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MEAN SQUARE STABILITY OF LINEAR DYNAMICAL SYSTEMS WITH MARKOV COEFFICIENTS

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Abstract. The linear dynamical systems in $\mathbf{R}^n$ with coefficients dependent on diffusion Markov process is studied. It is proven that covariance matrix of solutions form strong continuous semigroup of class $\mathcal{C}_0$. It permits to develop an algorithm for mean square stability analysis of equations with coefficients near to constant based on the second Lyapunov method and on decomposition of a Lyapunov functional in a Laurent series in terms of powers of a small parameter is proposed. The method and algorithm are illustrated using a Mathieu type stochastic oscillator.

1 Introduction

This paper deals with the linear differential equation in $\mathbb{R}^n$

$$\frac{dx}{dt} = (A_0 + \varepsilon\, y(t)\, A_1)\, x(t), \tag{1}$$

where ε is a small parameter, $y(t)$ is a diffusion Markov process defined by a stochastic differential Ito equation in $\mathbb{R}$ of Ornstein-Ulenbeck type

$$dy = -2y\, dt + \sqrt{2}\, dw(t) \tag{2}$$

and $w(t)$ is the standard Wiener process. We will use the continuous semigroup [4] of linear operators $\{T_\varepsilon(t), t \geq 0\}$

$$(T_\varepsilon\,(t)q)(y) := \mathbf{E}_y^{(s)}\{X^T(t+s,s,y)q(y(t+s))X(t+s,s,y)\}, \tag{3}$$

where $\{X(t+s,s,y),\ s \geq 0,\ t \geq 0\}$ is the Cauchy matrix-family of (1) under the condition $y(0) = y$. The infinitesimal operator of this family can be represented as the sum of two closed operators dependent on the parameter ε and we can use results of the perturbation operator theory in [6]. In the second Section the relative boundedness of the perturbation operator will be proven. Section three reviews some spectral properties of the sum of the commutative operators. The results of this Section will be using in the Sections 5 and 6.In Section four it is proven that the exponential mean square stability problem of equation (1) can be formulated as a spectral projector decomposition problem for the infinitesimal operator of the above mentioned semigroup. In Section five the algorithm for this spectral projector decomposition is derived. Section six presents an example involving mean square stability analysis of a stochastic oscillator with coefficients dependent on Gaussian Markov process with rational spectral density.

2 Relative boundedness of perturbed operator

The Markov process defined by Ornstein-Ulenbeck equation (2) is uniformly ergodic [3] with invariant measure corresponding to the stationary solution $\{\hat{y}(t)\}$ of (2) with random initial condition $\hat{y}(0)$ being Gaussian distributed with density $\frac{1}{\sqrt{\pi}}\, e^{-y^2}$. The infinitesimal operator of this process [3] has the form $Q = -2y\,\frac{d}{dy} + \frac{d^2}{dy^2}$ and can be considered as a linear operator on the space $\mathbf{L}_2$ with scalar product $[f,\, g] := \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} f(y)\, g(y)\, e^{-y^2}\, dy$ and bases formed by Hermite polynomials

$$H_{(k-1)}(y) = \frac{(-1)^{(k-1)}}{\sqrt{2^{(k-1)}\,(k-1)!}}\, e^{y^2}\, \frac{d^{k-1}}{dy^{k-1}}\, e^{-y^2}, \quad k \in \mathbb{N}.$$

