MODELING OF STRENGTH AND FATIGUE LIFE OF FIBER COMPOSITE MATERIAL

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Some simple statistical model for fatigue life of composite material consisting mainly of parallel rigid components (strands) is offered. The model, which can be in some way considered as extension and specification of Daniels's model for composite material, allows to get S-type curve of internal stress growth and, finally, to see the connection between static strength distribution parameters and S-N fatigue curve (Wholer curve).

Key words: composite, strength, fatigue, fatigue curve

Introduction

Solution of the problem to get more precise relationship between fatigue life of composite material and applied stress is an important input to design-for-reliability process. Fatigue data are usually presented in the form of a median S-N curve, a log-log plot of cyclic stress or strain S versus the median fatigue life N, which is expressed in cycles to failure. A great number of articles is devoted to the problem. Wide discussion on this matter took place on F.G. Pascual's and W.Q. Meeker's paper in Technometrics, November 1999 [3], in which 7 models of this relationship can be found. These models are offered by Bastenaire (in 1972), Little and Ekvall (in 1981 and once again in 1981), Spindel and Haibach (in 1981), Castillo et al. (in 1985), Castillo and Hadi (1995), F.G. Pascual's and W.Q. Meeker (in1999). So this problem is still the subject of active research interest. One of the shortcomings of all these models is that they are really some regression models, which have no substantiation by some fatigue damage accumulation theory. Here we mean substantiation, which was made, for example, for Wholer's curve in Malmaister, Tamuzh, Teters [12]. Parameters of these models have no connection with parameters of static strength distribution function parameters. For composite material this question is connected with prediction of its static strength characteristics by the use of characteristics of composite components. It seems that Daniels made the most significant steps in this direction [1], studying in details a model, which is now called the "classical model of bundle of n parallel fibres stretched between two clamps". Phoenix and Tailor [4], in1973, extended the model to cover certain types of inhomogeneity among the fibres, such as random slack, but still within the basic framework of equal load-sharing.

Similar approach was developed by Paramonov and Kleinhof [8, 9], in 1980, 1983. Some improvements of the original normal approximation of bundles of fibers strength distribution were made by Daniels himself [2], in1986, which have used this time the theory of Brownian Bridge. Local load-sharing and the chain-of-bundles model was studied by Pitt and Phoenix [5], in 1982, by

Smith [6 and earlier references therein], in 1986. Some new approach to the calculations of "the strength of rope composed from several strands" was considered in Wolf and Linka [7], in 1999, which have used the methods and relevant theory of counting processes to get solution of some mathematical aspect of this problem.

The objective of this paper is on the base of some specific version of Daniels model to develop some simple fenomenological model of fatigue damage accumulation by the use of which we can get S-N curve with parameters in some way connected with static strength distribution parameters. Some numerical examples are given.

Generalization and specification of Daniel's static strength model

The main idea of Daniels's model is uniform distribution of tension loads between parallel unbroken strands. Before test there were n parallel strands and at the tension load *s* (per one strand) the expected part of destroyed strands will be equal to *F(s),* where *F(x)* is a cumulative distribution function of a strand strength. Then the expected destruction load of bundle of n strands is equal to

$$s_b = \max_s n \cdot s\big(1 - F(s)\big).$$

Later we put here $n = 1$, then we can consider the value *s* as the stress in parallel unbroken strands and s_b as the mean breaking nominal strength.

The random value of the breaking strength (later we'll use more neutral word "parallel component" or just "component" instead of "strand") we denote by *W* and corresponding cumulative distribution function by $F_w(x)$. Some generalization of this Daniels's model can be developed if we take into account not only strength, but also relative rigidity distribution. Let us denote relative rigidity of *i*-th strand by $G_i = E_i L_m / Em L_i$, where (E_i, E_m) and (L_i, L_m) are true (for *i*-th strand) and mean values of elasticity modulus and length correspondingly. Let us denote by $F_G(x)$ a cumulative distribution function (c.d.f.) of random variable *G*, by $F_{w|g}(w \mid g)$ a conditional c.d.f. of ultimate strength (destruction stress) *W* at given rigidity *g*. Then for a mean value of a breaking (ultimate) strength we have following formula

