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RELIABILITY OF SERIES OF PARALLEL SYSTEM WITH DEFECTS. MINMAXDM DISTRIBUTION FAMILY APPLICATION TO COMPOSITE STRENGTH ANALYSIS

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Reliability of series and parallel systems with two types of structural items (defected and without defect) is studied. Two cases are considered: number of defected items is random variable and it is defined by random process. Numerical examples of application to the composite specimen (specifically, monolayer) and dry bundles strength analysis are presented.

Keywords: *composite, fibre, strength, weakest-link model, distribution function*

1. Introduction

This paper is a development and, in some way, revision of [1], where the general description of MinMaxDM distribution family is given. It was shown that this distribution family deserves to be studied much more thoroughly. It can be used for reliability of series and parallel systems with defect element analysis. In this paper we again consider application of some specific distributions from MinMaxDM distribution family to the composite specimen strength scatter analysis and, additionally, application to the strength analysis of dry bundles of carbon fibres.

Now we remind the main ideas of [1] (some details can be found also in [2-8]). In general case, a composite specimen for test of tensile strength is considered as a bundle of n_C longitudinal items (LI), i.e. fibres or strands immersed into composite matrix (CM), which is a composition of the matrix itself and all the layers with stacking different from the longitudinal one. We make very simplified assumption that only LI carry the longitudinal load but matrix only redistributes the loads after the failure of some longitudinal items. In fact, therefore, our model is a model of unidirectional (more specifically, monolayer) composite. We divide the composite into n_L parts of the same length l_1 (approximately, this length can be interpreted as the interval in which the load of failed LI is fully transmitted to the adjacent intact LI; the stronger the CM the smaller l_1). The total length of the composite specimens is equal to $l = n_L\ l_1$. We suppose that development of the process of fracture of a specimen takes place in one or in several of these parts ("links"). We call these links as "cross sections" (CS). So using this term we describe the composite as a *series system of CS*. For description of the development of fracture process of the series system it is appropriate to use the ideas on which the extended weakest link distribution family, described in the authors' papers [2-8], are based. Let the process of monotonous tensile loading (i.e. the process of increase of the nominal stress (or mean load of one LI) in the specimen cross section) be described by an ascending (up to infinity) sequence $\{x_1, x_2, ..., x_t, ...\}$, and let $K_{Ci}(t)$, $0 \le K_{Ci} \le n_C$, be the number of random failures of LI under the load x_t in *i*-th CS with n_C initial number of LI. Then *i*-th CS can carry the load

$$X_i^* = \max(x_t : n_C - K_{Ci}(t) \ge 0), \tag{1}$$

but the ultimate load of the specimen (which is the sequence of n_L CS) is

$$X = \min_{1\le i\le n_L} X_i^* = \min_{1\le i\le n_L} \max(x_t : n_C - K_{Ci}(t) \ge 0). \quad (2)$$

In [1] there are the applications of these ideas to the strength distribution analysis of monolayer and of composite (see experimental data in [9]). Here additionally we consider also the prediction of the strength distribution of dry bundles using data of test of fibres [10].

2. Models of Failure of a Parallel System with Redistribution of Load after Failure of Some LI

Statistical description of the development of the process of fracture of one CS (as loose bundle of LI (fibers or strands)) was initially studied by Daniels [11]. The respective model can be described in a following way. Let $(X_1,...,X_n)$ be random strengths of intact LI in some CS and X_j is the *j*-th order statistics in this CS. If there is a uniform distribution of load between n LI, and load increases uninterruptedly, then the ultimate strength of this CS

$$X^* = \max_{1\le j\le n} X_j (n-j+1)/n. \quad (3)$$

We consider the case when $n = n_C - K_C$. In this case mean strength of initial n_C LI

$$X^* = \max_{1\le j\le n} X_j (n-j+1)/n_C. \quad (3a)$$

Daniels studied the case K_C =0. In the general case for random value of K_C, (technological) failure number, there is a priori distribution $\pi_C = (\pi_{C1}, \pi_{C2},...,\pi_{C(n_C+1)})$ (here $\pi_{Ck} = P(K_C = k-1)$) . Then

$$F_{X^*}(x) = \pi_C \vec{F}(x), \quad (4)$$

where vector column $\vec{F}(x) = (F_1(x),...,F_{n_{C+1}}(x))'$, $F_k(x)$, $k = 1,...,n_C$, is cdf of X^* if $n = n_C + 1 - k$, $F_{n_C+1}(x)$ is identical with unity (there are no intact LI).

Much broader spectrum of models of the considered process can be developed using the theory of Markov chains. We consider the process of accumulation of failures as an inhomogeneous finite Markov chain (MC) with finite state space $I = \{i_1, i_2,...,i_{n_C+1}\}$. We say that MC is in state i if $(i-1)$ LI have failed, $i = 1,...,n_C+1$. State i_{n_C+1} is an absorbing state corresponding to the fracture of CS (fracture of all LI in this CS). The process of MC state change and the corresponding process $K_{Ci}(t)$ are described by transition probabilities matrix *P*.