We will describe some of the properties of this operator by using notations of [6]. Let $\mathbf{G}$ be closed operator and its resolvent exists in the sector $S(\omega,\, 0) := \{|arg\,\lambda| > \omega + \frac{\pi}{2}\}$ with $\omega \in (0,\, \frac{\pi}{2})$. We will write $\mathbf{G} \in \mathcal{H}(\omega,\, 0)$ if in addition for any sufficiently small $\delta > 0$ there exists some positive constant M such that for any $\lambda \in S(\omega - \delta,\, 0)$ the inequality $\|(\mathbf{G} - \lambda J)^{-1}\| \le \frac{M}{|\lambda|}$ is fulfilled. If $\mathbf{G}_1$ is closed operator and $\mathbf{G}_1 - \beta\, J \in \mathcal{H}(\omega,\, 0)$ for some $\beta \in \mathbf{R}$ we will write $\mathbf{G}_1 \in \mathcal{H}(\omega,\, \beta)$. It is easy to verify that operator Q has compact resolvent and discrete spectrum with eigenvalues given by $-2\,(k-1)$, $k \in \mathbb{N}$. Each such eigenvalue has multiplicity one, and its eigenfunction is a Hermite polynomial with corresponding number. Therefore Q is an operator of the class $\mathcal{H}(\omega,\, 0)$ for any $\omega \in (0,\, \frac{\pi}{2})$. Let us define on the space $\mathbf{L}_2$ the linear operator $(Y f)\,(y) := y\, f(y)$.

Lemma 1. *The operator Y is Q-relatively bounded [6], that is $D\,(Y) \subset D\,(Q)$; $\exists c > 0$, $\forall f \in D\,(Q) : ||Y f|| \le c\,(||f|| + ||Q\, f||)$.*

We shall extend the operator Q on *the Hilbert space $\mathbf{Q}_2$ of symmetric $n \times n$ matrix-valued functions with scalar product*

$$[p,\, q] := \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} Tr\{p(y)\, q(y)\} e^{-y^2}\, dy \tag{4}$$

using the same notation Q for this extension. It is clear that the extension on the space $\mathbf{Q}_2$ preserves all above indicated properties of the initial operator on the space $\mathbf{L}_2$. Let us define on $\mathbf{Q}_2$ the linear closed "unperturbed" operator $\mathbf{A}_0\, q(y) := A_0^T\, q(y) + q(y)\, A_0 + Q\, q(y)$ for any $q \in D\,(Q)$, and the linear closed "perturbing" operator $\mathbf{F}\, q\,(y) := \mathbf{A}_1\, Y\, q\,(y)$ for any

$q \in D\,(Q)$, where $\mathbf{A}_1\, q(y) := A_1^T\, q(y) + q(y)\, A_1$. Due to Lemma 1 and boundedness of the operator $\mathbf{A}_1$ one can write the inequality $||\mathbf{F}\, q|| \le c_1\,(||q|| + ||\mathbf{A}_0\, q||)$ with some constant c_1 for any $q \in D\,(\mathbf{A}_0) = D\,(Q)$. Therefore, *the operator* $\mathbf{F}$ *is* $\mathbf{A}_0$*-relatively bounded and* $\{\mathbf{A}_0 + \varepsilon\, \mathbf{F}\}$ *is a holomorphic family of operators [6] in some neighborhood of the point* $\varepsilon = 0$.

3 Integral identity for resolvent of closed operators

Let $\mathbf{L}(\mathbf{X})$ be the Banach algebra of linear continuous operators and $\mathfrak{C}(\mathbf{X})$ be the set of closed operators acting in the real Banach space $\mathbf{X}$. We shall denote $\sigma\,(A)$ the spectrum of $A \in \mathbf{L}(\mathbf{X})$ or $A \in \mathfrak{C}(\mathbf{X})$, and $R_\lambda\,(A)$ its resolvent. We will say that *a point* $\lambda \in \mathbf{C}$ *divides the spectrum of the operators* A *and* B *if* $\inf\{|z_1 - z_2 - \lambda| : \; z_1 \in \sigma(A), z_2 \in \sigma(B)\} > 0$.

