW model (see Table 1), the statistics OSPPt for prediction for L_4 = 80 mm is better than for LW but worse than for PW.

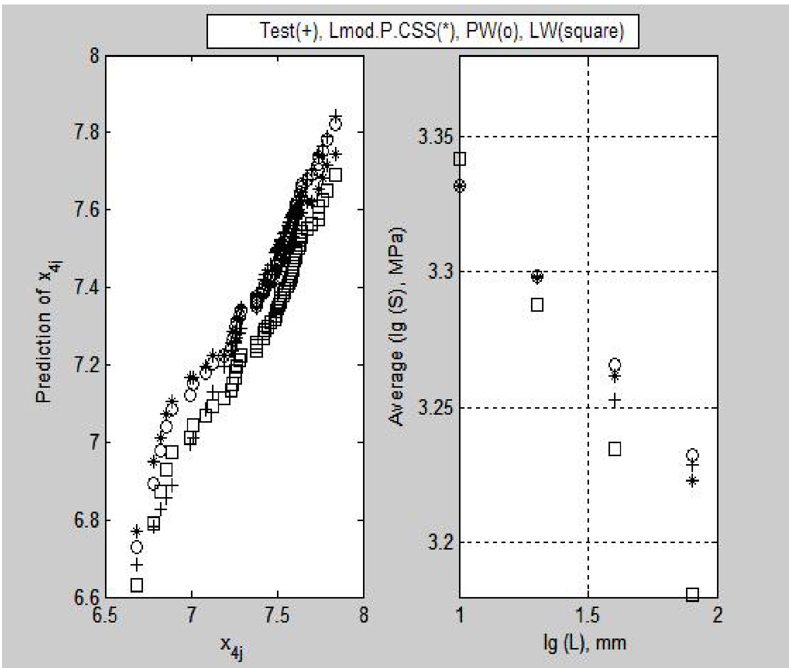

Figure 6

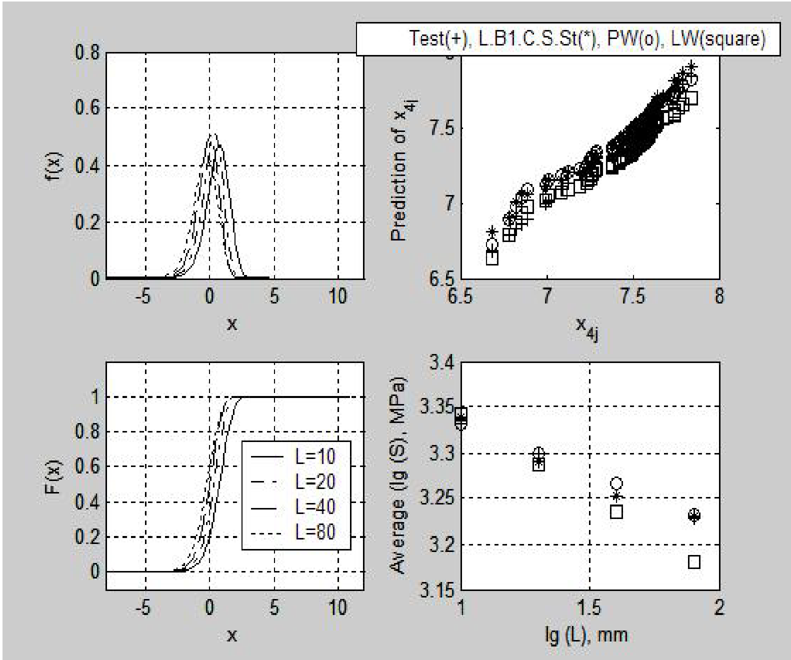

Figure 7

The Model L.B1.C.S.St (which in [1] for $\delta_0 = 0$ is denoted by D1) corresponds to equations (21) for $F(x)$ and (16) for $F_Z(x)$ with C = ∞. For parameter estimates: p = 0.9 (for $l_1 = 10$), $\theta_0 = 7.5398$, $\theta_1 = 0.2605$, $\delta_0 = 0.9$, corresponding results (which are very close to the results of Lmod.P.C.S.S.) can be seen in Fig. 7 and Table 1.

The same can be said concerning the Model L.π.MB.C.S.S (see equation (12) for $F(x)$ and (16) with C = ∞ for $F_Z(x)$), which was denoted by MB in [1]. The estimates of the model parameters are: l_1 = 5mm, $\theta_0 = 7.7578$, $\theta_1 = 0.236$, $\pi = (0,1,0,\ldots,0)$). The corresponding results (which are very close to the results of Lmod.P.C.S.S.) can be seen in Fig. 8 and Table 1.