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{23} & \cdots & p_{1(n_C+1)} \\ 0 & p_{22} & p_{23} & p_{24} & \cdots & p_{2(n_C+1)} \\ 0 & 0 & p_{33} & p_{34} & \cdots & p_{3(n_C+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & p_{n_C(n_C+1)} \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \quad (5.1)$$

At the *t*-th step of MC matrix *P* is a function of *t*, *t*=1,2,...
The cdf of strength of CS is defined on the sequence $\{x_1, x_2,...,x_t,...\}$ by equation

$$F_{X^*}(x_t) = \pi_C (\prod_{j=1}^{t} P(j)) u, \quad (5.2)$$

where $P(j)$ is the transition matrix for *t*=*j*, column vector $u = (0,...,0,1)'$.

We consider four main versions (hypotheses) of the structure of matrix *P*, denoted as P_a, P_{an_C}, P_b and P_c. In the simplest version we assume that in one step of MC only failure of one LI can take place and it is next LI. It is convenient (but not necessary) to think that the first failure appears in the boundary of CS and all the following failures can appear only in the adjacent LI. This version corresponds to a transverse crack growth in the monolayer. The stress concentration is supposed to be too small and it is not taken into account. For the corresponding matrix P_a we define $p_{ii} = 1 - F_C(x_t)$, where $F_C(x_t) = (F_0(x_t) - F_0(x_{t-1}))/(1 - F_0(x_{t-1}))$ is conditional cdf of strength of a LI, the failure of which did not take place under load x_{t-1}, $F_0(x)$ is the initial cdf of strength of a LI ; $p_{i(i+1)} = 1 - p_{ii}$, $i = 1,...,n_C$, $p_{(n_C+1)(n_C+1)} = 1$, but all the other p_{ij} are equal to zero.

We can consider hypothesis that again in one step of MC only failure of one LI can take place but now it is the weakest intact LI in these CS. Then $p_{ii} = (1 - F_C(x_t))^{n_C+1-i}$ and, again, $p_{i(i+1)} = 1 - p_{ii}$, $i = 1,...,n_C$, $p_{(n_C+1)(n_C+1)} = 1$, but all the other p_{ij} are equal to zero.

It can be assumed also that the number of failures in one step of MC has binomial distribution. Then for the corresponding matrix P_b we have $p_{i(i+r)} = b(r; p, k) = p^r (1-p)^{k-r} k!/r!(k-r)!$, $p = F_C(x_t)$, $k = n_c + 1 - i$, $r = 0,...,k$, $i = 1,...,n_C$; and again $p_{(n_C+1)(n_C+1)} = 1$, but all the other p_{ij} are equal to zero.

For previous versions of P, denoted by P_a, P_{an_C} and P_b, we suppose a uniform load distribution between intact LI. The matrix P_C again corresponds to a transverse crack growth in the monolayer but this time we take into account the stress concentration next to the tip of the crack. Let us denote by *j* the order number of LI in a CS (j=1 for the boundary LI) and let redistribution of CS load $x(t)$ between intact LI be defined by a "stress concentration" function $h(j; i, n_C)$. Then in the corresponding P_c matrix $p_{ij} = \prod_{i+1}^{j} F_C(x_{ij}(t)) \prod_{j+1}^{n_C+1} (1 - F_C(x_{ij}(t))$ for $j = i+1,...,n_C$; $p_{i(n_C+1)} = \prod_{i+1}^{n_C+1} F_C(x_{ij}(t))$ for $j = n_C$; $p_{ii} = 1 - \sum_{i+1}^{n_C+1} p_{ij}$, $p_{ij} = 0$ for $j < i$, $i = 1,...,n_C$;

where $x_{ij}(t) = h(j; i, n_C) x(t) n_C / (n_C + 1 - i)$ describes stress in *j*-th order LI after failure of *i*-th order LI.

3. Models of Failure of a Series System (Chain Of Links) with Damaged Items

In the framework of considered problem, there is a special case of n_C =1 (i.e. there is only one fibre, strand or thread). This case was studied in [7]. Below we remind the main ideas of [7], make the necessary corrections (appropriate for notation of this paper) and provide some generalization. We consider a specimen as a straight binary series system with n_L links of two types. There is a random number of "damaged" links , K_L, $0 \le K_L \le n_L$, with strength cdf $F_Y(x)$ (we say that they are Y-type links), and there are ($n_L - K_L$) links with strength cdf $F_Z(x)$ (we say they are Z-type links). "Damaged" links appear if stress in LI exceeds *defect initiation stress*. The probability of this event at the load (stress) x is defined by cdf of defect initiation stress $F_K(x)$.