$$s_b = \max_s \int \left(\int_{sg<w} sg \; dF_{w/g}(w/g) \right) dF_G(g) .$$

For processing of strand strength experimental data usually lognormal or Weibull distributions are used (both corresponding statistical hypotheses are accepted nearly at the same level of significance). If we'll use logarithm scale, then both distributions are distributions with location and scale parameters. We'll use later a normal $N\left(\theta_{0x}, \theta_{1x}^2\right)$ distribution of $X = ln\ W$, because at some additional assumption (independent normal $N\left(\theta_1, \theta_{1G}^2\right)$ distribution of $Y = ln\ G)$ all calculations can be done much easier, but nevertheless some results can be used and for Weibull distribution also.

$$\text{If } F(x) = F_0\left(\frac{ln\ x - \theta_0}{\theta_1}\right),$$

where $F_0(.)$ is some known c.d.f., then it can be shown that

$$s_b = \max_s s\left(1 - F(s)\right) = x^*\left(1 - F(x^*)\right),$$

where $\ln x^* = \theta_0 + \theta_1 t^*(\theta_1)$,
$t^*(z) = \lambda_0^{-1}(z)$, $\lambda_0(z) = f_0(z) / (1 - F_0(z))$,
$\lambda_0^{-1}(.)$ is inverse function .

Then $\zeta = \ln s_b = \theta_0 + \gamma(\theta_1)$,
where $\gamma(z) = z\, t^*(z) + \ln (1 - F_0(t^*(z)))$.

In order to get statistical characteristics of s_b we offer to use $x \cdot \left(1 - \hat{F}(x)\right)$ instead of $x \cdot (1 - F(x))$, where $\hat{F}(x)$ is approximation of empirical distribution function, corresponding to actual final sample of n component strengths $s_1, s_2, \ldots, s_n$

$$\hat{F}(x) = F_0\left(\frac{\ln x - \hat{\theta}_0}{\hat{\theta}_1}\right),$$

$\hat{\theta}_0, \hat{\theta}_1$ are (for example, maximum likelihood (ML)) estimations of θ_0 and θ_1, which can be obtained by processing $s_1, \ldots, s_n$ (we imagine, that we can see these values).

Linear approximation of $\gamma(z)$ (by the use of only linear terms of Taylor's formula) gives us following asymptotic formula (if n is large enough)

$$\hat{\zeta} = \hat{\theta}_0 + \gamma(\theta_1) + t^*(\theta_1)\left(\hat{\theta}_1 - \theta_1\right),$$

where random variable $\hat{\zeta}$ corresponds to random sample $s_1, \ldots, s_n$.

ML estimations of parameters $\hat{\theta}_0, \hat{\theta}_1$ have asymptotic normal distribution. So random variable $\hat{\zeta}$ also has asymptotic normal distribution $N\left(\theta_{0\zeta}, \theta_{1\zeta}^2\right)$, where

$$\theta_{0\zeta} = \theta_0 + \gamma(\theta_1),$$

$$\theta_{1\zeta}^2 = D(\hat{\theta}_0) + \left(t^*(\theta_1)\right)^2 D(\hat{\theta}_1) + 2\, t^*(\theta_1) \cdot Cov\left(\hat{\theta}_0, \hat{\theta}_1\right),$$

$D(X)$ is variance of r.v. X, $Cov(X,Y)$ is a covariance of r.v. X and Y.

For normal distribution, as it is known, we have

$$D(\hat{\theta}_0) = \theta_1^2 / n; \quad D(\hat{\theta}_1) = 2\,\theta_1^2 / n; \quad Cov\left(\hat{\theta}_0, \hat{\theta}_1\right) = 0.$$

All this formulae can be used for the case, when $G = Const$. But if we take into account distribution of r.v. G, then it can be shown that we'll get

$$\zeta = \theta_{0X} - \theta_{0Y} + \gamma(\theta_1);$$
$$\theta_1^2 = \theta_{1X}^2 + \theta_{1Y}^2.$$

And if again we'll use linear approximation for $\gamma(\theta)$, then $\hat{\zeta}$ again will have normal distribution with $\theta_{0\zeta} = \zeta$ and

$$\theta_{1\zeta}^2 = D(\hat{\theta}_{0X}) + k^2 D(\hat{\theta}_{1X}^2) + \left(k - \frac{1}{2}\right)^2 D(\hat{\theta}_{1Y}^2),$$

where $k = t^*(\theta_I) / 2\theta_I$

and variances $D(\hat{\theta}_{0X}), D(\hat{\theta}_{1X}), D(\hat{\theta}_{1Y}^2)$ are given earlier.

