NORMAL, LOGNORMAL, WEIBULL OR P-SEV DISTRIBUTION OF ULTIMATE STRENGTH OF STRANDS AND COMPOSITE SPECIMENS

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Analysis of ultimate strength of strands (bundles) and specimens of composite material is considered with the use of simple goodness-of-fit test for cumulative distribution function (cdf) with location and scale parameters. The test, offered for preliminary analysis, is based on comparison of ordered statistics with their standard distribution expected values (when location parameter is equal to zero and scale parameter is equal to one). For specific W-L dilemma testing (**W**eibull against **L**ognormal distribution) some approximation of uniformly most powerful invariant test (UMPIT) is offered. The critical boundary of the region of rejection of null-hypothesis and power of the tests can be calculated using modern PC. Numerical examples (processing of ultimate strength dataset) are given. As a subject of further research new p-sev family distribution is offered for ultimate strength distribution description. The weakest link model can be combined in this family with lognormal or some other distribution law.

Keywords: *distribution function, composite, static strength*

1. INTRODUCTION. INVESTIGATION OF DISTRIBUTION FUNCTION OF STATIC STRENGTH OF COMPOSITE ITEMS

The significant dependence of static strength of composite on scatter of static strength of its items can be seen in the following example. Let us consider three parallel items, which have the following strengths: 10 N, 15 N, and 30 N. It is a surprise in some way that the total loads 30 N will destroy them, as if the strength of every item is equal to 10 N. Why?

The reason is that by the total load 30 N at first will be destroyed the weakest item because its strength is equal to 10 N and, at the uniform distribution of total loads, its load is equal to 10 N also. Now uniform load of every "survived" item is equal to 15 N. So the second item, the strength of which is equal to the same value 15 N, will be destroyed. Now the load for the last strongest item will be equal to 30 N. It will be destroyed also because its strength is just equal to this load. This process is shown in Fig.1. The same "domino phenomenon" for n items take place if there strengths are proportional to the items of harmonic series: 1, ½, 1/3,…, 1/n. So it is obvious, that it is necessary to study distribution of strength of components of composite. Usually Weibull distribution and its modification are used for processing of tensile ultimate strength [1]. But sometimes-lognormal distributions are considered appropriate also [2]. It is not easy to differentiate one from another [3]. In logarithm scale both these distributions are distributions of location-scale families and for corresponding distribution identification a graphical technique is used usually. "It allows check visually how closely the data follow the hypothetical distribution function. Furthermore, it provides a quick estimation of distribution parameters" [4]. We offer to make some change of this technique so at first we'll give it's short description. Let $x_{(1)}, x_{(2)}, \ldots x_{(n)}$ be the ordered observations from population with cdf of the type $F(x) = F_0((x-\theta_0)/\theta_1)$. The main idea of this graphical technique is: the values $x_{(i)}$ are plotted against $F_0^{-1}((i-C_1)/(n+C_2))$, where $F^{-1}(.)$ is the inverse function of $F(.)$, C_1, C_2 are some constants. The using of these constants allows to avoid the following problem: if i/n is close to zero then $F^{-1}(.)$ is close to $-\infty$ and when i/n is close to 1 then $F^{-1}(.)$ is close to $+\infty$ but real $x_{(i)}$ are finite and for *i*=1 and *i* =n.

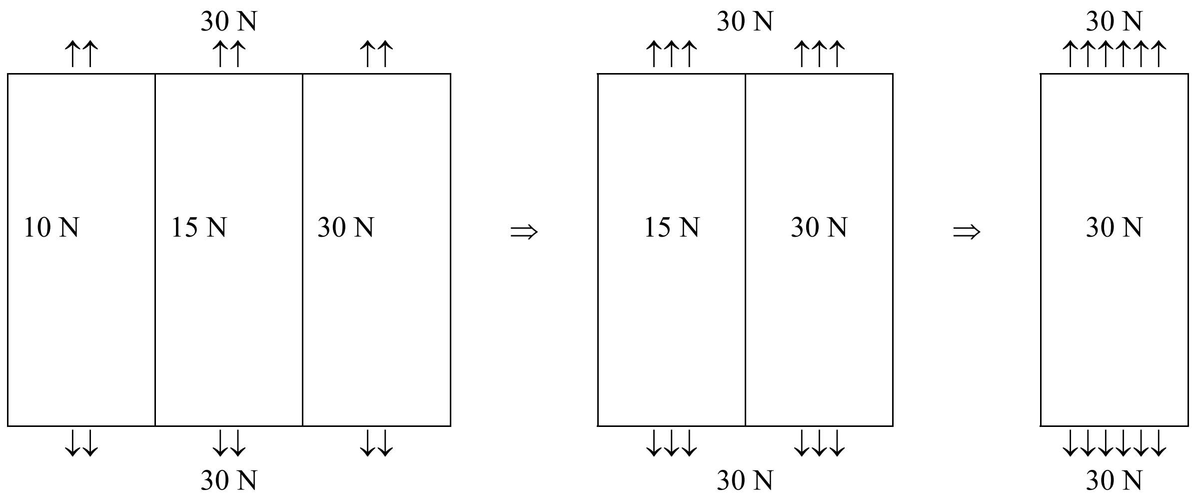


Figure 1. *The ultimate strength of "composite" of three items is defined by strength of the weakest item*