We suppose (see [7]) that the failure process of considered system has two-stages. In the first stage, the process develops along the specimen and damage appear in K_L, $0 \le K_L \le n_L$, links (K_L links of Y-type appear). Then the second stage takes place: the process of accumulation of elementary damages in crosswise direction up to specimen failure. We consider three levels of accuracy of description of the second stage and three corresponding probability models (probability structure). Level A: the development of fracture process takes place in every link (containing or not some initial defects) and the strength of the weakest link defines the strength of the specimen. Level AB: the strength of the link without defects can be (relatively) so high and probability of its fracture before fracture of the damaged link so small that independence of failure probability of any Z-type CS on n_L can be assumed (only the probability that $K_L > 0$ depends on the number of links, n_L). And finally, level B: in addition to the assumption of the level AB it is assumed that the cdf of strength of the critical link does not depend on this number also. Correspondingly we have three probability structures.

A : $X = \min(Y_1,...,Y_{K_L}, Z_1,...,Z_{n_L-K_L})$;

AB: $X = \begin{cases} \min(Y_1,...,Y_{K_L},Z), & K_L > 0, \\ Z, & K_L = 0; \end{cases}$ B : $X = \begin{cases} Y, & K_L > 0, \\ Z, & K_L = 0. \end{cases}$.

Two different versions of the first stage can be considered also. First version: (technological) defects appear before the loading and their number does not depend on the subsequent loading. Second version: defects appear during loading (instantly or gradually) and their number depends on the load.

3.1. For "instant fracture" version for structures A, AB, B we have correspondingly

$$F(x) = 1 - (1 - F_Z(x))^{n_L} \sum_{k=0}^{n_L} p_k \delta^k(x), \quad \delta(x) = (1 - F_Y(x))/(1 - F_Z(x)), \tag{6}$$

$$F(x) = 1 - \sum_{k=0}^{n_L} p_k \left(1 - F_Y(x)\right)^k \left(1 - F_Z(x)\right) = 1 - (1 - F_Z(x)) \sum_{k=0}^{n_L} p_k \left(1 - F_Y(x)\right)^k, \tag{7}$$

$$F(x) = p_Y F_Y(x) + (1 - p_Y) F_Z(x), \tag{8}$$

where (in equations (6, 7)) binomial probability mass function (pmf) $p_k = b(k; p_L, n_L) =$ $= p_L^k (1 - p_L)^{n_L - k} n_L!/k!(n_L - k)!$ is probability that there is k links of Y-type; $p_Y = 1 - p_0 = 1 - (1 - p_L)^{n_L}$ is the probability that there is at least one link of Y-type (in this case, actually, it is enough to know only p_Y; we should not know two parameters n_L and p_0 separately).

Binomial or Poisson pmf can be used for random number of links of Y-type, K_L. In the latter case equations (6, 7) (approximately, if n_L is sufficiently large) can be written in the following way

$$F(x) = 1 - (1 - F_Z(x))^{n_L} \exp(-\lambda(1 - \delta(x)), \tag{9}$$

$$F(x) = 1 - (1 - F_Z(x)) \exp(-\lambda F_Y(x)), \tag{10}$$

where $\lambda = n_L p_L$ or it is just independent parameter of Poisson pmf. If initiation of the defects depends on the applied load, then it can be assumed that $p_L = F_K(x)$, where $F_K(x)$ is the cdf of defect initiation load.

In the numerical example considered in this paper it was assumed that the strength of defected link *S* has Weibull distribution; then $Y = \log(S)$ has the smallest extreme value (sev) distribution

$$F_Y(x) = 1 - \exp(-\exp((x - \theta_{0Y})/\theta_{1Y})). \tag{11}$$

And it was assumed also that for link without defects

$$F_Z(x) = 1 - \exp(-\exp((x - \theta_{0Z})/\theta_{1Z})) \tag{12}$$

but for the logarithm of defect initiation stress

$$F_K(x) = 1 - \exp(-\exp((x - \theta_{0K})/\theta_{1K})). \tag{13}$$

In some numerical examples it was considered that if $\theta_{0Z} = C$, but $\theta_{1Z} \to 0$, then

$$F_Z(x) = \begin{cases} 0, & x < C, \\ 1, & x \geq C. \end{cases} \quad (14)$$

3.2. The process of gradual (during loading) accumulation of defects along the chain of n_L links again can be considered as a Markov chain (MC). In this case MC is in state i if there are $(i-1)$ of Y-type links, i=1,...,n_L+1. State i_{n_L+2} is an absorbing state corresponding to the fracture of specimen. The matrix of transition probabilities has the same form as in (5.1). The initial distribution of K_L is represented now by some row vector $\pi_L = (\pi_{L1}, \pi_{L2}, ..., \pi_{L,n+1,} \pi_{L,n+2})$. In the new approach the number of CS of Y-type and the strength of specimens are random functions of time, $K_L(t)$ and $X(t)$. Now the three main structures we denote by MA, MAB and MB. They have the same description but instead of K_L we should write $K_L(t)$. For example, for the MA we have $X(t) = \min(Y_1, Y_2, ..., Y_{K_L(t)}, Z_1, Z_2, ..., Z_{n_L - K_L(t)})$. In similar way $X(t)$ is defined for the other structures.