Experimental Data Processing

Special tests were made for the purpose of the model check [9]. The values of 64 carbon-fiber strand strengths and 64 special specimen strengths were obtained. Every specimen was made of 10 strands. For processing of test results the lognormal distribution was used. Parameter estimations are given in table 1. On the base of fiber test information [10] the prediction for 1-strand test result was made (only for parameter θ_0 (for mean strength logarithm), prediction for parameter θ_1 is evidently very bad, because a number of fibers in one strand is very large). On the base of 1-strand test results the prediction for 10-strand specimen test results was made. In model 1 we did not take into account the randomness of rigidity G, in model 2 we suppose that θ_{0Y} is equal to zero but θ_{IY} is equal to standard deviation of elasticity modulus [10]. We see that prediction of mean strength is not too bad, but prediction much worse for standard deviation.

The standard deviation does not decrease proportionally to $1/\sqrt{n}$ or we should take into account another value of n: relatively small number of components in some random small microvolume. It seams, that after distraction of this volume some " wave of distraction" begins to move across the cross section similar to the phenomenon, which we see, when we try to tear some fabric. In considered case we'll get coincidence with test results if put n=5.

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Table 1

Source \ Parameters	θ_{1G}	S_b, *MPa*	θ_0	θ_I
1 fibre				
Test		1491	7.198	0.467
1 strand				
Test		702	6.554	0.181
Model 1	0	733	6.598	-
Model 2	0.332	733	6.598	-
10 strands				
Test		525	6.263	0.194
Model 1	0	501	6.217	0.120
Model 2	0.332	501	6.217	0.136

Brittle Fatigue Model

Now consider connection between composite component static strength distribution parameters and fatigue life. Let s_1, s_N are stresses in unbroken components at the beginning of first and N - th cycles correspondingly (it is assumed, that s_N is a maximum stress of pulsating cycle). If again we assume uniform distribution of tension loads between parallel unbroken components (more exactly, the same absolute value of strain in all unbroken components), then mean value of cross section part of unbroken components will be equal to

$$\psi(s_N) = \int (1 - F_{W|g}(gs_N \mid g))dF_G(g) \, .$$

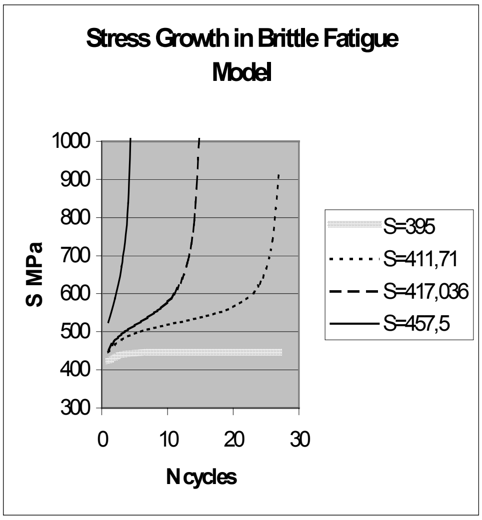

Fig.1

Then stress at the beginning of (N+1) -th cycle will be equal to $s_1/\psi(s_N)$. So we can calculate a sequence of stress (in unbroken components). Examples of the calculation sequences { s_N } for θ_0=6.55 and θ_l=0.378 are shown in Fig. 1. Value θ_0=6.55 is taken just from table 1, value θ_l=0,378 is result of calculations: $\theta_l^2=\theta_{lX}^2+\theta_{lY}^2$, where θ_{lX}=0.181, θ_{lY}=0.332 are also taken from table 1.

We see that at the smallest stress S=395MPa intensive decreasing (up to zero) of stress grows rate. So this stress is lower than predicted fatigue limit. At the others stress levels the internal stress grows rate relatively high at the beginning process, then relatively small and finally increases very drastically. As it is well known,that different specimen parameters have the same behavior during fatigue test.