But choice of C_1, C_2 is very obscure and really it should depend on *n* and $F(.)$. The values C_1=0.5, C_2=0 are recommended, for example in [4], the values C_1=0.3, C_2=0.4 are used in [1].

The other disadvantage of this graphical technique is that it is used only for ***visual*** testing hypotheses. We offer to avoid both disadvantages. Much more precise preliminary analysis of the considered problem is the use of the offered here OSPPTest (**T**est based on **P**robability **P**lot of **O**rdered **S**tatistics, $x_{(i)}$, against expected values of standard order statistics, $E(\overset{o}{X}_{(i)})$, corresponding to θ_0=0, θ_1=1). It is discussed in section 2. In section 3 for specific W-L dilemma testing (**W**eibull against **L**ognormal distribution) some approximation of uniformly most powerful invariant test (UMPIT) is offered. Numerical examples are provided for both tests. In section 4 analysis of "classical model of bundle of n parallel fibres are stretched between two clamps", which is considered as a model of unidirectional composite. It is shown that Daniels prediction of normal distribution should be used only with great caution because uniform distribution of stresses between fibres probably does not take place. In section 5 as a subject of further research a new family of ultimate strength is offered in which the weakest link model can be combined with lognormal or some other distribution law.

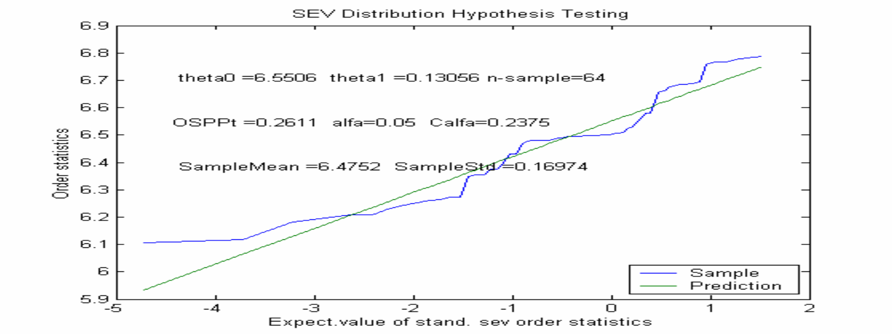
2. OSPPTEST FOR PRELIMINARY ANALYSIS

So it is offered to plot $x_{(1)}, x_{(2),} ... x_{(n)}$, the ordered observations from population with cdf of the type $F(x) = F_0((x-\theta_0)/\theta_1)$ against $E(\overset{o}{X}_{(i)})$, expected values of standard order statistics, corresponding to θ_0=0, θ_1=1.

Corresponding plot again can be used for visual analysis. But the critical region for the test of the hypothesis in question is offered to define by inequality

$$OSPPt = \overline{R}_{LR} = (1-R^2)^{1/2} = \left(\sum_{i=1}^{n} (\hat{x}_{(i)} - x_{(i)})^2 / ns^2 \right)^{1/2} > C_{alfa},$$

where *OSPPt* is statistic of OSPPTest, $\overline{R}_{LR}$ is specific notation of *OSPPt*, if linear regression analysis is used for θ_0 and θ_1 estimation (another index must be used if another method of estimation is used), R^2 is the standard statistic of linear regression analysis (it is so called coefficient of determination), $\hat{x}_{(i)} = \hat{\theta}_0 + \hat{\theta}_1 E(\overset{o}{X}_{(i)})$, $\hat{\theta}_0$, $\hat{\theta}_1$ are estimates of θ_0, θ_1, $\overline{x} = \sum_{i=1}^{n} x_i / n$, $s^2 = \sum_{i=1}^{n} (x_i - \overline{x})^2 / n$.