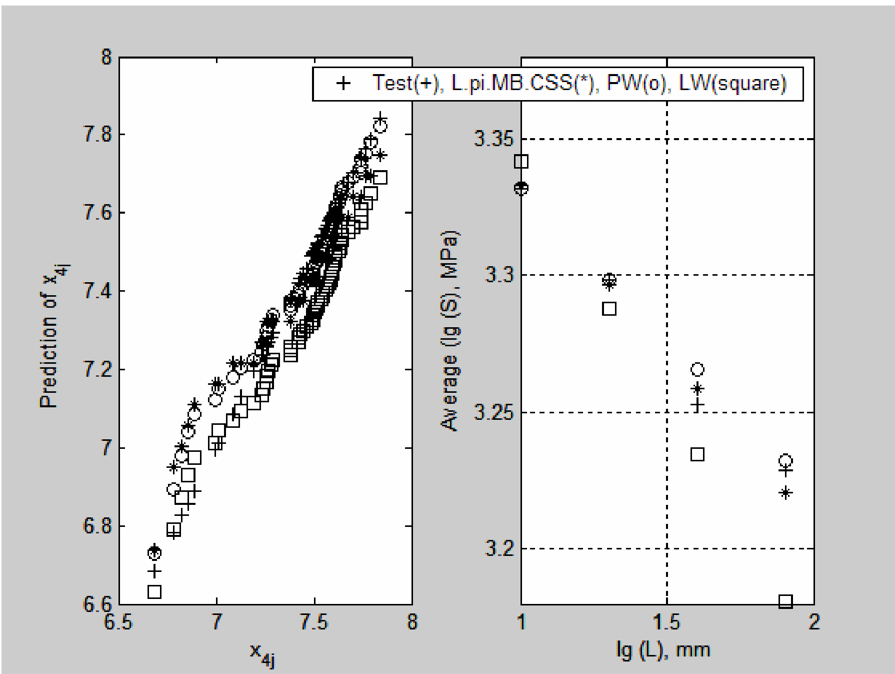

Figure 8

The best results, which are better than results of both LW and PW models (see Fig. 9 and Table 1), we obtained using L.Pm.MBm.C.S.S model (see equation (12) for $F(x)$ and (16) with C = ∞ for $F_Z(x)$). For this model a prior distribution of defect number is the (truncated at m = 2) Poisson distribution with $\lambda = \lambda_1 L / l_1$, where λ_1 is the defect intensity (defect number per specimen length unit), Parameter estimates of the model are λ_1= 0.15, l_1 = 5, θ_0= 7.7578, θ_1= 0.2346. In this paper we did not estimate the parameter δ_0. It was assumed that δ_0 = 0.

We see that the L.Pm.MBm.C.S.S model ensures the minimum of all three statistics.

TABLE 1. The comparison of models

Statistics	L.Pm.MBm.C.S.S	Lmod.P.S.S	L.B1.C.S.St	L.π.B.C.S.S	PW	LW
OSSPt	0.1574	0.3094	0.2630	0.3202	0.2155	0.4760
Q_1	0.1032	0.1279	0.1441	0.1303	0.1644	0.6702
$\overline{R}_{LR}$	0.1479	0.1509	0.2274	0.1545	0.1525	0.1855

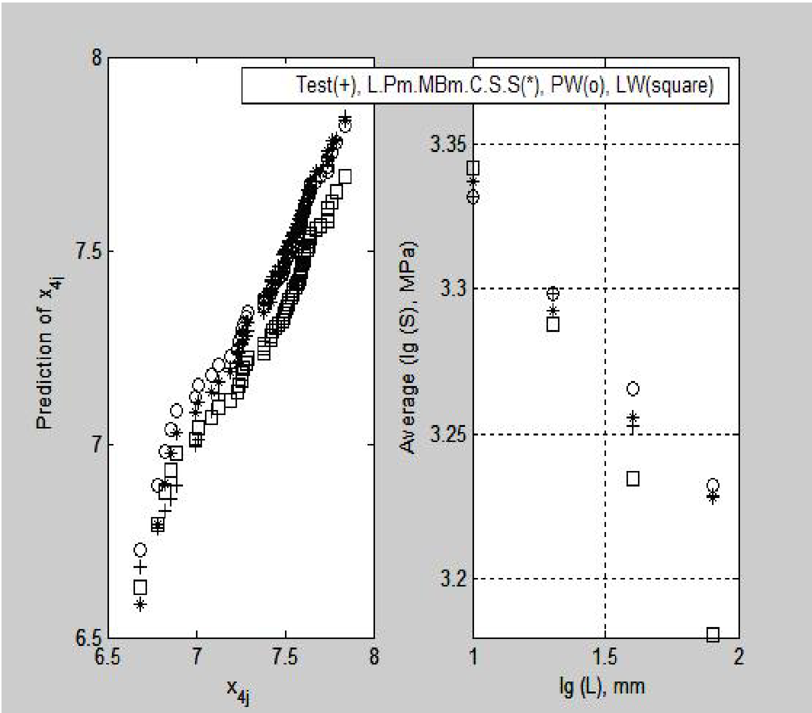

Figure 9

4. Resume

The Model L.Pm.MBm.C.S.S provides the best estimates of fiber strength for L = 80 mm using test data for L = 10 mm and L = 20 mm. All four WLMF models, including those in the Table 1, are providing better estimates of fiber strength dependence on specimen length than both LW and PW models (see statistics Q_1). Common feature of these models is the presence of some form of limitation of this dependence. It is obvious for Model L.B1.C.S.St (C $=\infty$), where (see (21)) only the probability of defect initiation depends on specimen length (or number of elements, $n = L / l_1$). It can be seen also for Model Lmod.P.S.S. The equation (20 a) corresponds to the assumption that initially there is one defect in specimen regardless of its length. The same is true for L.π.B.C.S.S where $\pi = (0,1,0,...,0)$. For the Model L.Pm.MBm.C.S.S the number of possible defects is deliberately limited by the number ($m+1$). Both models T.B.Zt.S and T.P.C.S have no similar limitation and fail to capture the strength dependence on specimen length.

The Model L.Pm.MBm.C.S.S provides the best agreement with the experimental dataset among the considered models. But we should take into account that it has five unknown parameters: θ_0, θ_1, l_1, λ_1 and *m*. PW model has three parameters only. Evidently we have random conclusions because we have random dataset. But it seems that the presented distribution family has great potential (for example, we have wide choice of $F_Z(x), F_Y(x), F_0(x), \ldots$) and deserves to be studied much more thoroughly using much more test data. We should mention also that the considered distribution family can be applied not only to the fiber strength analysis but to the analysis of reliability of any series system with two types of elements as well.

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