Now the ultimate strength of specimen is defined by equation

$$X = x_{T^*}, \quad (15)$$

where

$$T^* = \max(t : X(t) > x_t). \quad (16)$$

The cdf of ultimate strength , X, is defined again by an equation similar to equation (5.2):

$$F_X(x_t) = \pi_L \left(\prod_{j=1}^{t} P(j)\right) u .$$

Specifying the matrix *P* for probability structures A and AB. The probability that in some element a defect appears at the stress x_t under the condition that it has not appeared at the stress x_{t-1} is

$$b(t) = (F_K(x_t) - F_K(x_{(t-1)})) / (1 - F_K(x_{(t-1)})) .$$

Consider the case of s defects present. The probability that r new defects appear, $0 \leq r \leq k = n - s$, and the total number of defects is equal to $m=s+r$

$$\tilde{p}_{sm}(t) = (b(t))^r (1 - b(t))^{k-r} k! / r!(k-r)!$$

Conditional probability of Y-type link fracture at the nominal stress x_t

$$q_Y(t) = (F_Y(x_t) - F_Y(x_{(t-1)})) / (1 - F_Y(x_{(t-1)})) .$$

Conditional probability of Z-type link fracture at the nominal stress x_t

$$q_Z(t) = (F_Z(x_t) - F_Z(x_{(t-1)})) / (1 - F_Z(x_{(t-1)})) .$$

Corresponding probability that none of the links (of both types) fails when there are defects in m links for probability structure MA is

$$u_m(t) = (1 - q_Y(t))^m (1 - q_Z(t))^{n_L - m} ,$$

and for probability structure MAB

$$u_m(t) = (1 - q_Y(t))^m (1 - q_Z(t)) .$$

The probability of coincidence of these events, which we consider as independent, and the probability of transition from state *i=s+1* to state *j=i+r*

$$p_{ij}(t) = \tilde{p}_{(i-1)(j-1)}(t)u_{j-1}(t),$$

where $i \le j \le (n+1)$.

It is worth to note that if equation (14) is used and C is large enough (this means that only damaged CS define the strength) then it can be assumed that $q_Z(t)$ =0.

Conditional fracture probability (for both probability structure MA and MAB) at state *i*

$$p_{i(n+2)}(t) = 1 - \sum_{j=i}^{n+1} p_{ij}(t).$$

Of course, $p_{ij}(t) = 0$, if $j < i$, and $p_{(n+2)(n+2)}(t) = 1$.

Specifying the matrix *P* for probability structures MB. The corresponding Markov chain has only three states. The first state corresponds to the absence of defective links; the second one means the presence of at least one defective link, and the third, absorbing one, means failure of the specimen. The corresponding probabilities at a *t*-th step are determined by the formulae

$$p_{11}(t) = [1-b(t)]^{n_L}, \quad p_{12}(t) = (1-p_{11}(t))(1-q_Y(t))(1-q_Z), \quad p_{13}(t) = 1 - p_{11}(t) - p_{12}(t),$$

$$p_{21}(t) = 0, \quad p_{22}(t) = (1-q_Y(t))(1-q_Z(t)), \quad p_{23}(t) = 1 - p_{22}(t), \quad p_{31}(t) = p_{32}(t) = 0,$$

$$p_{33}(t) = 1.$$

4. MinMaxDM Distribution Family

Clearly, all the ideas considered in the previous section can be used also for the series system of CS if instead of the word "link" now we use the word CS. Instead of cdf $F_Y(x)$ and $F_Z(x)$, which were defined by (11-12) now we should use cdf of CS strength of Y-type or Z-type correspondingly. For building these cdf in the following numerical examples we again suppose that logarithm of strength of one LI (in one CS) without defect has the smallest extreme value (sev) distribution: $F_0(x) = 1 - \exp(-\exp((x-\theta_{0Z1})/\theta_{1Z1}))$. We use the logarithm scale and in this case the cdf of specimen strength also has location and scale parameters θ_0 and θ_1: $F_X(x) = F_{\overset{0}{X}}((x-\theta_0)/\theta_1))$. Of course it is not the only possible assumption. Different assumptions about the distribution of strength of bundles within the frame of one CS (one "link") , a priori distribution of initial (technological) defects, the influence of length and width of specimens compose a family of the distributions of ultimate composite tensile strength. Taking into account (2) and (3) we denote this family by abbreviation MinMaxD (in memory of Daniels) if the strength $F_{X^*}(x)$ is defined by equation (4) and by abbreviation MinMaxM (because of connection with Markov chain theory), if it is defined by equation (5), and for unified family we suggest an abbreviation MinMaxDM.