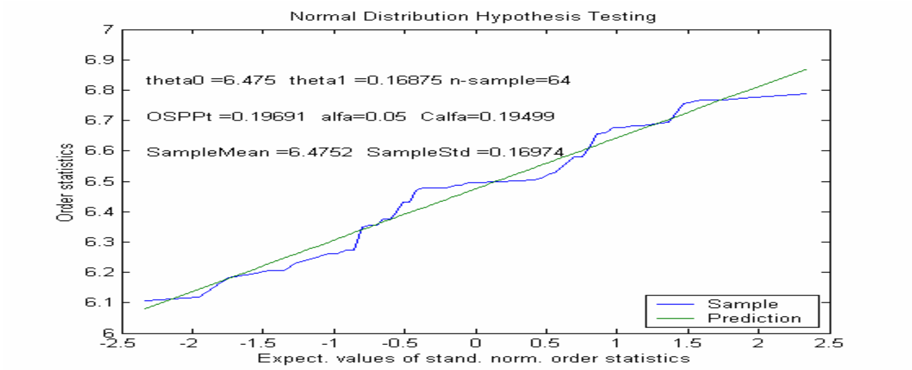


Figure 2. *OSPPT plots for the 1-strand-specimen ultimate strengths. Null hypotheses: sev (Weibull) in upper part and normal(lognormal) distributions in lower part*

Results of processing of sample, $(y_1,..., y_{64})$, of ultimate strengths of 64 carbon-fibre strands for testing of Weibull and lognormal hypotheses are shown in Fig.2 $x_{(i)}$, against $E(\overset{o}{X}_{(i)})$. It should be mentioned that in considered problem $X = \log(Y)$ has smallest extreme values (sev) or normal distribution with cdf

$$F_X(x) = 1 - \exp(-\exp((x - \theta_0)/\theta_1)) \quad \text{or} \quad F_X(x) = \Phi((x - \theta_0)/\theta_1)$$

if random variable *Y* has Weibull or lognormal distribution correspondingly.

We see that in accordance with Monte Carlo estimate of the (1-α)-quintile (where α is significance level) of *OSPPt*- statistic, Calfa, for α=.05 (see Fig.2) both hypotheses should be rejected. But really this decision is very doubtful because the power of this test for considered specific W-L dilemma is equal to 0.55 (see Fig.3).

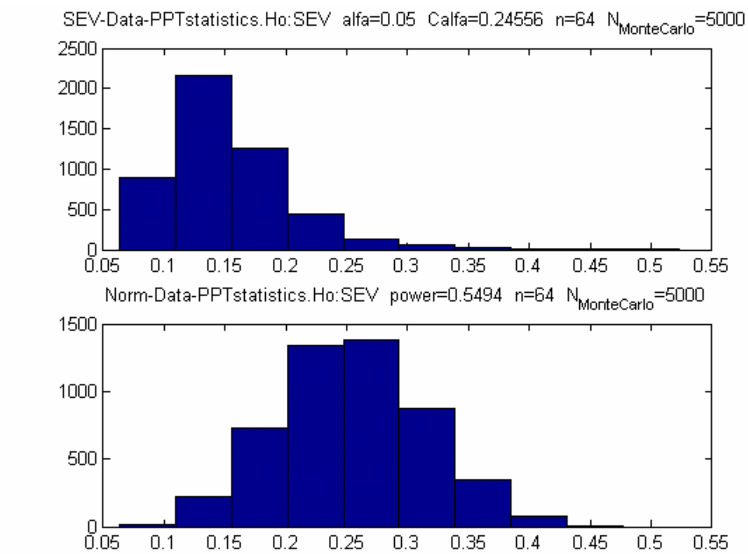


Figure 3*. Histogram of OSPPt statistics; hypothesis H_0 is sev distribution alternative H_1 is normal distribution*

Probably it can be increased if the best linear unbiased estimates will be used instead of estimates of linear regression analysis. In this case estimates of parameters θ_0 and θ_1 are defined by formula

$$(\hat{\theta}_0, \hat{\theta}_1)^T = (A^T V^{-1} A)^{-1} A^T V^{-1} X ,$$

where $X = (X_{(1)}, ..., X_{(n)})^T$,

$$A = \begin{bmatrix} 1 & E(\overset{o}{X}_{(1)}) \\ 1 & E(\overset{o}{X}_{(2)}) \\ \multicolumn{2}{c}{\dots\dots\dots\dots} \\ 1 & E(\overset{o}{X}_{(n)}) \end{bmatrix},$$

$V = \| Cov[\overset{o}{X}_{(i)}, \overset{o}{X}_{(j)}] \|$, $i, j = 1,2,...,n$ is variance-covariance matrix of standard order statistics, symbol $(.)^T$ is transposition sign.

