Computer Modelling and New Technologies, 2010, Vol.14, No.1, 15–25
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RELIABILITY OF A SERIES OF PARALLEL SYSTEMS WITH DEFECTS

Application of MinMaxDM Distribution Family to Composite Strength Analysis

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Reliability of series and parallel systems with two types of structural items (damaged and without defects) is studied. Two cases are considered: number of damaged items is a random variable, or it is defined by a random process. Numerical examples of application to a composite specimen (specifically, monolayer) and the strength analysis of dry bundles are presented.

Keywords: *composite, fibre, strength, weakest-link model, distribution function*

1. Introduction

This paper is a development and, in some way, revision of [1–6], where the general description of the extended family of weakest-link distributions (EFWLD) is given. It was shown that this distribution family deserves to be studied much more thoroughly. It can be used for a series of parallel systems with defect element reliability analysis. In this paper we again consider application of some specific subfamily of distributions, which we call MinMaxDM distribution family, to the specific composite specimen strength scatter analysis. In general case, a composite specimen for test of tensile strength is considered as a chain-of-bundles [7]. But from the reliability theory point of view it is a series of parallel systems. We divide the composite into n_L parts of the same length l_1 (the total length of the composite specimen is equal to $L = n_L\, l_1$). We call these parts as links. Every link is a parallel system with n_C longitudinal items (LI) – fibres or strands. Some LI may be damaged. For example, the strength of some of them has another cumulative distribution function (cdf) or just is equal to zero.

In section 2 we correct the description of the set of probability structures (p.s.), which initially is described in [3] in application to the series system. In section 3 we consider two versions of describing the strength of n_C parallel LI with redistribution of load after failure of some LI. In section 4 the description of MinMaxDM distribution family is presented. In section 5 we consider and analyse some numerical examples.

2. Models of Failure of a Series System (Chain of Links) with Damaged Items

In the framework of the considered problem, there is a special case of $n_C = 1$ (i.e. there is only one fibre, strand or thread). This case was studied in [1–4]. The extended family of weakest link distributions (EFWLD) was introduced in [4]. Below we recapitulate the main corresponding ideas, make the necessary corrections (appropriate for notation of this paper) and provide some generalization. Here we consider a specimen as a straight binary series system with n_L links of two types. There is a random number, K_L, of "damaged" links, $0 \le K_L \le n_L$, with strength cdf $F_Y(x)$ (we say that there are K_L Y-type links), and there are ($n_L - K_L$) links with strength cdf $F_Z(x)$ (we say they are Z-type links). Damaged links can appear before loading in accordance with some initial distribution, $\pi_L = (\pi_{L1}, \pi_{L2}, ..., \pi_{L(n_L+1)})$, where $\pi_{Lk} = P(K_L = k-1)$, or during load increase, in case the stress in LI exceeds *a defect initiation stress* with cdf $F_K(x)$ [3]. We suppose that the failure process of the considered system has two stages. In the first stage the process develops along the specimen and K_L links of Y-type appear, $0 \le K_L \le n_L$. Then the second stage takes place: the process of accumulation of elementary damages in crosswise direction up to specimen failure.

Correcting the description of the set of the probability structures (p.s.) given in [3] we consider two levels of differences between LI with and without defects and three groups (levels) of accuracy of description of the difference of strength inside these groups. Six types of the corresponding p.s. are shown in Table 1.

Table 1. Probability structures of specimen strength dependence on the strength of single links

A 1 :	$X = \min(Y_1,...,Y_{K_L}, Z_1,...,Z_{n_L-K_L})$	B1:	$X = \begin{cases} \min(Y_1,...,Y_{K_L}), & K_L > 0, \\ \min(Z_1,...,Z_{n_L}) & K_L = 0; \end{cases}$
A 2 :	$X = \min(Y_1,...,Y_{K_L}, Z)$	B2:	$X = \begin{cases} \min(Y_1,...,Y_{K_L}), & K_L > 0, \\ Z, & K_L = 0; \end{cases}$
A 3 :	$X = \min(Y, Z)$	B3:	$X = \begin{cases} Y, & K_L > 0, \\ Z, & K_L = 0. \end{cases}$

In p.s. of type A it is assumed that the difference between the strength of links of Y and Z types is relatively small and the failure of the specimens can be caused by the failure of a link of either type.

In p.s. of type B it is assumed that the difference between the strength of links of Y and Z types is very large and we must take into account the strength of the link of Z type only if there are no links of Y-type. In some way, the description of the group of type B is a limit of the description of the group of type A if the difference between the cdf of strength of Y and Z type links increases. Really, all the description of p.s. different from A1 is some form of approximation of the description of the group A1. We suppose that the usefulness of considering this set of different p.s. is defined by the difference of materials, different requirement to accuracy of calculation and different size of the test sample.

2.1. A Connection of the cdf of the Strength of the Specimens and the cdf of the Strength of Single LI when n_C =1

2.1.1. The Case When the cdf of the Strength of Single LI, $F_Y(x)$ and $F_Z(x)$, are Known

In accordance with these structures we have the following equations.