5. Specific Distributions of Cross-Section-Strength

In this paper we consider processing of experimental data using two simplest versions of cross-section-strength distributions.

Cross-section-strength distributions based on Daniels's model

If the number n is large enough then for X^* in equation (3) there is convergence in probability to some constant μ defined by equation

$$\mu = \max_x x(1-F_X(x)), \tag{3b}$$

where $F_X(x)$ is cdf of strength of one LI. We consider the case of Weibull distribution, and then using logarithm scale (in order to use the advantage of sev distribution with the location and scale parameters) we can write equation (3b) in following form

$$\mu = \max_x \exp(x)\exp(-\exp((x-\theta_0)/\theta_1)) \,, \tag{3c}$$

We have following solution of this equation

$$\mu = \theta_1^{\theta_1} \exp(\theta_0 - \theta_1) \,. \tag{17}$$

Daniels has shown that for enough large n X^* in equation (3) has approximately normal distribution. For Weibull distribution in considered case, when K_C =0, the parameters of this distribution are μ and $\sigma = \mu(\exp(\theta_1 - 1)/n_C)^{1/2}$. But if there are K_C defected LI (there are only ($n_C - K_C$) intake LI) then we should us $\mu_n = \mu(n_C - K_C)/n_C$ and $\sigma_n = \sigma(n_C - K_C)/n_C$ (the denominator is equal to n_C instead of ($n_C - K_C$) because the specimen strength is calculated taking into account all initial number of LI, n_C).

So if $n = n_C - K_C$, where random variable K_C has truncated binomial distribution (remind, we should eliminate the case $K_C = n_C$) with parameter (n_C, p_C) then cdf of X^* approximately defined by equation

$$F_{X^*}(x) = \sum_{n=1}^{n_C} F_{X_n^*}(x) b(n_C - n, n_C, p_C) / (1 - b(n_C, n_C, p_C)) \,, \tag{18}$$

where $F_{X_n^*}(x) = \Phi((x-\mu_n)/\sigma_n)$; $\Phi(.)$ is cdf of standard normal distribution, $b(k,m,p) = p^k (1-p)^{m-k} m!/k!(m-k)!$.

Of course binomial distribution can be approximated by normal distribution but in following numerical example we did not used this approximation.

And on the contrary, along with the normal approximation for $F_{X_n^*}(x)$ we have made calculation of this function using Monte Carlo method, equation (3a) and Weibull distribution of strength of one strand. If we use logarithm scale for strength then this equation has following form

$$X^* = \max_{1 \le j \le n} (n - j + 1) \exp(\theta_0 + \theta_1 \overset{0}{X}_j) / n_C \,, \tag{19}$$

where $\overset{0}{X}_j$ is j-th order statistic of sample of the size n from standard sev distribution with cdf

$$F(x) = 1 - \exp(-\exp(x)) \,, \tag{20}$$

that is to say, from sev distribution with θ_0 =0 and θ_1 =1.

Comment. The distribution (19) is not distribution with scale and location parameter. But appears temptation to study also the distribution, corresponding to equation

$$X^* = \theta_0 + \theta_1 \max_{1 \le j \le n} (n - j + 1) \exp(\overset{0}{X}_j) / n_C \,. \tag{21}$$

If we have distribution of random variable

$$\overset{0}{D} = \max_{1 \le j \le n} (n - j + 1) \exp(\overset{0}{X}_j) / n_C \,, \tag{22}$$

(we can get it using Monte Carlo method) then we can use the distribution (21) for fitting experimental data and estimate parameter using linear regression analysis).

Cross-section-strength distributions based on Markov chains theory model

In this case we suppose that random variable X^* has structure $X^* = \theta_0 + \theta_1 \overset{0}{X}*$, where cdf of $\overset{0}{X}*$ is defined by equation (5.2)

$$F_{X^*}(x_t) = \pi_C \left(\prod_{j=1}^{t} P(j)\right) u$$

with matrix P in the form P_a with probabilities { p_{ij} } defined in assumption that $F_0(x)$ is standard sev distribution (20).

6. Processing of Test Data

Here we consider two sets of experimental data. Data_A1, data_A2 and B are presented in [9]. Data_C1 and data_C2, presented in [10].