Corresponding *OSPPt* can be denoted by $\overline{R}_{BU}$ ("best unbiased"). Similarly *OSPPt* based on using the method of moments, $\overline{R}_{MM}$, or maximum likelihood, $\overline{R}_{ML}$, and any others (but with "true" structure: $\hat{\theta}_0(a+bx_1,...,a+bx_n) = a + b\hat{\theta}_0(x_1,...,x_n)$, $\hat{\theta}_1(a+bx_1,...,a+bx_n) = b\hat{\theta}_1(x_1,...,x_n)$) can be considered. Investigation of power of corresponding tests is subject of special paper. Here we limit ourselves by standard regression analysis just because of its simplicity. But for solution of specific W-L dilemma testing (**W**eibull against **L**ognormal distribution) some approximation of uniformly most powerful invariant test (UMPIT) is discussed in section 3.

3. APPROXIMATION OF UNIFORMLY MOST POWERFUL INVARIANT TEST

In [4,5] can be found uniformly most powerful invariant test for testing hypothesis $f(x) = (1/\theta_1) g((x-\theta_0)/\theta_1)$ when the alternative $f(x) = (1/\theta_1) h((x-\theta_0)/\theta_1)$. But the calculation of distribution of corresponding statistics is very labour consuming.

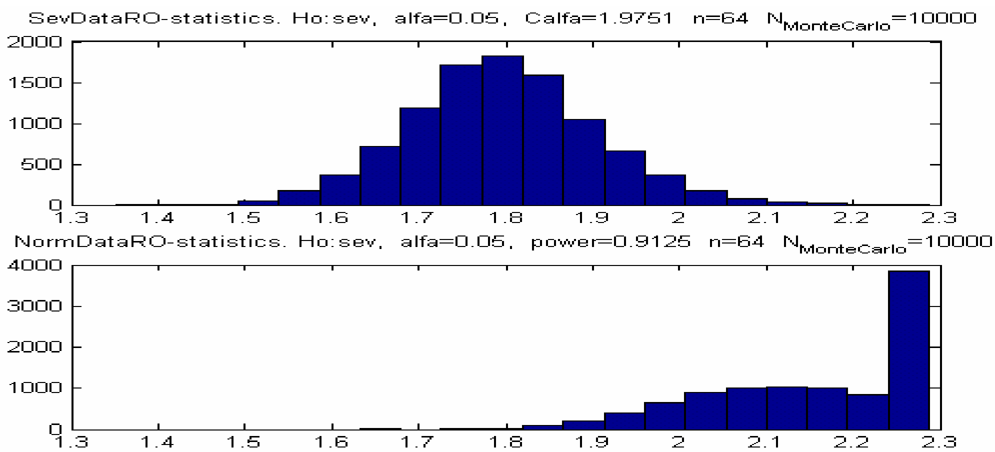


Figure 4. *The histograms of ρ-statistics for samples from sev and normal distributions*

For the case, when g(.) is sev probability mass function (pmf) but h(.) is normal pmf the approximation of UMPIT with the critical region

$$\rho = \sum_{i=1}^{n} \exp(\pi (x_i - \bar{x})^2 / s\sqrt{6}) > C_{alfa}$$

can been used [3,6]. It was shown that it has nearly the same power as the power of UMPIT. Now the rejection of Weibull distribution for α=0.05 (for processed dataset ρ=2.03>C_{alfa}=1.98) is much more convincing: the power of the ρ-test in this case appears to be equal to 0.91 (see Fig.4). Although really we never know the "truth" in this "random world" it seems that the lognormal distribution is more appropriate. This means that the model of the weakest item in *this case* does not work properly. The process of accumulation should be taken into account also. And it is the reason why the lognormal distribution appears more adequate for considered dataset.

4. RELATIONSHIP BETWEEN THE DISTRIBUTIONS OF ULTIMATE STRENGTH OF UNIDIRECTIONAL FIBRE COMPOSITE AND ITS COMPONENTS

At every step of development of complex composite material (from fibre to strands, from strands to film and then to multi layer composite) we see the change of ultimate strength probability distribution function and its parameters.

Solution of the problem to get more precise relationship between static strength of unidirectional fibre composite material and static strength of its rigid items (parallel rigid components (strands or fibres) is an important input to design-for-reliability process. A great number of articles are devoted to the problem. It seems that Daniels [7] made the most significant steps in this direction, studying in details a model, which is now called the "classical model of bundle of n parallel fibres stretched between two clamps". Phoenix and Tailor [8] extended the model to cover certain types of inhomogeneity among the fibres, such as random slack, but still within the basic framework of equal load sharing. Paramonov and Kleinhof [2] developed similar approach. Daniels himself [9] made, which have used this time the theory of Brownian Bridge. Local load sharing and the chain-of-bundles model were studied by Pitt and Phoenix [10] and by Smith [11] and earlier references therein]. Wolf and Linka [12] considered some new approach to the calculations of "the strength of rope composed from several strands". They have used the methods and relevant theory of counting processes to get solution of some mathematical aspect of this problem. Experimental data show that mean strength decreases, standard deviation decreases also when complexity of structure increases: ultimate strength of fibres lesser than ultimate strength of strand (bundle) and ultimate strength of strand lesser than ultimate strength of specimens.