For the structures A1, A2, A3 we have respectively

$$F(x) = 1 - \sum_{k=0}^{n_L} p_k (1 - F_Y(x))^k (1 - F_Z(x))^{n_L - k}, \tag{1}$$

$$F(x) = 1 - \left(1 - F_Z(x)\right) \sum_{k=0}^{n_L} p_k \left(1 - F_Y(x)\right)^k, \tag{2}$$

$$F(x) = 1 - \left(1 - F_Y(x)\right)\left(1 - F_Z(x)\right). \tag{3}$$

For the groups B1, B2, B3

$$F(x) = 1 - \sum_{k=1}^{n_L} p_k \left(1 - F_Y(x)\right)^k - p_0 \left(1 - F_Z(x)\right)^{n_L}, \tag{4}$$

$$F(x) = 1 - \sum_{k=1}^{n_L} p_k \left(1 - F_Y(x)\right)^k - p_0 (1 - F_Z(x)), \tag{5}$$

$$F(x) = (1 - p_0) F_Y(x) + p_0 F_Z(x), \tag{6}$$

where $\{p_k, k = 0,1,...,n_L\}$ is the probability distribution for the r.v. K_L.

For binomial distribution

$$p_k = b(k; p_L, n_L), \tag{7}$$

where $b(k; p_L, n_L) = p_L^k (1 - p_L)^{n_L - k} n_L!/k!(n_L - k)!$, p_L is the distribution parameter.

Let us note that in this case

$$p_0 = (1 - p_L)^{n_L} . \tag{8}$$

For a conditional Poisson distribution (under condition that $K_L \le n_L$):

$$p_k = (\exp(-\lambda)\lambda^k / k!) / (\exp(-\lambda)\sum_{r=0}^{n_L} \lambda^r / r!) = (\lambda^k / k!) / (\sum_{r=0}^{n_L} \lambda^r / r!) \tag{9}$$

and $p_0 = 1 / (\sum_{r=0}^{n_L} \lambda^r / r!)$. (10)

If n_L is large enough, instead of binomial the Poisson distribution, but instead of equations (1, 2) the following equations can be used

$$F(x) = 1 - (1 - F_Z(x))^{n_L} \exp(-\lambda(1 - \delta(x)) , \ \delta(x) = (1 - F_Y(x))/(1 - F_Z(x)) , \tag{11}$$

$$F(x) = 1 - (1 - F_Z(x)) \exp(-\lambda F_Y(x)) , \tag{12}$$

where $\lambda = n_L p_L$ or it is an independent parameter of the Poisson distribution. If the defects appear during the process of loading it may be assumed that $p_L = F_K(x)$, where $F_K(x)$ is a cdf of stress of initiation of a link of *Y*-type.

We should pay special attention to the case when r.v. K_L can take only two values. For example $K_L = n_L$ with probability p_{KL} and $K_L = 0$ with probability (1 – p_{KL}). Then, for example, for the structure B2 we have

$$F(x) = p_{KL}(1 - (1 - F_Y(x))^{n_L}) + (1 - p_{KL})F_Z(x) . \tag{13}$$

In the following numerical examples the Weibull distribution is used for single LI strength, S. Then the smallest extreme value (sev) distribution takes place for $\log(S)$ with cdf

$$F(x) = 1 - \exp(-\exp((x - \theta_0) / \theta_1)) . \tag{14}$$

The same type of distribution (but with specific parameters) can be used also for initiation stress of *Y*-type link and also for $F_Y(x)$ and $F_Z(x)$ in case of n_C=1 (see [3,4]). For the case of n_C>>1 in this paper we consider using of randomised and non-randomised Daniels's models (RDM and NRDM) and models, based on Markov chain (MC) theory, are used also (see section 3).

2.1.2. The Process of Gradual (During Loading) Accumulation of Defects and Failure of a Series System

Now for notation of the p.s. we use additional letter M: MA1, MA2 and so on. Let the process of monotonous tensile loading (i.e. the process of increase of the nominal stress (or mean load of one LI)) be described by an ascending (up to infinity) sequence $\{x_1, x_2, ..., x_t, ...\}$. Now the number of links of *Y*-type and the strength of specimens are random functions of time, $K_L(t)$ and $X(t)$. For example, for the MA1 we have $X(t) = \min(Y_1, Y_2, ..., Y_{K_L(t)}, Z_1, Z_2, ..., Z_{n_L - K_L(t)})$. Let us consider a MC with ($n_L + 2$) states. MC is in state *i* if there are $(i-1)$ of *Y*-type links, *i* = 1,...,n_L + 1. State i_{n_L+2} is an absorbing state corresponding to the fracture of specimen. The process of MC state change and the corresponding process $K_L(t)$ are described by the transition probabilities matrix *P*.

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{23} & \cdots & p_{1(n_C+1)} \\ 0 & p_{22} & p_{23} & p_{24} & \cdots & p_{2(n_C+1)} \\ 0 & 0 & p_{33} & p_{34} & \cdots & p_{3(n_C+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & p_{n_C(n_C+1)} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{15}$$

At the *t*-th step of MC the matrix *P* is a function of *t*, *t* = 1, 2, ... A prior or initial distribution of K_L is represented by some row vector $\pi_L = (\pi_{L1}, \pi_{L2}, ..., \pi_{L,n+1,} \pi_{L,n+2})$, where $\pi_{L(n+2)} = 0$.