In [9] there are the test results of both 64 carbon fibre strands with length 20 mm (data_A1) and the same number of strips of 10 strands of the same length (data_A2). There are also the test results of 14 specimens of carbon fibre reinforced specimens. We attempt to obtain the statistical description of data_A2, and then of data B, using results of processing of data_A1. Let random variable X has location and scale parameters, θ_0 and θ_1, but x_i be *i*-th order statistic, $i = 1, 2, ..., n$, of a sample of size n; $E(X_i)$ is the expected value of ith order statistic, $E(\overset{0}{X}_i)$ is the same but for θ_0=0 and θ_1=1. Then for estimation of θ_0 and θ_1, if all the other parameters are fixed, we have the following linear regression model: $E(X_i) = \theta_0 + \theta_1 E(\overset{0}{X}_i)$. Using these ideas we perform fitting (expected value of “standard” order statistics $E(\overset{0}{X}_i)$ versus data order statistics) of the data_A1 and get linear regression parameter estimates $\widehat{\theta}_0$=6.554 and $\widehat{\theta}_1$=0.1243 (corresponding mean and standard deviation of $X = \log(S)$, where S is strength in MPa, are μ_X = 6.4769, σ_X =0.17322) assuming that sev distribution of X takes place.

Then we consider prediction of data_A2 using models described in section 5.1., more precisely, model described by equation (19) . Parameters θ_0 and θ_1 should be replaced by estimates $\widehat{\theta}_0$ and $\widehat{\theta}_1$ and then we make calculation of cdf of X^* and, using equation (18) for different p_C, we can get different cdf $F_Y(.)$. Using the same calculation algorithm but for p_C =1 we get cdf $F_Z(.)$, then cdf of X, defined by (8), corresponding “predicted” order statistics $\hat{x}_i$, mean and standard value of strength and, finally, statistic $OSPPt = (\sum_{i=1}^{n}(x_i - \hat{x}_i)^2 / \sum_{i=1}^{n}(x_i - \overline{x})^2)^{1/2}$, the “measure of prediction quality” [5].

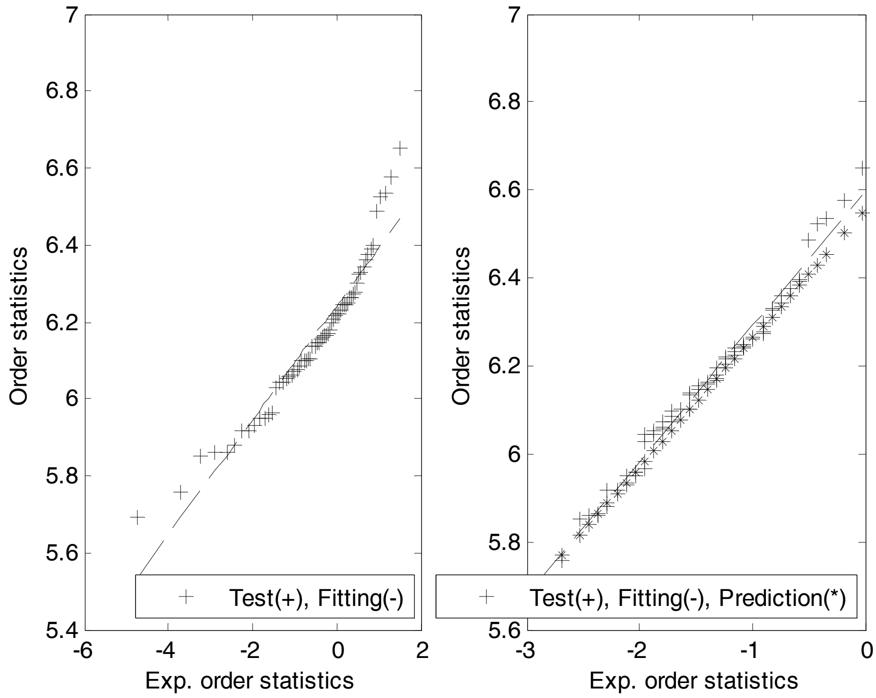
For p_C =0.05 , p_Y =0.5 the calculated predicted mean value of data_A2 appears to be equal to 489.8 MPa . This value do not differ too much from experimental value, 480.6 MPa. Much more difference between predicted standard deviation, 59.7 MPa, and experimental value, 95.7 MPa. Value of statistic of $OSPPt$ is equal to 0.399. This result is better then $OSPPt$ =0.512, calculated for the “pure” Daniels’s model, which corresponds to p_C =0 and p_Y =0, but of course it is not good enough. Pity, we did not find parameters for lesser value of $OSPPt$ and better prediction of mean and standard deviation.