The main idea of Daniels's model is uniform distribution of tension loads between parallel-unbroken items (strands or fibres). The main result of Daniels is formulated in following way: " If all the fibres have the same load-tension curve and $b(s)$ is probability of failure of one fibres under load s and (1-$b(s)$) converges to 0 faster than 1/s, then the strength s of strand of enough larger number $\boldsymbol{n}$ of fibres has normal distribution with expectation value

$$S_r = n \cdot s_r \cdot [1 - b(s_r)],$$

standard deviation

$$\sigma = s_r \cdot \sqrt{n \cdot b(s_r) \cdot [1 - b(s_r)]},$$

where s_r corresponds to maximum of $s \cdot [1 - b(s)]$".
So the mean strength is defined by formula

$$\bar{S}_r = s_r \cdot [1 - b(s_r)],$$

and its standard deviation by formula

$$\bar{\sigma} = s_r \cdot \sqrt{b(s_r) \cdot [1 - b(s_r)]} / \sqrt{n}.$$

In previous section we have chosen the lognormal distribution as the most appropriate for the static strength distribution of composite components. In this case we are interested in studying the function

$$y(x) = x(1 - \Phi_0((\log(x) - \theta_0)/\theta_1)),$$

where $\Phi_0(\cdot)$ – is normal standard distribution function.

Results of calculations of this function for several values of θ_0 = 6,0; 6,5; 7 and θ_1 = 0,15; 0,2; 0,25; 0,3; 0,35 are given in Fig.5.

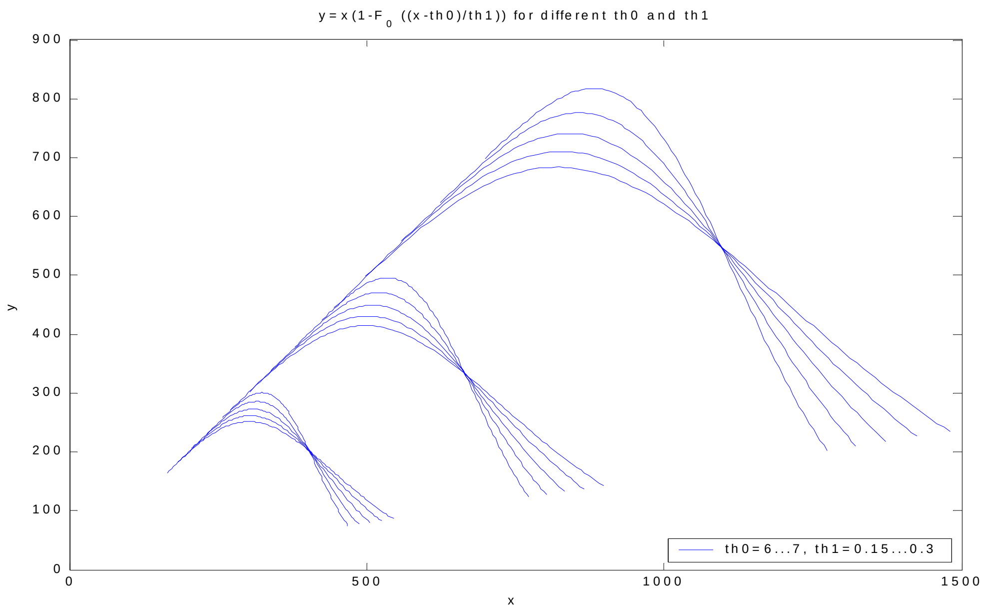


Figure 5. *Results of calculations of function* $y(x) = x(1 - \Phi_0((\log(x) - \theta_0)/\theta_1))$ *for* θ_0 *= 6,0; 6,5; 7 and* θ_1 *=0,15; 0,2; 0,25; 0,3; 0,35.*

Maximal values of these functions correspond to strengths of bundle of items with corresponding initial parameters of static strength distribution if the number of items is large enough. But as it will be showed later really we should consider the destruction of relatively small number of items. And we need to consider this case more attentively.