Stress ceases to grow, if $s_{(N+1)}=s_N$ or stress s is such, that

$$s_1/\psi(s)=s \quad or \quad s_1=s\,\psi(s).$$

Maximum value of s_1, for which such stress s exists, can be considered as fatigue limit. After comparison with expression for ultimate strength s_b we see, that fatigue limit for brittle fatigue model coincides with ultimate strength, if we use the same distribution parameters of static strength of components. In literature, indeed, there are notes, that fatigue strength is very close to static strength (see pp. 363, 367 in [11]).

By the use of this model we can calculate the function $N(s_1, s)$, which gives the cycles number, needed to increase initial stress s_1 to some large value s. “Brittle” fatigue curve, the function $N_b{}^*(S)$, is defined then by formula

$$N_b{}^*(S)= N(S, S^*)= \min \{N:\ s_{(N+1)}=S/\psi(s_N)>S^*\}.$$

Here S* is enough great stress value, corresponding to full destruction of specimen (for example $\log(S^*)= \theta_0 + \theta_l F_0^{-1}(0,999)$).

If N is large enough, we can make assumption that N is a continuous variable and put derivative ds/dN to be proportional to stress increase in one cycle:

$$ds/dN = (s_{(N+1)} - s_N)/\kappa.$$

Then dN/ ds=1/(ds/dN) and number of cycles required to increase s_1 up to s is equal approximately to

$$N(s_1, s) = \kappa \int_{s_1}^{s} (1/h(s))ds,$$

where $h(s)=s_1/\psi(s) - s$. If s more than fatigue limit, then denote by s_h the value of s at which we have minimum of $h(s)$. The use of expansion in Taylor's series of function $h(s)$, neglecting terms above the second order ($h(s)=a_0+a_2(s-s_h)^2+\ldots$), gives the following approximation

$$N(s_1, s) = K(\arctan(D) - \arctan(D_1)),$$

where $D=(s - s_h)(a_2/a_0)^{1/2}$, $D=(s_1 - s_h)(a_2/a_0)^{1/2}$, $K=k(a_2 a_0)^{-1/2}$.

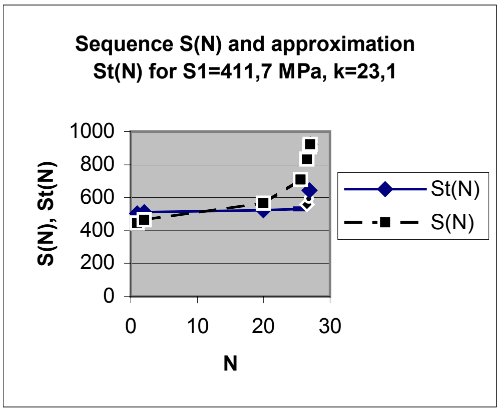

Fig. 2

For the log-normal distribution $a_0=s_1/R - s_h$, $a_2= (F''+2F'/R)\, s_1/2R^2$, $F'=\varphi(t)/\,\theta_1 s_h$, $F''=(\theta_1+t)F'/\theta_1 s_h$, $R=1-\Phi(t)$; $\Phi(.)$,$\varphi(.)$ are cumulative distribution function and distribution density function of standard normal distribution ; $\log(s_h) = \theta_0 + \theta_1 t$, t is solution of equation

$$\log(\theta_1) + \theta_0 + \theta_1 t - \log(R/(1-\Phi(t)) = \log(s_1).$$

For fixed s_1 we can get inverse function $s_t(N)$: stress as function of N, which is, in fact, approximation of sequence s_N. Example of comparison of sequence s_N and function $s_t(N)$ for the same parameters, which were used for Fig.1 but for s_1= 411,7 MPa, is shown in Fig.2.