Now the ultimate strength of specimen is defined by equation

$$X = x_{T^*}, \tag{16}$$

where

$$T^* = \max(t : X(t) > x_t). \tag{17}$$

The cdf of ultimate strength, X, is defined by equation

$$F_X(x_t) = \pi_L(\prod_{j=1}^{t} P(j))u \cdot \tag{18}$$

As an example, we consider now the specifying of the matrix P for p.s. M A1 and MB3, for $n_C = 1$. The probability that in some link a defect appears at the stress x_t under the condition that it has not appeared at the stress x_{t-1} is $b(t) = (F_K(x_t) - F_K(x_{(t-1)}))/(1 - F_K(x_{(t-1)}))$.

Consider the case of s defects present. The probability that r new defects appear, $0 \le r \le k = n - s$, and the total number of defects is equal to $m = s + r$

$$\tilde{p}_{sm}(t) = (b(t))^r(1 - b(t))^{k-r} k!/r!(k-r)!.$$

Conditional probability of Y-type link fracture at the nominal stress x_t

$$q_Y(t) = (F_Y(x_t) - F_Y(x_{(t-1)}))/(1 - F_Y(x_{(t-1)})).$$

Conditional probability of Z-type link fracture at the nominal stress x_t

$$q_Z(t) = (F_Z(x_t) - F_Z(x_{(t-1)}))/(1 - F_Z(x_{(t-1)})).$$

Corresponding probability that none of the links (of both types) fails when there are defects in m links for probability structure MA1 is $u_m(t) = (1 - q_Y(t))^m(1 - q_Z(t))^{n_L - m}$.

The probability of coincidence of these events, which we consider as independent, and the probability of transition from state $i = s + 1$ to state $j = i + r$

$$p_{ij}(t) = \tilde{p}_{(i-1)(j-1)}(t)u_{j-1}(t),$$

where $i \le j \le (n+1)$.

Conditional fracture probability for the structure MA1 at state i $p_{i(n+2)}(t) = 1 - \sum_{j=i}^{n+1} p_{ij}(t)$.

Of course, $p_{ij}(t) = 0$, if $j < i$, and $p_{(n+2)(n+2)}(t) = 1$.

The corresponding Markov chain for probability structures MB3 has only three states. The first state corresponds to the absence of damaged links; the second one means the presence of at least one damaged link, and the third, absorbing one, means failure of the specimen. The corresponding probabilities at a t-th step are determined by the formulae

$$p_{11}(t) = [1 - b(t)]^{n_L},\ p_{12}(t) = (1 - p_{11}(t))(1 - q_Y(t))(1 - q_Z),\ p_{13}(t) = 1 - p_{11}(t) - p_{12}(t),$$

$$p_{21}(t) = 0,\ p_{22}(t) = (1 - q_Y(t))(1 - q_Z(t)),\ p_{23}(t) = 1 - p_{22}(t),\ p_{31}(t) = p_{32}(t) = 0,\ p_{33}(t) = 1.$$

3. Models of Failure of a Parallel System with Redistribution of Load after Failure of Some LI

3.1. Statistical Description of the Development of the Process of Fracture of One Link

A connection of the cdf of the strength of the link and the cdf of the strength of single LI for the case when $F_Y(x)$ and $F_Z(x)$ are known.

Statistical description of the development of the process of fracture of one link (as a loose bundle of LI (fibres or strands) or as a parallel system without initial defects with redistribution of load after failure of some LI) was studied by Daniels [8–9]. The respective model can be described in a following way. Let $(X_1, ..., X_n)$ be random strengths of intact LI in some link and X_j be the j-th order statistics.

If there is a uniform distribution of load between intact LI and total specimen load increases monotonically, then the ultimate strength of this link

$$X^* = \max_{1\le j\le n} X_j(n-j+1)/n \cdot \tag{19}$$

We consider the case when $n = n_C - K_C$. In this case mean strength of initial n_C LI

$$X^* = \max_{1\le j\le n} X_j(n-j+1)/n_C \cdot \tag{20}$$

Daniels studied the case $K_C = 0$. In the general case for r.v. K_C, (technological) failure number, there is a priori distribution $\pi_C = (\pi_{C1}, \pi_{C2}, ..., \pi_{C(n_C+1)})$ (here $\pi_{Ck} = P(K_C = k-1)$, $\pi_{C(n_C+1)} = 0$). Then

$$F_{X^*}(x) = \pi_C \overrightarrow{F}(x), \tag{21}$$

where vector column $\vec{F}(x) = (F_1(x), ..., F_{n_{C+1}}(x))'$, $F_k(x)$, $k = 1, ..., n_C$, is cdf of X^* if $n = n_C + 1 - k$, $F_{n_C+1}(x)$ is identical with unity (there are no intact LI).