But we get much better result using Markov chains model with cdf defined by (5.2) and matrix P in the form P_a. Preliminary we make the fitting of the data_A2 using sev model, see Figure 1a. On the Figure 1b the fitting of the same data_A2 using $E(\overset{0}{X}_i)$ of cdf corresponding to MinMaxMa-Bsev model are shown (for P_a type of matrix P, $F_0(x)$ is sev distribution, probability structure B (see equation (8) where n_C =5; π_C is a binomial a priori distribution of K_C with p_C =0.01 , $n = n_C$ =5; $p_Y = 0.9048$)). Then the “regression prediction”, $\hat{x}_i = \hat{\theta}_0 + \hat{\theta}_{1C} E(\overset{o}{X}_i)$, was made using the obtained estimates of structure parameters (n_C, p_C, p_Y), the mentioned above parameter estimate $\widehat{\theta}_0$ (estimates of parameter of data_A1) and corrected estimate $\widehat{\theta}_{1C}$, which was calculated taking into account the standard deviation of Young’s modulus, σ_E =0.332 [9]. (Corrected value $\widehat{\theta}_{1C}$ was calculated using the connection between scale parameter and standard deviation for sev distribution: $\hat{\theta}_{1C} = (6(\hat{\sigma}_X^2 + \hat{\sigma}_E^2))^{1/2} / \pi$ ==0.29197). Results of “prediction” (*) are shown on Figure 1b also. Of course, really it is not PREDICTION but “PREDICTION”, because in fact we have used also the estimates of “structure parameters” p_C, n_C and p_Y which was found fitting data_A2. It would be real prediction if “structure parameters” are **parameters of technology, which are nearly the same for**

different specimens with the same type of technology and are known in advance. Unfortunately, it is only hope, but it is not the fact.

The statistic $OSPPt$, as the measure of fitting for Figure1a is equal to 0.267 (for sev distribution) and as the measure of fitting and prediction quality for Figure 1b (for MinMaxMa.sev-B structure model) is equal to 0.161 and 0.192 correspondingly.

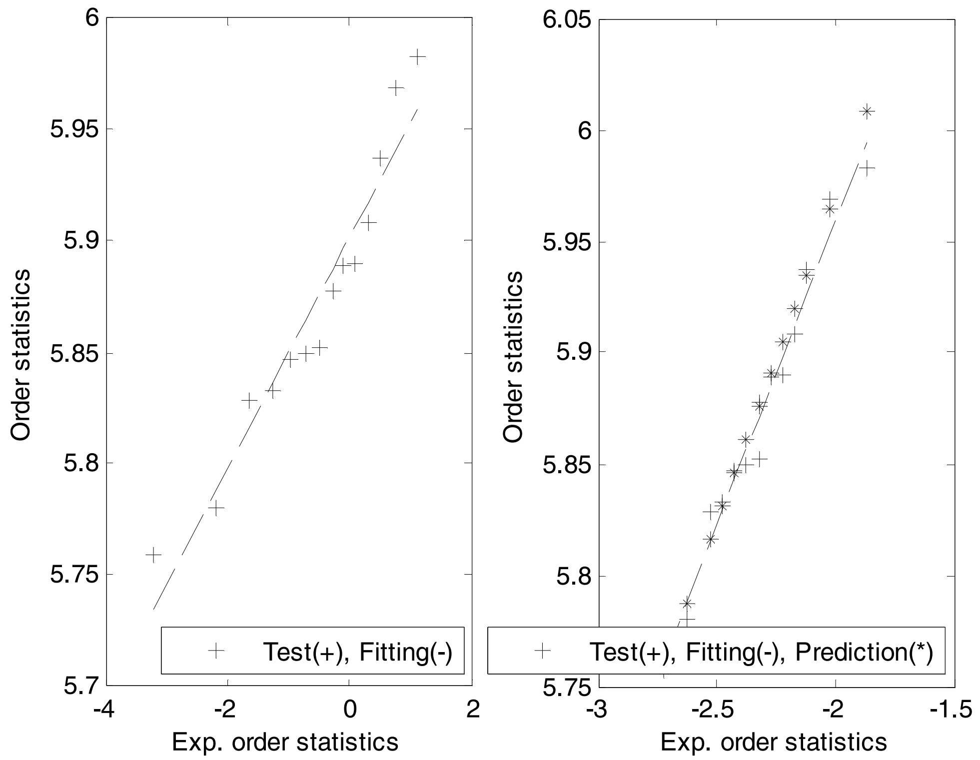
Although the considered models are models of unidirectional monolayer composite in framework of "very simplified assumption that only LI carry the longitudinal load but matrix only redistributes the loads after the failure of some longitudinal items" we can try to make processing of the data_B [9], which are the test results of carbon reinforced composite specimens (($0_6^o / +-45_4^o / 90_3^o$)$_s$, length : 250 mm, width : 38 mm, thickness : 1.7 mm, number of observation is 14). On Figure 2a we see fitting of these data (+) using sev distribution (statistics OSPPt=0.2504). On Figure 2b we see fitting of the same data using MinMaxMa-Bsev model (statistics OSPPt=0.1548). "Prediction" of these data using MinMaxMa-Bsev model (*) and the same algorithm as for prediction data_A2 is shown also. Corresponding statistics OSPPt is equal to 0.1879. This time the following structure parameter was used: π_C is a binomial priori distribution of K_C with p_C =0.325 , $n = n_C$ =50; $p_Y = 1$.



a b

Figure 1. Fitting (-) (expected value of "standard" order statistics $E(\overset{0}{X}_i)$ versus data order statistics (+)) and "prediction" (*) of results of tensile strength test of carbon fibre strip of 10 strands using sev distribution (a) and MinMaxMa-Bsev model (b) (see explanation in text)