Theorem 1. *Let* $A \in \mathbf{L}(\mathbf{X}), B \in \mathbf{L}(\mathbf{X})$ *be commutative and let the spectrum of the operator* A *be represented in the form* $\sigma(A) = \bigcup\limits_{k=1}^{m} \sigma_k$*, where* $\sigma_k \cap \sigma_j = \emptyset$ *if* $k \ne j$. *If* λ *divides the spectrum of* A *and* B*, then* $\lambda \notin \sigma(A - B)$ *and*

$$R_\lambda\,(A - B) = \frac{1}{2\pi\, i} \oint\limits_{C} R_\mu\,(A) R_{\lambda-\mu}\,(-B)\, d\mu,$$

where $C = \bigcup\limits_{k=1}^{m} C_k$ *with simply rectifiable Jordan closed contours* C_k*, so that* $C_k \cap C_j = \emptyset$ *if* $k \ne j$.

Let $\mathbf{X} = \mathbf{X}_1 \times \mathbf{X}_2$, $dim\mathbf{X} < \infty$, A_1, $A_2 \in \mathbf{L}\,(\mathbf{X})$, and J_1, J_2 be the units of the algebras $\mathbf{L}\,(\mathbf{X}_1)$ and $\mathbf{L}\,(\mathbf{X}_2)$ respectively. Remember that the tensor sum of these operators is defined as $\mathbf{A} := A_1 \oplus A_2 := A_1 \otimes J_2 + J_1 \otimes A_2$.

Corollary 1. *The spectrum of the operator* $\mathbf{A}$ *has the form* $\sigma(\mathbf{A}) = \{\lambda_1 + \lambda_2 \;:\; \lambda_1 \in \sigma(A_1),\, \lambda_2 \in \sigma(A_2)\}$ *and for any* $\lambda \notin \sigma(\mathbf{A})$ *the resolvent of* $\mathbf{A}$ *may be represented in the form*

$$\begin{aligned} R_\lambda\,(\mathbf{A}) = \sum_{h=1}^{s_1} \sum_{k=1}^{s_2} (\lambda - \lambda_{1h} - \lambda_{2k})^{-1}\, [P_{1h} \otimes P_{2k} + \sum_{m=1}^{m_{2k}-1} (\lambda - \lambda_{1h} - \lambda_{2k})^{-m}\, P_{1h} \otimes D_{2k}^m + \\ + \sum_{n=1}^{m_{1h}-1} \sum_{m=1}^{m_{2k}-1} (\lambda - \lambda_{1h} - \lambda_{2k})^{-n-m}\, \frac{m+n-1}{m} D_{1h}^n \otimes D_{2k}^m] + \\ + \sum_{n=1}^{m_{1h}-1} (\lambda - \lambda_{1h} - \lambda_{2k})^{-n}\, D_{1h}^n \otimes P_{2k}, \end{aligned}$$

where P_{1h}, P_{2k} *are eigen-projectors,* D_{1h}, D_{2k} *are eigen-nilpotent operators corresponding to the eigenvalues* $\lambda_{1h} \in \sigma(\mathbf{A}_1)$, $\lambda_{2k} \in \sigma(\mathbf{A}_2)$, m_{1h}, m_{2k} *are the corresponding indices of nilpotency, and* s_1, s_2 *are the numbers of points in the sets* $\sigma(\mathbf{A}_1)$ *and* $\sigma(\mathbf{A}_2)$ *respectively.*

Theorem 2. *Let* $A \in \mathcal{H}\,(\omega,\, \beta)$ *and* $B \in \mathcal{H}\,(\omega,\, \beta)$ *be commutative operators acting in the Hilbert space* $\mathbf{H}$*, and let the operator* A *be normal [6] and having a compact resolvent. If* λ *divides the spectra of the operators* A *and* $-B$*, then* λ *is a regular point of the resolvent of the* *operator* $A+B$ *and*

$$R_\lambda(A+B)=\frac{1}{2\pi i}\oint\limits_C R_\mu(A)R_{\lambda-\mu}(B)\,d\mu,$$

where $C=\bigcup\limits_{\delta\in\sigma(A)}\{\mu\in\mathbf{C}:\,|\,\delta-\mu\,|=\varepsilon\}$ *and the positive constant* ε *satisfies the condition* $\varepsilon<\rho(\,\sigma(A),\,\lambda-\sigma(B)\,)$.