3.2. The Process of Gradual (During Loading) Accumulation of Defects And Failure of a Parallel System

Let us recall that the process of monotonous tensile loading (i.e. the process of increase of the nominal stress (or mean load of one LI)) is described by an ascending (up to infinity) sequence $\{x_1, x_2, ..., x_t, ...\}$, and let $K_{Ci}(t)$, $0 \le K_{Ci} \le n_C$, be the number of random failures of LI under the load x_t in *i*-th link with n_C – initial number of LI. There is the failure of *i*-th link if $K_{Ci} = n_C$. We again consider the process of accumulation of failures as an inhomogeneous finite Markov chain (MC) with finite state space $I = \{i_1, i_2, ..., i_{n_C+1}\}$. We say that MC is in state i if $(i-1)$ of LI have failed, $i = 1, ..., n_C + 1$. State i_{n_C+1} is an absorbing state corresponding to the fracture of the link. The process of MC state change and the corresponding process $K_{Ci}(t)$ are described by transition probabilities matrix *P*, which has again the form defined by (15), and at the *t*-th step of MC the matrix *P* is a function of *t*, *t* = 1, 2, ...
The cdf of strength of link, X^*, is defined on the sequence $\{x_1, x_2, ..., x_t, ...\}$ by equation

$$F_{X^*}(x_t) = \pi_C\left(\prod_{j=1}^{t} P(j)\right)u, \tag{22}$$

where $P(j)$ is the transition matrix for *t*=*j*, column vector $u = (0,...,0,1)'$.

We consider four main versions (hypotheses) of the structure of matrix *P*, denoted as P_a, P_{an_C}, P_b and P_c. In the simplest version we assume that in one step of MC failure of only one LI can take place. It is convenient (but not necessary) to think that the first failure appears in the boundary of the link and all the following failures can appear only in the adjacent LI. This version corresponds to a transverse crack growth in the monolayer. The stress concentration is supposed to be negligibly small in the so-called Global Load Sharing [10–12], leading to uniform distribution of load between intact LI. We assume that a very small stress concentration is present at a break and do not take it into account.

For the corresponding matrix P_a we define $p_{ii} = 1 - F_C(x_t(i))$, where $x_t(i) = (x_t n_C/(n_C - i + 1))$, $F_C(x_t(i)) = (F_0(x_t(i) - F_0(x_{t-1}(i))/(1 - F_0(x_{t-1}(i)))$ is the conditional cdf of strength of a LI, the failure of which did not take place under load x_{t-1}, $F_0(x)$ is the initial cdf of strength of a LI; $p_{i(i+1)} = 1 - p_{ii}$, $i = 1, ..., n_C$, $p_{(n_C+1)(n_C+1)} = 1$, but all the other p_{ij} are equal to zero.

In second hypothesis we suppose again that in one step of MC failure of only one LI can take place but now it is the weakest intact LI in the link. Then for the matrix P_{an_C} $p_{ii} = (1 - F_C(x_t(i)))^{n_C+1-i}$ and, again, $p_{i(i+1)} = 1 - p_{ii}$, $i = 1, ..., n_C$, $p_{(n_C+1)(n_C+1)} = 1$, but all the other p_{ij} are equal to zero.

In the third hypothesis it is assumed also that the number of failures in one step of MC has a binomial distribution. Then for the corresponding matrix P_b we have

$p_{i(i+r)} = b(r; p,k) = p^r(1-p)^{k-r} k!/r!(k-r)!$, $p = F_C(x_t(i))$, $k = n_c + 1 - i$, $r = 0,...,k$, $i = 1,...,n_C$; and again $p_{(n_C+1)(n_C+1)} = 1$, but all the other p_{ij} are equal to zero.

For previous versions of P, denoted by P_a, P_{an_C} and P_b, we suppose uniform load distribution between intact LI. In the fourth hypothesis it is assumed again that the matrix P_C corresponds to a transverse crack growth in the monolayer but this time we take into account the stress concentration at the tip of the crack. Let us denote by *j* the order number of LI in a LINK (*j* = 1 for the boundary LI) and let redistribution of load $x(t)$ between intact LI be defined by a "stress concentration" function $h(j;i,n_C)$.

Then in the corresponding P_c matrix we have $p_{ij} = \prod_{i+1}^{j} F_C(x_{ij}(t)) \prod_{j+1}^{n_C+1} (1 - F_C(x_{ij}(t))$ for $j = i+1,...,n_C$; $p_{i(n_C+1)} = \prod_{i+1}^{n_C+1} F_C(x_{ij}(t))$ for $j = n_C$; $p_{ii} = 1 - \sum_{i+1}^{n_C+1} p_{ij}$, $p_{ij} = 0$ for $j < i$, $i = 1,...,n_C$; where $x_{ij}(t) = h(j;i,n_C)x(t)n_C/(n_C + 1 - i)$ describes stress in *j*-th order LI after failure of *i*-th order LI.

4. MinMaxDM Distribution Family

For the case n_C = 1 the type of cdf $F_Y(x)$ and $F_Z(x)$ should be chosen "a priori" [3]. But clearly, that all the ideas considered in section 2 can be used also for the series system in which links are parallel systems with n_C >1. Cdf $F_Y(x)$ and $F_Z(x)$ define now cdf of strength of parallel systems of *Y*-type or *Z*-type correspondingly with n_C > 1. Taking into account (20) and definition of a more general p.s. A1, we can describe the strength of specimen by equation

$$X = \min_{1 \le i \le n_L} X_i^* = \min_{1 \le i \le n_L} \max_t (x_t : n_C - K_{Ci}(t) \ge 0), \tag{23}$$

where $K_{Ci}(t)$ is $K_C(t)$ for i-th link.