Relevant "brittle" fatigue curve, the function $N_{bt}*(S)$, can be approximated by formula

$$N_{bt}*(S) = N(S,\infty) = K(\pi/2 - \arctan(D_1)) = K(\pi/2 + \arctan((s_h - S)(a_2/a_0)^{1/2}).$$

Functions $N_b*(S)$ and $N_{bt}*(S)$ are very similar to real S-N curve. For example, they predict existence of fatigue limit as function of parameters θ_0, θ_1. But numerical distinction from real S-N curve is very large. In Fig. 1 we see that $N_b*(S)$ gives very small number of cycles to failure or ∞. So we can use these functions only for regression analyses. In [9] some results of fatigue test of carbon-fibre composite specimens are given. To fit model to this dataset we used θ_0=5.98 (instead of 6.55; this means, that we assume θ_{0Y} is equal to 0.57) and coefficient k=2 126 525. Calculation results (*N*) and experimental data (*Nexp*) are shown in Fig.3.

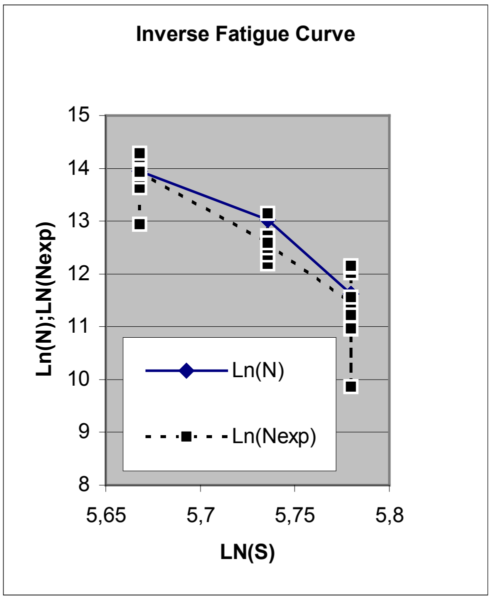

Fig.3

We see calculation results are in a range of experimental data scatter.

Main advantage of the offered model is that *N(S)* function includes parameters of component static strength distribution function and fatigue limit is not parameter of regression function, but itself is function of the same static strength parameters.

Statistical characteristics of S-N curve can be obtained in the same manner as statistical characteristics of static strength (by the use of asymptotic normal distribution of ML parameters estimations $\hat{\theta}_0, \hat{\theta}_1$, corresponding to processing of real strength of components in relevant micro volume). Here we again should take into account that fatigue damage accumulation process takes place in some random small micro volume, with relatively small number of components n, as it can be seen from the fact, that variation coefficient of fatigue life, which should be approximately proportional to the $1/\sqrt{n}$, is not too small usually and (it is very important), in fact, does not decrease, when the size of tested specimens cross section increases.

Conclusions and Areas for Further Research
Some Ideas of Plastic Fatigue Model

The considered brittle fatigue model offers the explanations of some phenomena's, which we see in fatigue test of composite materials (S-form curve of internal stress growth, high fatigue limit) and can give some prompts for prediction of fatigue life changes if there are some static strength changes, but it fails to get adequate numerical results:

1) usually the fatigue limit is much lower than predicted one;

2) the model offers too drastic increasing of fatigue life if stress is near fatigue limit;

3) it did not take into account a range of cycle stresses.

Some improvement of the model can be done if we take into account that fatigue is connected with some damage accumulation in some plastic part of material. Suppose that there are two parts of specimen body. One of them consists of large number of parallel brittle elastic items with random ultimate strength, logarithm of which has normal distribution with parameters θ_{0E}, θ_{IE}. This part is characterized by initial relative cross section f_{0E}, $0 < f_{0E} < 1$. All these items have the same elasticity modulus E_E. The other part consists of large number of parallel plastic items with random yield limits. We'll suppose the lognormal distribution of yield limit with parameter θ_{0Y}, θ_{IY}. All of these items have the same elasticity modulus E_Y. If stresses lower than corresponding yield limit, then these items are elastic (with elasticity modulus $E_Y < E_E$). The initial relative area of cross section of plastic part is equal to $f_{0Y} = 1 - f_{0X}$. All these components are working together (this means, that all of them always have the same length). So if in a brittle elastic unbroken components there is some stress s_E , then mean value of nominal stress in cross section will be defined by formula

$$s(s_E) = s_E f_{0E}(1 - F_E(s_E)) + (1 - f_{0E}) \int s_{YE}(s_E, s\ , l)\, dF_{S_Y}(s),$$

where

$$s_{YE}(s_E, s_Y, l) = \begin{cases} \varepsilon_Y E_Y, & if \quad \varepsilon_Y \le s_Y / E_Y, \\ s_Y, & if \quad \varepsilon_Y > s_Y / E_Y, \end{cases}$$

is a stress in plastic components with yield limit s_Y and length l, when in elastic components stress is equal to s_E,