Just after destruction of r items the total load, which is applied to the still survived items, is equal to

$$S_{(r)} \cdot (n - r),$$

where $S_i, i = 1,...,n$ is random strength of ith item, $S_{(r)}$ – *rth* order statistic

$$S_{(1)} \leq S_{(2)} \leq ... \leq S_{(r-1)} \leq S_{(r)} \leq ... \leq S_{(n)},$$

of item strengths. So average strength of one item in bundle is defined by formula

$$\bar{S} = \max_r S_{(r)} \cdot \left(1 - \frac{r}{n}\right). \quad (1)$$

Ultimate strength tests of 64 one-strand specimens (bundle) and of 64 10-strands specimens (every specimen comprised 10 strands) were made in order to study the connection of corresponding parameters of ultimate strength distribution functions. The following lognormal distribution parameter and mean value of strength ($\hat{\theta}_0$, $\hat{\theta}_1$, $\hat{\mu}$) was obtained: (6.476, 0.169, 658 MPa) for one-strand and (6.156, 0.195, 481 MPa) for 10-strands specimens.

Monte Carlo modelling of ultimate strength of 5 - and 10-strands specimens was made with and without taking into account the scatter of the Young's modulus. Standard deviation of logarithm of Young's modulus (for one-strand specimen) is equal to 0.332. So total standard deviation was taken equal to 0.3723.

The best prediction of mean strength we get under assumption that if in10-strands specimens there are really only 5 strands and if we take into account the scatter of Young's modulus. In this case the predicted mean strength is equal to 482.39 MPa but real mean strength of 10-strands specimen is equal to 481 MPa. Predicted standard deviation of logarithm of strength is equal to 0.1904 but real standard deviation of logarithm of strength is equal to 0.1945. We see that prediction of strength by the use of Daniels model (really, formula (1)) is not too bad, but to get it we need to decrease the number of strands in Monte Carlo calculation.