For building the cdf of X in the following numerical examples we suppose that the logarithm of strength of one LI (in one link) without defect has a 'sev' distribution (14). Of course, it is not the only possible assumption. Different assumptions about the distribution of strength of one link, a priori distribution of initial (technological) defects compose a family of the distributions of ultimate composite tensile strength.

Taking into account (23) we denote the corresponding family of distributions of X by abbreviation MinMaxD (in memory of Daniels). If for calculation of cdf the MC theory is used then it is appropriate to use the abbreviation MinMaxM. For the unified family we use the abbreviation MinMaxDM.

Really, MinMaxDM distribution family is a special subfamily of the EFWLD when the strength of damaged LI is equal to zero and the difference between $F_Y(x)$ and $F_Z(x)$, which in this case are cdf of strength of parallel system with different a priori distribution of the number of damaged LI. For *Y*-type link this number is equal to zero.

5. Processing and analysis of test data

In [4] there are examples of processing of fibre tension test data using some specific versions of models from EFWLD and their comparison with the Weibull and the so-called Power-Weibull models [14–16]. Here we study two groups of test data. In [17] carbon fibre test data (for every specimen) are reported for $(L_1,...,L_4)$ = (1, 10, 20, 50 mm) (Data_A1) and in [16] there are mean values, μ_X, and standard deviations, σ_X, for dry bundle (of the same fibres) tests with $(L_1,...,L_4)$ = (5, 20, 100, 200 mm) (Data_A2). In [18] there are tension test data of carbon fibre strands (Data_B1) and of tapes of 10 of the same carbon fibre strands of the same length 20 mm (Data_B2). We perform fitting and parameter estimation using data_A1 and try to predict the strength of bundles (Data_A2). And then we do the same with Data_B1 and Data_B2.

We cannot consider all the versions of models in the framework of MinMaxDM distribution family. We consider two models of strength of one link: randomised Daniels's model (MinMaxDM_RDM) and Markov chains theory model with matrix of the type P_a (MinMaxDM_Ba). For description of specimen length influence we use the simple p.s. B3.

5.1. Link-Strength Distribution Specification Based On a Randomized Daniels's Model

If the number n in equation (19) is sufficiently large then for X^* there is a convergence in probability to a constant μ defined by equation

$$\mu = \max_x x(1 - F_X(x)),$$

where $F_X(x)$ is the cdf of strength of one LI. We consider the case of Weibull distribution of the single LI strength (without defect), then using logarithmic scale (in order to use the advantage of sev distribution with the location and scale parameters) we can write the equation for μ in the following form

$$\mu = \max_x \exp(x)\exp(-\exp((x-\theta_0)/\theta_1)),$$

We have the following solution of this equation

$$\mu = \theta_1^{\theta_1} \exp(\theta_0 - \theta_1).$$

Daniels has shown that for sufficiently large n r.v. X^* in equation (19) has approximately normal distribution. For the considered case, when K_C=0, the parameters of this distribution are μ and $\sigma = \mu(\exp(\theta_1 - 1)/n_C)^{1/2}$ [16]. But if there are K_C damaged LI (i.e. there are only ($n_C - K_C$) intact LI) then we should use $\mu_n = \mu(n_C - K_C)/n_C$ and $\sigma_n = \sigma(n_C - K_C)/n_C$ (the denominator is equal to n_C instead of ($n_C - K_C$) because the specimen strength is calculated taking into account the initial number of LI, n_C).

So if $n = n_C - K_C$, where random variable K_C has a truncated binomial a priori distribution (note that, we should eliminate the case of $K_C = n_C$) with parameters (n_C, p_C), is large enough then cdf of X^* is approximately defined by the equation

$$F_{X^*}(x) = \sum_{n=1}^{n_C} F_{X_n^*}(x) b(n_C - n, n_C, p_C)/(1 - b(n_C, n_C, p_C)), \tag{24}$$

where $F_{X_n^*}(x) = \Phi((x-\mu_n)/\sigma_n)$; $\Phi(.)$ is the cdf of standard normal distribution, $b(k,m,p) = p^k(1-p)^{m-k} m!/k!(m-k)!$.

Let us call this model as a randomised Daniels's model and denote it by MinMaxDM_RDM or just RDM. Let us denote by NRDM the usual non-randomised Daniels's model.

5.2. Link-Strength Distribution Specification Based On the Model of Markov Chains Theory

In this case we suppose that the random variable X^* has a structure $X^* = \theta_0 + \theta_1 \overset{0}{X}{}^*$, where the cdf of r.v. $\overset{0}{X}{}^*$ is defined by equation (22) with matrix P in the form P_a with probabilities $\{p_{ij}\}$ defined at an assumption that $F_0(x)$ is equal to cdf of a standard sev distribution with θ_0 = 0, θ_1 = 1: $F_0(x) = 1 - \exp(-\exp(x))$. And we suppose that in the definition $F(x) = (1-p_0)F_Y(x) + p_0 F_Z(x)$ the difference between $F_Y(x)$ и $F_Z(x)$ is defined only by the a priori distribution. For $F_Z(x)$ the initial number of defects is equal to 0. But for $F_Y(x)$ it has the conditional binomial distribution with parameters (n_C, p_C) (under the condition that the initial number of defects is less than n_C). In this paper we assume the binomial distribution for the number of links of Y-type, it is for the r.v. K_L, and we use B3 – type p.s. Then $p_0 = (1-p_L)^{n_L}$.