$\varepsilon_Y = (1 + s_E / E_E) / l - 1$ is a strain of plastic component, the length of which before considered cycle is equal to l, but after the maximum of cycle stress will be applied, then new length

$$l'(l, s_Y) = l + \max\,[0, (\varepsilon_Y - s_Y / E_Y)]\,.$$

Calculating function $l'(l, s_Y)$ we can find residual stress Δs_E in elastic part of specimen. For pulsating cycle Δs_E is a root of equation

$$s\ (\Delta s_E) = 0.$$

We make also following assumption: in next cycle the new stress of elastic part will be equal to $s'_E = s_E + \Delta s_E$. This assumption defines the rate of elastic part stress growth. Example of calculation shows, that behavior of corresponding function of stress s_E growth has the same form as in Fig.1.

If we make such calculation for every cycle and define condition of destruction by inequality $s_E > F_E^{-1}(q)$, where probability of failure $q = 1 - \varepsilon$, ε is a very small, $F_E^{-1}(q)$- q-quantile of elastic part strength distribution function (can be made additional limitation of length l also), then we can calculate cycle number to failure and, finally, to get, a fatigue curve, S-N. But it appears that such calculations require too much time, because in every cycle we need to get solution of special equation and should calculate some integral at every step of iterations. So solution of the problem is a subject of another paper.

References

1. **H.E. Daniels,** “The statistical theory of the strength of bundles of threads,” *Mathematical and Physical Sciences,* Series A, v. 183, #995, 405-435 (1945).
2. **H.E. Daniels,** The distribution of bundle strength under general assumptions. Paper presented at the First World Congress of the Bernoulli Society, Tashkent, USSR, September 1986, -3p.
3. **F.G. Pascual and W.Q. Meeker,** “Estimating Fatigue Curves With the Random Fatigue-Limit Model,” *Technometrics*, v.41, 277-302 (1999).
4. **S.L. Phoenix and H.M. Taylor,** The asymptotic strength of a general fiber bundle. – *Adv. Appl. Prob.* 1973, vol. 22, p. 200-216.
5. **R.E. Pitt and S.L.Phoenix,** Probability distributions for the strength of composite materials, III and IV. Int. J. Fracture, v. 20, 1982, 291-311 and v.22, 243-276.
6. **R.L. Smith,** Statistical models for composite materials. Paper presented at the First World Congress of the Bernoulli Society, Tashkent, USSR, September 1986, -11p.
7. **P. Volf and A. Linka,** On reliability of system composed from parallel units. Paper presented at the First International Symposium on Industrial Statistics, Linkoping, Sweden, August 1999, -9p.
8. **Клейнхоф М.А., Парамонов Ю.М.** Статистическая модель разрушения композиционного материала. Научные труды Московского ин-та инж. ГА, Динамика, выносливость и надежность авиационных конструкций и систем. – Москва,1980. – С. 57-61.
9. **Клейнхоф М.А.** Исследование статической и усталостной прочности композитных материалов, используемых в конструкции летательных аппаратов / Диссертация на соискание ученой степени к.т.н. – Рига, 1983.
10. **Кобец Л.П.** Исследование стабильности физико-механических свойств углеродных волокон . 1.Зависимость прочности модуля Юнга от площади поперечного сечения волокна; 2. Зависимость прочности при растяжении от площади поперечного сечения волокна; Механика полимеров, 1975, №3, с.430-435; №6, с. 1005-1010.
11. **Композиционные материалы.** Том 5. Разрушение и усталость. – Москва: Мир, 1978 – 484 с.
12. **Малмейстер А.К., Тамуж В.П., Тетерс Г.А.** Сопротивление полимерных и композитных материалов. – Рига: Зинатне, 1980. – 572 с.