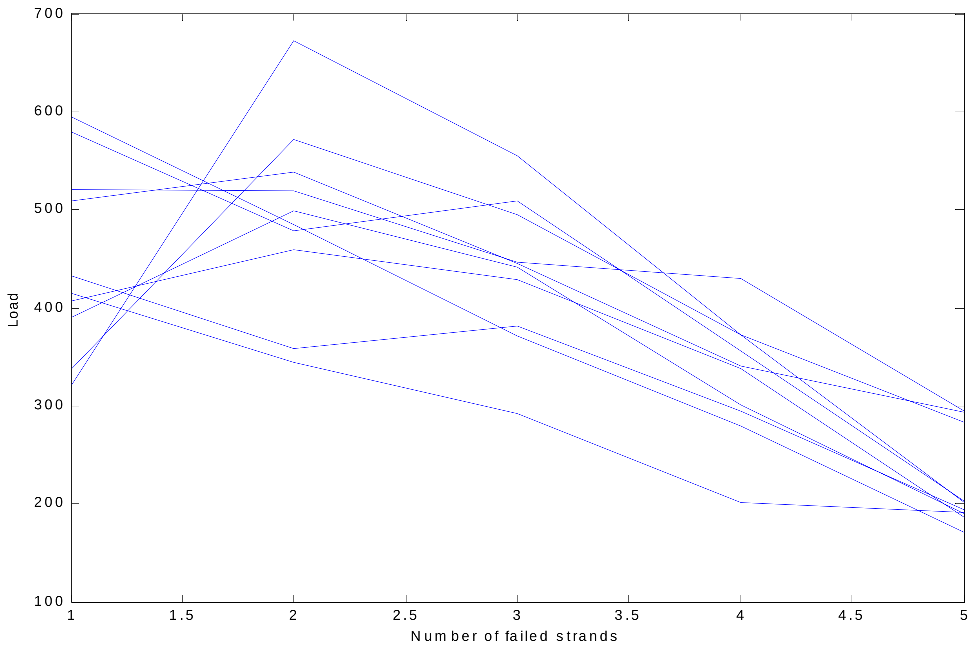


Figure 6. *Results of 10 Monte Carlo trials of strength of 5-strands specimens*

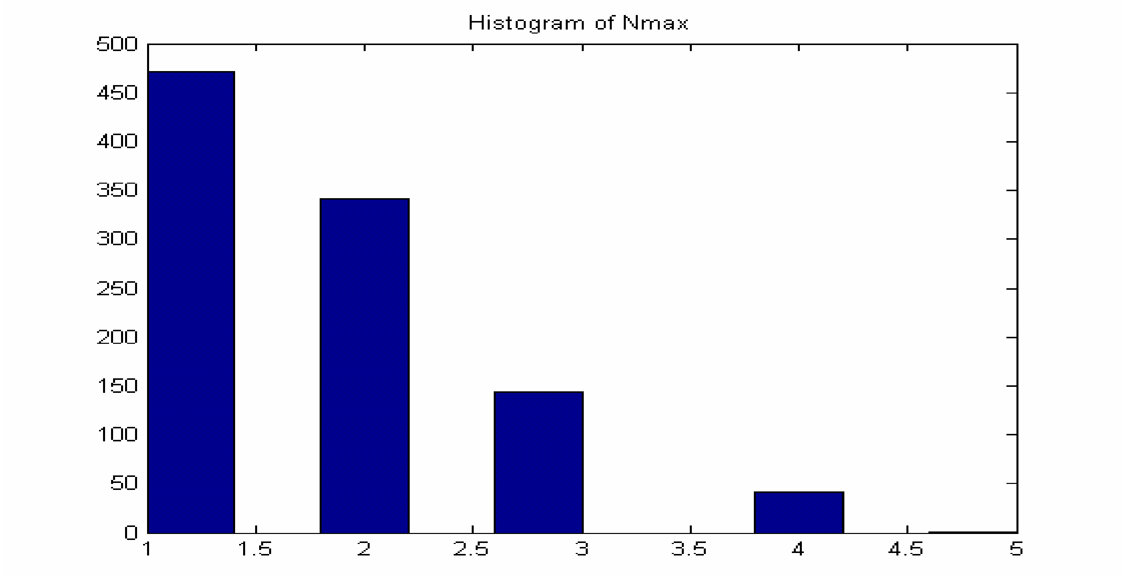


Figure 7. *Histogram of order numbers of fibre (Nmax) after failure of which the failure of specimen take place*

In Fig. 6 we see the samples of the random process of destruction of 5-strands specimens. We see that some times at the beginning of this process we can increase the load despite the failure of the weakest items but then we see that the load, which can be carried by survived specimens, decreases. The order number of the failure corresponding to the maximum of load is shown in Fig. 8. We see that in nearly 50% of trials the catastrophic failure take place just after failure of only one strand ("domino phenomenon'). This is the reason that during Monte Carlo modelling we should decrease the real number of strands in order to minimize the discrepancy between the result of calculation and test data.

10 realizations (10 Monte Carlo trials) of this type process for 5-fibers specimens are shown in Fig. 6, but total 1000 trials were made. For every realization of this process we can find maximum. It is the strength of specimen.

The histograms of strengths (maximums in Fig. 6) in natural and in logarithm scale and corresponding OSPPT plots are shown in Fig. 8.

Analysis of Fig. 8 shows that lognormal distribution much better fit these data than normal distribution. Daniels's theorem predicts normal distribution only if we have uniform distribution between n items and n is large enough. But really we have not uniform distribution. It can be true for bundle of disconnected threads but in composite new damage appears near previous damage. This process begins, as a rule, in the surface, then grows along cross-section of specimens and finishes by catastrophic failure of specimens. This means that even approximate uniform distribution of stress can be expected only in very limited volume with drifting location. For example, as it was mentioned already) the result of calculation of standard deviation of strength of 10-strands specimens (by Monte Carlo method) is very near to the previously mentioned experimental data if we assume (for Monte Carlo modelling) that in this specimens there are only 5 strands.

5. AREAS FOR FURTHER RESEARCH. P-SEV DISTRIBUTION FAMILY

The main result of presented investigation is the great doubt that the widely used Weibull distribution is the only one, which should be used for processing of ultimate strength data of fibre and strands. At least sometimes lognormal distribution is more appropriate. Should be mention however some indisputable advantage of Weibull distribution: it offers very natural explanation of decreasing of mean ultimate strength if the fibre length increases [1]. But similar model can be offered and with the use of lognormal distribution. Let us considered a fibre as a chain of n elementary items with length l_1.