We denote the MC theory model considered here as MinMaxDM_Ba. In the general case in the model parameter there are six components: two parameters of the cdf of strength of single LI, θ_0 and θ_1, and four structure parameters: n_C, p_C, n_L (or l_1, then $n_L = L/l_1$), p_L. For solution of some specific problems, part of the structure parameters can be known. For example, for NRDM p_C = 0, p_L = 0; n_C – is the known number of fibers in a dry bundle; l_1 can be accepted equal to two stress recovery lengths (ineffective length). But in general case for fitting of test data we can use all 6 parameters. It is clear, the more parameters are used for fitting the better the fitting (the less $OSPPt$ [1]), but the less the hope to get better prediction for test results with a different structure parameter.

5.3. Processing of Data_A1 and Data_A2

In [16] Weibull cdf is used in the following form $F(y,L)=1-\exp[-(y/y_L)^{\rho_L}]$. Corresponding parameter estimates are given in Table III. Using parameter estimates of single fibres with length 20 mm ($y_L = 2650$ MPa, $\rho_L = 5.5$), the prediction was made for the strength of a dry bundle of 1000 fibres (mean value, μ_S , and standard deviation of strength, σ_S , of strength, were predicted (see Table 2)) using the usual NRDM for different lengths (see Table V in [14]). We do the same but using the two mentioned above models from MinMaxDM distribution family and corresponding parameters of sev distribution $\hat{\theta}_0 = \log(y_L) = 7.8823$, $\hat{\theta}_1 = 1/\rho_L = 0.1818$. In Table 1 experimental data, calculation results and structure parameters are given (note that $n_L = L/l_1$), $p_0 = (1-p_L)^{n_L}$).

In [16] the good agreement of NRDM prediction of mean strength of dry bundle with the same length, 20 mm, is shown, but this agreement does not extend to other values of L and there is a significant decrease of σ_S for all four lengths.

Using of MinMaxDM_RDM, for which the equation (24) for $F_Y(x) = F_{X^*}(.)$ (all links are *Y*-type link) and equation (6) for $F_X(.)$ are used, allows to get much better estimations of σ_S and not too bad estimation of μ_S at least for L ≥ 20 mm if we use mentioned above strength parameters of single fibres and specific structural parameters (see Table 1). Much better prediction of σ_S is a consequence of definition of $F_X(.)$ in (6) as a mixture of two cdf, $F_Y(.)$ and $F_Z(.)$, corresponding to the presence and absence of defects (see Fig. 1), and , of course, it is a consequence of using specific structural parameters.

In Table 2 in the line denoted as MinMaxDM_Ba_2, we see very good agreement of prediction of μ_S with experimental data if again we use some specific structural parameters. But estimations of σ_S for these parameters are not satisfactory. (It is a good lesson: good prediction of the mean by a given model does not mean good prediction of the standard deviation!). Some mean quality of prediction we have in line denoted by MinMaxDM_Ba_1. Let us note that the choice of parameters is made taking into account "the quality criterion"

$$Q_{ms} = w\sum_{i=1}^{4}(\overline{x}_i - \hat{x}_i)^2 / \sum_{i=1}^{4}(\overline{x}_i - \overline{x})^2 + (1-w)\sum_{i=1}^{4}(s_i - \hat{s}_i)^2 / \sum_{i=1}^{4}(s_i - \overline{s})^2,$$

where for $L = L_i$, $i = 1,...,4$, $\overline{x}_i$ and s_i are test-estimates of μ_S and σ_S; $\overline{x}$ and $\overline{s}$ are the corresponding mean values; $\hat{x}_i$ and $\hat{s}_i$ model-estimates of μ_S and σ_S; w is the relative weight of first part of the sum, $0 \le w \le 1$. Structural parameters MinMaxDM_Ba_1 and MinMaxDM_Ba_2 correspond to $w = 0.5$ and $w = 1$.

Table 2. Summary of processing Data_A

L (mm)		5	20	100	200	Parameter estimates					
Number of observations		28	25	29	27	$\hat{\theta}_0$	$\hat{\theta}_1$	n_C	p_C	l_1	p_L
Mean (GPa)	Observed	1.92	1.68	1.58	1.38	7.883	0.182	1000	0	20	0
	NRDM[16]	2.19	1.71	1.28	1.14	7.883	0.182	1000	0	20	0
	MinMaxDM_RDM	1.61	1.57	1.48	1.4	7.883	0.182	1000	0.2	20	0.1
	MinMaxDM_Ba_1	1.99	1.83	1.73	1.70	7.883	0.182	30	0.3	20	0.9
	MinMaxDM_Ba_2	1.9	1.75	1.57	1.44	7.883	0.182	30	0.9	20	0.7
Std (GPa)	Observed	0.07	0.1	0.13	0.11	7.883	0.182	1000	0	20	0
	NRDM[16]	0.031	0.024	0.018	0.016	7.883	0.182	1000	0	20	0
	MinMax	0.057	0.114	0.161	0.156	7.883	0.182	1000	0.2	20	0.1
	MinMaxDM_Ba_1	0.182	0.106	0.096	0.1	7.883	0.182	30	0.3	20	0.9
	MinMaxDM_Ba_2	0.303	0.435	0.534	0.498	7.883	0.182	30	0.9	20	0.7