Example 1. Let $\mathbf{X}=\mathbb{R}^n$, $A\in\mathbf{L}(\mathbb{R}^n)$ and $\hat{\mathbf{M}}(\mathbb{R}^n)$ be the set of symmetric $n\times n$-matrices. The operator $\hat{\mathbf{A}}\in\hat{\mathbf{M}}(\mathbb{R}^n)$, defined by $\hat{\mathbf{A}}\,q:=A^T\,q+q\,A$ can be considered as a tensor sum of operators acting in the space $\mathbb{R}^n\otimes\mathbb{R}^n$. Then, $\sigma(\hat{\mathbf{A}})=\{\lambda_1+\lambda_2:\,\lambda_1\in\sigma(A),\,\lambda_2\in\sigma(A)\}$ and

$$\begin{aligned}R_\lambda(\hat{\mathbf{A}})\,q\;=\;&\sum_{n=1}^{m_h-1}\sum_{m=1}^{m_k-1}(\lambda-\lambda_h-\lambda_k)^{-n-m}\begin{pmatrix}m+n-1\\ m\end{pmatrix}(D_h^T)^n\,q\,D_k^m+\\&+\sum_{n=1}^{m_h-1}(\lambda-\lambda_h-\lambda_k)^{-n}\,(D_h^T)^n\,q\,P_k+\\&+\sum_{h=1}^{s}\sum_{k=1}^{s}(\lambda-\lambda_h-\lambda_k)^{-1}\,[P_h^T\,q\,P_k+\sum_{m=1}^{m_k-1}(\lambda-\lambda_h-\lambda_k)^{-m}\,P_h^T\,q\,D_k^m\end{aligned}\qquad(5)$$

where $\lambda_h,\,\lambda_k\in\sigma(A)$, $h=\overline{1,s}$, $k=\overline{1,s}$.

Example 2. Let us consider the Hilbert space $\mathbf{Q}_2$ of the symmetric real $n\times n$-matrix-valued functions from Section 2 and let Q be the operator described there. We shall consider the space $\mathbf{Q}_2$ as the tensor product $\mathbf{Q}_2=\hat{\mathbf{M}}(\mathbb{R}^n)\otimes\mathbf{L}_2$. It can be easily verified that the operator Q is self-adjoint. The operator $\mathbf{A}_0$ of Section 2 can be considered as a tensor sum of the operators $\hat{\mathbf{A}}$ and Q. Hence, $R_\lambda(\mathbf{A}_0)=\sum\limits_{j=1}^{\infty}\hat{P}_{j-1}R_{\lambda-2(j-1)}(\hat{\mathbf{A}})$, where $\hat{P}_{j-1}$ are self-adjoint projectors corresponding to the eigenvalues $2(j-1)$, $j\in\mathbb{N}$ and $R_{\lambda-2(j-1)}(\hat{\mathbf{A}})$ is defined by (5). The spectrum of the operator $\mathbf{A}_0$ has the form

$$\sigma(\mathbf{A}_0)=\{\lambda_1+\lambda_2+2(j-1),\,\lambda_1\in\sigma(A_0),\,\lambda_2\in\sigma(A_0),\,j\in\mathbb{N}\}.\qquad(6)$$