Now let us study the data_C1 and the data_C2 (see Tables I, III and V in [10]). As the data_C1 we consider the results of test of carbon fibres with the length L=20 . In [10] there is example of using this data for prediction of the strength of dry bundles (1000 fibres) of the same length. In [10] the Weibull cdf is presented in the form $F(y,L) = 1-\exp[-(y/y_L)^{\rho_L}]$ and the parameter estimates are given in Table III. For single fibres of 20 mm these estimates are: $y_L = 2650$ MPa, $\rho_L = 5.5$ .Corresponding parameter estimates in sev form are $\hat{\theta}_0 = \log(y_L)$ = 7.8823, $\hat{\theta}_1 = 1/\rho_L$ =0.1818. For the length L=[5 20 100 200] mm using these estimates and both equation (18) for $F_{X^*}(.)$ and equation (8) for $F_X(.)$ with p_C =0.2, $p_Y = 1-(1-\overset{0}{p}_Y)^{L/\overset{0}{L}}$, $\overset{0}{L}$ =20, $\overset{0}{p}_Y$ =0.1 we make calculations of mean strength and standard deviation of the strength of dry bundles. Results of these calculations in comparison with experimental data and calculations are presented in [10] (in accordance with Daniels normal approximation) are shown in Table 1.



a b

Figure 2. Fitting (expected value of "standard" order statistics $E(\overset{0}{X}_i)$ versus data order statistics) and "prediction" of the tensile strength of carbon reinforced composite specimens test results (+) using sev distribution (a) and MinMaxMa-Bsev model (b) (see explanation in text)

Table 1. Summary of data and results of calculation

L (mm)		5	20	100	200
Number of observations		28	25	29	27
Mean (GPa)	Observed	1.92	1.68	1.58	1.38
	[10]	2.19	1.71	1.28	1.14
	MinMax	1.61	1.57	1.48	1.4
Std (GPa)	Observed	0.07	0.1	0.13	0.11
	[10]	0.031	0.024	0.018	0.016
	MinMax	0.057	0.114	0.161	0.156

Conclusions

Analysis of the data_A and data_B shows that MinMaxMa-Bsev model can provide better (than sev distribution) fitting of results of tensile strength test of carbon fibre strip of 10 strands (but only if we preliminary make estimation of "structures parameters" (p_C and p_Y (or p_L) ,…) and take into account variation of Young's modulus!). Better fitting is not surprising, of course, because MinMaxMa-Bsev model has much more parameters. For the same reason this model **can not be used for prediction if the structure parameters are not known.** But estimates of this parameter (during fitting of experimental data) can be useful for the **numerical estimation of the "quality" of the structure.**

Analysis of the data_C shows that considered MinMaxDM version of distribution also provides better (than Daniels model) description of the strength distribution of dry bundles using data of test of fibres. This advantage is particularly evident for calculation of standard deviation. Daniels model gives too small value of standard deviation. The advantage of considered MinMaxDM distribution version allows to explain increasing of standard deviation in comparison with theoretical value (corresponding to Daniels's model) by assuming that really we have the mixture of test specimens with defect and without defects. As an example, the corresponding pdf of dry bundles strength for L=20 mm, p_C =0.2, p_Y =0.1 is shown on Figure 3.

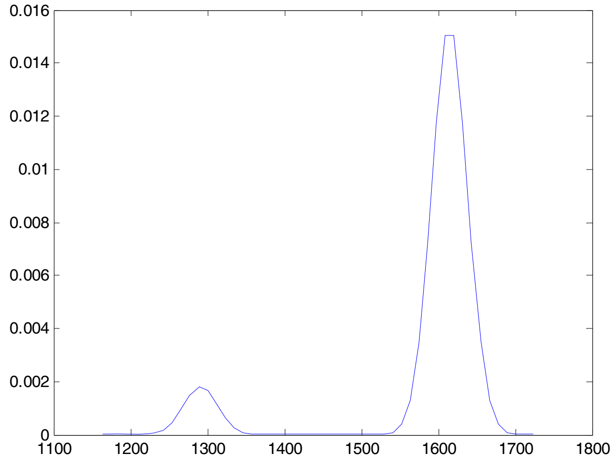


Fig.3. The "predicted" pdf of strength of dry bundles for L =20 mm

As a whole, it seems that MinMaxDM distribution family deserves to be studied much more thoroughly using much more test data. Interpretation of the parameters of the corresponding models allows to make comparison of different composite structures and explanation of some specific features of failure process of composite. For example, the value p_C =0.325 for data_B indicates that at least 32.5% of the critical cross section of specimens does not carry the longitudinal load.

Of course the MinMaxDM distribution family can be used also for reliability analysis of series and parallel systems with defect elements with structure different from structure of monolayer.

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