5.4. Processing of Data_B1 and Data_B2

In [18] test results are reported of both 64 carbon fibre strands (Data_B1) and the same number of tapes of 10 strands with the same length 20 mm (Data_B2). We attempt to analyse the statistical description of Data_B2, using results of processing of Data_B1. Processing Data_B1, assuming that sev distribution of X takes place, we determined the parameters of corresponding cdf: $\widehat{\theta}_0 = 6.55$ and $\widehat{\theta}_1 = 0.133$ (corresponding mean and standard deviation of $X = \log(S)$, where S is strength in MPa, are: $\mu_X = 6.4769$, $\sigma_X = 0.17322$).

Using standard NRDM and these estimates (for strength of one strand) we have following prediction for strength distribution of 10 strands (Data_B2): $(\hat{\mu}_X, \hat{\sigma}_X)$=(6.245, 0.095) in comparison with sample values (6.156, 0.195); quality of prediction of orders statistics is estimated by $OSPPtP = 0.695$ (here we use the definition given in [1], namely $OSPPt = (\sum_{i=1}^{n}(x_i - \hat{x}_i)^2 / \sum_{i=1}^{n}(x_i - \overline{x})^2)^{1/2}$, where x_i is a sample order statistics, $\hat{x}_i$ is a prediction of its expected value, $\overline{x} = \sum_{i=1`}^{n} x_i / n$ (in this paper we use abbreviation $OSPPtP$ for prediction and $OSPPtF$ for fitting)). We see not too large mistake in prediction of μ_X, but significant mistake of prediction of σ_X and large enough value of $OSPPtP$.

Results of conditional prediction of corresponding order statistics using RDM (under condition that $(p_C, p_L, n_C, n_L) = (0.2,\ 0.4, 10, 1)$) we see on Figure 1. This time we get $OSPPtP$ = 0.3, $(\hat{\mu}_X, \hat{\sigma}_X) = (6.154, 0.195)$.

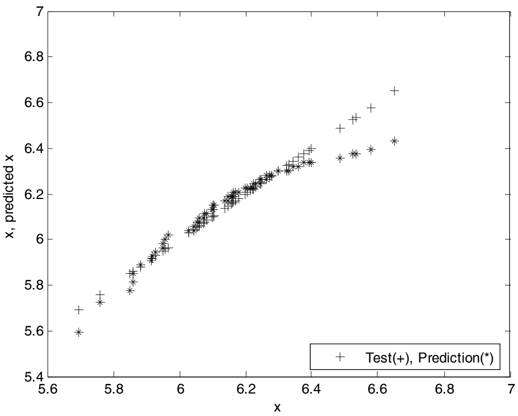

Figure 1. Processing of Data_B2 using RDM. $OSPPtP = 0.3$ for $(p_C, p_L, n_C, n_L) = (0.2, 0.4, 10, 1)$

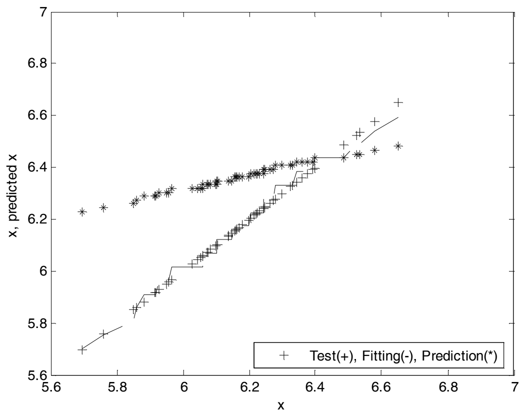

Figure 2. Processing of Data_B2 using MinMaxDM_Ba. For (p_C, p_L, n_C, n_L) =(0.0, 0.0, 10, 1), $OSPPtF = 0.132$; $OSPPtP = 1.27$

Here parameter n_L is equal to one because length of the specimen of a single strand is equal to the length of tape-specimen; n_C is equal to 10, because in the tape-specimen there are 10 strands. The choice of estimates $(p_C, p_L) = (0.2, 0.4)$ correspond to the smallest $OSPPtP$.

For MinMaxDM_Ba at a fixed $n_C = 10$ and $n_L = 1$ for $(p_C, p_L) = (0,0)$ we have enough good fitting with $OSPPtF = 0.132$ but significant increasing of both parameter estimate: for tape-specimen of 10 strands $(\widehat{\theta}_0, \widehat{\theta}_1) = (6.86, 0.473)$ instead of, remind, $(\widehat{\theta}_0, \widehat{\theta}_1) = (6.55\ ,0.133)$ for one strand. So it is not wonder that for prediction, using parameter of one strand, we have $OSPPtP = 1.27$, see Fig. 2.

It appears, that at the same $n_C = 10$ and $n_L = 1$ the smallest value of $OSPPtF = 0.098$ we get at $(p_C, p_L) = (0.3, 0.8)$ and $(\widehat{\theta}_0, \widehat{\theta}_1) = (6.94, 0.359)$. For prediction with the same structural parameters but one strand parameter $(\widehat{\theta}_0, \widehat{\theta}_1) = (6.55\ ,0.133)$ the corresponding value of $OSPPtP = 0.85$, see Fig. 3.