4 An asymptotic of second moment

Let $\mathbf{K}\subset\mathbf{Q}_2$ be the cone [8] of nonnegative definite matrix-valued functions. For any $q_1\in\mathbf{Q}_2$ $q_2\in\mathbf{Q}_2$ we will write $q_1\ll q_2$ if $q_2-q_1\in\mathbf{K}$. It is well know that for $q\in\mathbf{Q}_2$ and any $y\in Y$ there exists an orthogonal matrix $U(y)$ such that $q(y)=U^T(y)\,\Lambda(y)\,U(y)$ where $\Lambda(y):=diagonal\{\gamma_j(y)\}$ is a diagonal matrix, and $\gamma_j(y)$, $j=\overline{1,n}$ are eigenvalues of the matrix $q(y)$. For any $q\in\mathbf{Q}_2$ using the definition (4) of scalar product as well as the definition of the stationary Markov process $\hat{y}(t)$ one can write $||q||^2=\mathbf{E}\{tr\,(q(\hat{y}(t)))^2\}$ for any $t\in\mathbb{R}$. Hence, $||q||^2=||\Lambda||^2=\sum\limits_{j=1}^{n}\mathbf{E}\{(\gamma_j\,(\hat{y}(t)))^2\}$. It is clearly to see that there exists decomposition $\gamma_j(y)=\gamma_j^+(y)-\gamma_j^-(y)$ with nonnegative functions $\gamma_j^+(y)$ and $\gamma_j^-(y)$ defined by equalities $\gamma_j^+(y)=max\{\gamma_j(y),\,0\}$, $\gamma_j^-(y)=max\{-\gamma_j(y),\,0\}$. Let us put $\Lambda^+(y):=diagonal\{\gamma_j^+(y)\}$, $\Lambda^-(y):=diagonal\{\gamma_j^-(y)\}$ and $q^+(y)=U^T(y)\,\Lambda^+(y)\,U(y)$, $q^-(y)=U^T(y)\,\Lambda^-(y)\,U(y)$. Due to the identity $\Lambda(y)\equiv\Lambda^+(y)-\Lambda^-(y)$ one can write $q(y)\equiv q^+(y)-q^-(y)$ and $q^+\in\mathbf{K}$, $q^-\in\mathbf{K}$, that is *the cone* $\mathbf{K}$ *is reproducing [8] in* $\mathbf{Q}_2$. Besides by definition $\gamma_j^+(y)\,\gamma_j^-(y) \equiv 0$ for all $j = \overline{1,\, n}$, Therefore $q^+(y)\,q^-(y) \equiv 0$ $||q||$ defined by the scalar product (4) is equivalent to the norm $||q^+|| + ||q^-||$ defined by decomposition according to the cone $\mathbf{K}$. Let $\hat{\mathbf{Q}}_2 \subset \mathbf{Q}_2$ be the subspace of matrix-valued functions satisfying the condition $\lim_{t\to 0} \mathbf{E}_y\, q(y(t)) = q(y)$ for any $y \in \mathbf{Y}$. We should deal with operator family defined on $\hat{\mathbf{Q}}_2$ by (3). The expectation in the right part of the equality (3) exists [7] because a stochastic process $\{y(t)\}$ with any initial condition $y(s) = y$ is a Gaussian process, which has constant variance and exponentially decreasing correlation function. Due to homogeneity of the Markov process $\{x(t),\, y(t)\}$ the definition (3) does not depend on $s \in \mathbb{R}$.

Theorem 3. *There exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0,\, \varepsilon_0)$ *under the above assumptions the family of operators* $\{T_\varepsilon(t),\, t \geq 0\}$ *forms a strong continuous semigroup* [4] *with infinitesimal operator defined by*

$$\mathbf{A}(\varepsilon)\, q(y) = \mathbf{A}_0\, q(y) + \varepsilon\, Fq(y) = A_0^T\, q(y) + q(y)\, A_0 + Q\, q(y) + \varepsilon\, y\, (A_1^T\, q(y) + q(y)\, A_1).$$

Due to the results of Section 3 one can confirm that for a sufficiently small positive ε all points of the spectrum $\sigma(\mathbf{A}(\varepsilon))$ of the operator family $\mathbf{A}(\varepsilon)$ are situated in small neighborhoods of the corresponding points of the spectrum $\sigma(\mathbf{A}_0)$ of the operator $\mathbf{A}_0$.This information helps us in the analysis of the asymptotic of the second moments of the solutions of (1) as $t \to +\infty$. It is easy to show that for any $y(0) = y \in \mathbf{Y}$ and $t \geq 0$ one can write the inequality $|y(t) - \hat{y}(t)| \leq (|y| + |\hat{y}(0)|)\, e^{-2t}$ where $\hat{y}(t)$ is the corresponding stationary process. Therefore, we will assume that (1) has the form