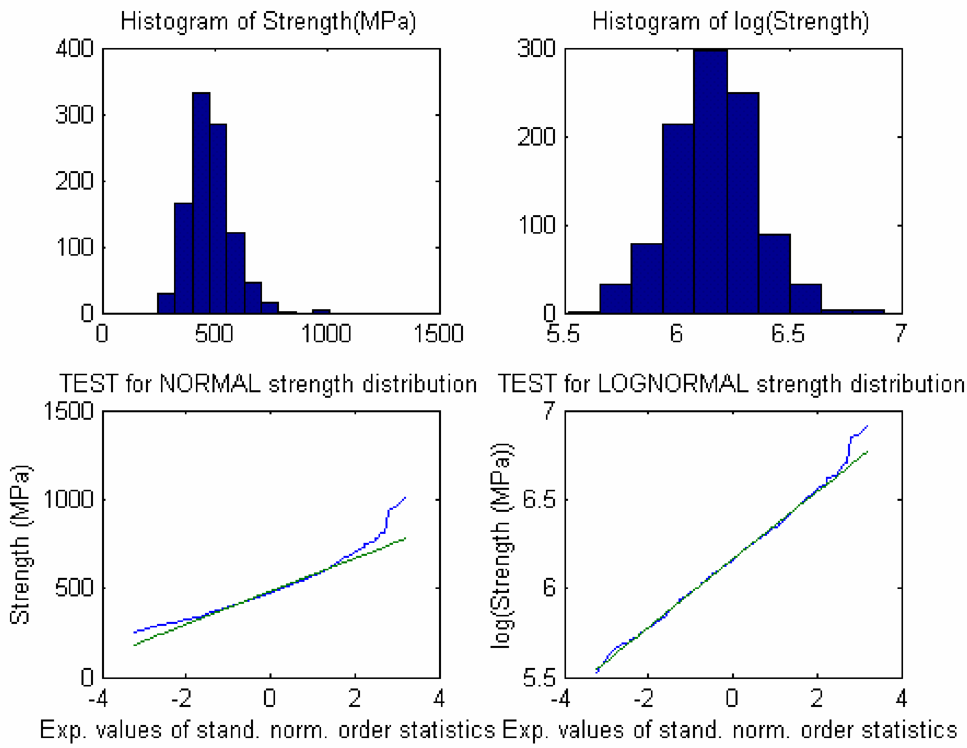


Figure 8. *Histograms (1000 Monte Carlo trials) of strength of 5-fibers specimens and OSPPT plots in natural and logarithm scale*

Let us suppose that fibre destruction process has two stages. First, in K items, $K \le n$, some flaws appear. Then in the weakest item the destruction process develops up to fibre specimen failure. Stress is uniformly distributed along the fibre so binomial distribution of random variable K is expected. Corresponding probability mass function

$$p_k = b(k;n,p) = (n!/(k!(n-k)!))p^k(1-p)^{n-k},$$

where $n = [l/l_1]+1$, $[x]$ rounds the x to the nearest integers towards minus infinity,

$p = F_0(s)$, s is stress , $F_0(x)$ is some cumulative distribution function (cdf).

Then cdf of specimen strength

$$F(x) = \sum_{k=0}^{n} p_k(1-(1-F_1(x))^k),$$

where $F_1(x)$ is cdf of strength of elementary item (with length l_1).

If n is large enough then instead of Binomial the Poisson distribution can be used with parameter $\lambda = (l/l_1)\ F_0(x)$. The number of elementary items should be more or equal to 1, so equation

$$F(x) = \sum_{k=0}^{\infty} \exp(-\lambda)(\lambda^k/k!)(1-(1-F_1(x))^{k+1}) \quad (2)$$

is more appropriate.

For different $F_0(x)$ and $F_1(x)$ we have the whole family of models. The lognormal distribution (and Weibull also!) can be used here for describing cdf $F_0(x)$ and $F_1(x)$. The calculation of $F(x)$ is not too difficult if we use Monte Carlo (MC) method. Let N is the number or random variables (rv) with Poisson distribution and $\{K_1, K_2, ..., K_N\}$ is the set of rv from corresponding population. If N is large enough then approximately

$$F(x) = (1/N)\sum_{i=0}^{N}(1-(1-F_1(x))^{K_i+1}).$$

This family of distributions may be called ***p-sev family***, because it is connected with Poisson and sev (smallest extreme value) distributions. In the simplest case we can assume that cdf $F_1(x) = F_0(x) = \overset{0}{F}((x-\theta_0)/\theta_1)$, where $\overset{0}{F}(\cdot)$ is some known (standard) cdf, then for ***fixed*** θ_2 =(l/l$_1$) the cdf $F(x)$ again has location and scale parameters: θ_0 and θ_1. So all the methods of mathematical statistics appropriate to the location-scale cdf family can be used and for offered $F(x)$ (but only for fixed θ_2 !). For example, for fixed θ_2 using method of moment we have following estimates of θ_0 and θ_1:

$$\hat{\theta}_1 = (s^2)^{1/2}/\overset{0}{\sigma},\ \hat{\theta}_0 = \bar{x} - \hat{\theta}_1\overset{0}{\mu},$$

where $\overset{0}{\mu}$, $\overset{0}{\sigma}$ are mean and standard deviation of X for θ_0=0 and θ_1=1, as functions of θ_2 these values can be found using (for example) Monte Carlo method.

Parameter θ_2 defines the dependence of mean ultimate strength on the length of fibre. Preliminary investigations of the dataset described in [1] show that this dependence is nearly the same as the one provided by the cdf

$$F(x) = 1-\exp(-(l/l_0)^{\gamma}(x/\beta)^{\alpha}), \quad (3)$$

which was offered [13,14, 15] especially for better fit of experimental data with different length. It seems that model (2) has more natural ground and allow much more general approach than the model (3) and models described in [16-19]. At least it has right to exist and deserves more checking. But this is area for further research.

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