It is still worse than for NRDM ($OSPPtP$ = 0.695). The smallest $OSPPtP$ = 0.129 we have got for (p_C, p_L, n_C, n_L) =(0.1, 0.3, 4, 2), see Fig. 4.

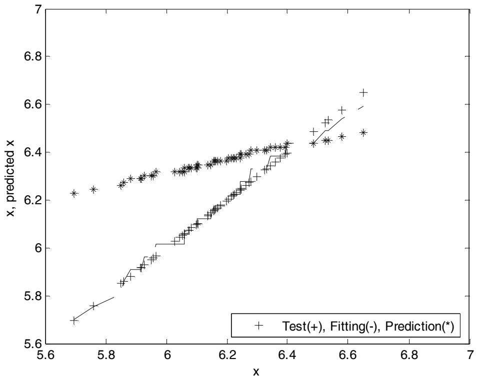

Figure 3. Processing of Data_B2 using MinMaxDM_Ba. For (p_C, p_L, n_C, n_L) = (0.3, 0.8,10,1), $OSPPtF$ = 0.098; $OSPPtP$ = 0.85

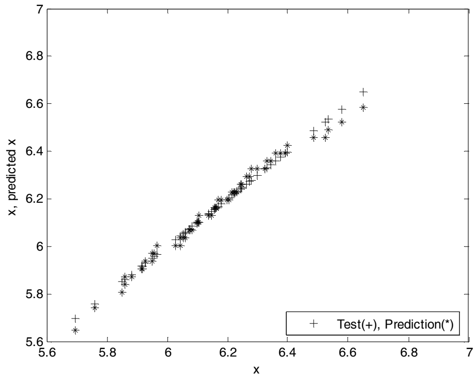

Figure 4. Prediction of Data_B2 using MinMaxDM_Ba. For (p_C, p_L, n_C, n_L) = (0.1,0.3,4,2), and corrected θ_1; $OSPPtP$ = 0.129

But to get this small value of $OSPPtP$ we need additionally to correct $\hat{\theta}_1$. The corrected value of $\hat{\theta}_1$ was calculated taking into account the standard deviation of Young's modulus, $\sigma_E = 0.332$ [18], using the connection between scale parameter and standard deviation for 'sev' distribution: $\hat{\theta}_1 = (6(\hat{\sigma}_X^2 + \hat{\sigma}_E^2))^{1/2} / \pi = 0.29197$).

Of course, demonstrated "prediction" is not a real prediction, but it is actually a "known-structural-parameter" conditional prediction. Really, it is a conditional fitting under condition that the strength parameters of LI in the framework of tape-specimens are the same as strength parameters of a single LI. But there is some usefulness of such fitting. The obtained structural parameter p_C, p_L, n_C and n_L show us how the parameters of single LI strength change when it is part of some complex structure. For example in the last case we see that LI keeps its strength parameter only if we suppose that failure of specimen takes place, when failure of some critical cluster (with four LI (n_C = 4), with the length two time less then "nominal" length (n_L = 2)) takes place; there are 51% of *Y*-type links in which some defects can appear ($p_0 = (1 - p_L)^{n_L} = 0.49$); there is a priori probability p_C = 0.1 that there is defect of LI in *Y*-type link (failure or too large length of single LI in framework of tape). Of course we should not trust too much this interpretation of structural parameter if the sample size is not large enough. But it provides information for thought.

Conclusions

We see that considered models as part of MinMaxDM distribution family provide good fitting of the results of tensile strength tests of carbon fibre dry bundles and tapes of 10 strands and can explain some specific features of strength of LI in framework of more complex structure.

Analysis of the Data_A show us that if we have an inclination to think that strength of fibre in framework of bundle does not change than we should accept that technology of making of bundle specimen is the reason of some damages defined by structural parameter : (p_C, p_L, n_C, n_L). And we have numerical estimate of this technology quality. For example, RDM allows explaining the increase of standard deviation in comparison with the theoretical value (corresponding to usual Daniels's model).

Analysis of the Data_B show us that really the strength parameter of one strand changes in framework of tape of 10 strands: we see increasing of both mean strand strength and its variance as if the length of strand decreases.

Good fitting is not surprising, of course, because considered models have large number of parameters which can be used for fitting of experimental data. But although for the same reason this model really cannot be used for prediction the estimates of this parameter during fitting of dataset can be useful for the **numerical estimation of the "quality" of the structure.** Interpretation of the parameters of the corresponding models enables numerical comparison of different composite structures and explanation of some specific features of the failure process of composite.

The MinMaxDM distribution family is initially developed for analysis of reliability of series and parallel systems with damaged elements. We have studied two versions of the cdf types from this family for analysis of the strength of monolayer. But of course, using this family the analysis of the strength of structure different from the structure of a monolayer can be made. For example, we can assume that only LI carry the longitudinal load but the matrix only redistributes the loads after the failure of some longitudinal items. Then again we have structure similar to structure of monolayer. As a whole, it seems that MinMaxDM distribution family deserves to be studied much more thoroughly using more test data.

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Received on the 1st of December, 2009