$$\frac{dx}{dt} = (A_0 + \varepsilon\, \hat{y}(t)\, A_1)\, x(t), \tag{7}$$

and we will call equation (1) as *exponentially mean square stable if there exist positive constants* c *and* M *such that the solution* $x(t)$ *of (7) with initial condition* $x(0) = x$ *satisfies the inequality*

$$x \in \mathbb{R}^n :\ \mathbf{E}\, |x(t)|^2 \leq M\, |x|^2\, e^{-ct}.$$

Theorem 4. *Equation (1) is exponentially mean square stable for all* $\varepsilon \in (0,\, \varepsilon_0)$ *and sufficiently small* $\varepsilon_0 > 0$ *if and only if*

$$\sigma(\mathbf{A}(\varepsilon)) \cap \{\lambda \in \mathbb{R} :\ \lambda \geq 0\} = \emptyset.$$

5 The spectral projection decomposition

Let $\sigma_\varepsilon \subset \sigma(\mathbf{A}(\varepsilon))$ be the part of the spectrum of the operator $\mathbf{A}(\varepsilon)$ which satisfies the condition

$$\lim_{\varepsilon\to 0} \sigma_\varepsilon = \sigma_0 := \{\lambda_1 + \lambda_2 :\ \lambda_1 \in \hat{\sigma}_0 \subset \sigma(A_0),\ \lambda_2 \in \hat{\sigma}_0 \subset \sigma(A_0)\}, \tag{8}$$

with some subset $\hat{\sigma}_0$ of the spectrum of the matrix A_0. In this Section we shall describe an algorithm for analyzing the spectrum σ_ε for sufficiently small positive ε by using the formula (6) for representing the spectrum $\sigma(\mathbf{A}_0)$ of the operator $\mathbf{A}_0$. The root subspace $\mathbf{Q}(\varepsilon) \subset \hat{\mathbf{Q}}_2$ corresponding to the part of spectrum given by (8) has the same dimension $N = dim\, \mathbf{Q}(0)$ for all sufficiently small $\varepsilon \geq 0$. One can construct [6] in $\mathbf{Q}(\varepsilon)$ a basis $\mathbf{B}^\varepsilon = \{\vec{b}_1(\varepsilon,\, y),\, \cdots,\, \vec{b}_N(\varepsilon,\, y)\}$ of the form $\mathbf{B}^\varepsilon = P(\varepsilon)\, \mathbf{B}^0$, where $P(\varepsilon)$ is the total projector in $\mathbf{Q}(\varepsilon)$ and $\mathbf{B}^0 \subset \hat{\mathbf{M}}_n(\mathbb{R})$ because all eigen-elements of the operator $\mathbf{A}_0$ are symmetric $n\times n$-matrices (see example 2 of Section 3). This projector is an analytic function of ε [6] and one can look for the basis as a decomposition

$$\mathbf{B}(\varepsilon) = \mathbf{B}^{\,0} + \varepsilon\,\mathbf{B}^{\,1} + \varepsilon^{\,2}\,\mathbf{B}^{\,2} + \cdots \tag{9}$$

where

$$\mathbf{B}(\varepsilon) = \{\vec{b}_1\,(\varepsilon, y),\, \vec{b}_2\,(\varepsilon, y), \cdots, \vec{b}_N\,(\varepsilon, y)\} \subset \hat{\mathbf{Q}}_2,$$
$$\mathbf{B}^{\,0} = \{\vec{b}^0_1,\, \vec{b}^0_2, \cdots, \vec{b}^0_N\} \subset \hat{\mathbf{M}}_n(\mathbb{R}),$$
$$\mathbf{B}^{\,j} = \{\vec{b}^j_1\,(y),\, \vec{b}^j_2\,(y), \cdots, \vec{b}^j_N\,(y)\} \subset \hat{\mathbf{Q}}_2$$

for any $j \in \mathbb{N}$. This means that all $\mathbf{B}^j$ are matrices with $\vec{b}^j_k$, $k = \overline{1, N}$ as columns,. For the shortening of the computations we next define some special operations involving *a row* $G = \{g_1, g_2, \cdots, g_N\}$ and *a column* $H = \{h_1, h_2, \cdots, h_N\}^T$ of elements of the space $\hat{\mathbf{Q}}_2$. The first of this operations is

$$H \bullet G := \begin{pmatrix} [h_1, g_1] & [h_1, g_2] & \cdots & [h_1, g_N] \\ [h_2, g_1] & [h_2, g_2] & \cdots & [h_2, g_N] \\ \vdots & \vdots & \ddots & \vdots \\ [h_N, g_1] & [h_N, g_2] & \cdots & [h_N, g_N] \end{pmatrix},$$

where $[p, q]$ is inner product (4) in $\mathbf{Q}_2$. The second operation with row G is defined by $\mathbf{S} \star G := \{\mathbf{S}g_1, \cdots, \mathbf{S}g_N\}$ *for any* $\mathbf{S} \subset \mathfrak{C}(\mathbf{Q}_2)$ *if* $G \subset D(\mathbf{S})$. The same operation is defined for column H. Let

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{NN} \end{pmatrix}$$

be a real matrix with N rows and N columns. We also will write for above G, H, C

$$G \diamond C := \{\sum_{k=1}^{N} g_k\, c_{1k},\, \sum_{k=1}^{N} g_k\, c_{2k}, \cdots, \sum_{k=1}^{N} g_k\, c_{Nk}\}$$

and

$$C \diamond H := \begin{pmatrix} \sum_{k=1}^{N} h_k\, c_{k1} \\ \sum_{k=1}^{N} h_k\, c_{k2} \\ \vdots \\ \sum_{k=1}^{N} h_k\, c_{kN} \end{pmatrix}$$

Let $\Lambda(\varepsilon)$ be the matrix of restriction of the operator $\mathbf{A}(\varepsilon)$ on the subspace $\mathbf{Q}_2(\varepsilon)$. This matrix can be obtained [6] from the expression

$$\mathbf{A}(\varepsilon) \,\star\, \mathbf{B}(\varepsilon) = \mathbf{B}(\varepsilon) \,\diamond\, \Lambda(\varepsilon), \tag{10}$$

where for matrix $\Lambda(\varepsilon)$ we also can use the decomposition $\Lambda(\varepsilon) = \Lambda_0 + \varepsilon\,\Lambda_1 + \varepsilon^{\,2}\,\Lambda_2 + \cdots$ Therefore (10) can be rewritten in the form

$$(\mathbf{A}_0 + \varepsilon\, F) \star (\mathbf{B}^{\,0} + \varepsilon\, \mathbf{B}^{\,1} + \varepsilon^{\,2}\, \mathbf{B}^{\,2} + \cdots) =$$
$$= (\mathbf{B}^{\,0} + \varepsilon\, \mathbf{B}^{\,1} + \varepsilon^{\,2}\, \mathbf{B}^{\,2} + \cdots) \diamond (\Lambda_0 + \varepsilon\, \Lambda_1 + \varepsilon^{\,2}\, \Lambda_2 + \cdots)$$

and we can look for $\Lambda_0,\, \Lambda_1,\, \Lambda_2,\, \cdots$ by equating the coefficients near the same powers of ε. At the same time we must look for the components $\mathbf{B}^{\,0}, \mathbf{B}^{\,1}, \mathbf{B}^{\,2}, \cdots$ of the basis's decomposition from (9).
We start with the system of N equations

$$\mathbf{A}_0 \star \mathbf{B}^{\,0} - \mathbf{B}^{\,0} \diamond \Lambda_0 = 0 \tag{11}$$

for the elements $\vec{b}_1^0,\, \vec{b}_2^0, \cdots, \vec{b}_N^0$ of the basis $\mathbf{B}^{\,0}$. In the space of symmetric matrices $\hat{\mathbf{M}}_n(\mathbb{R})$ the operator $\mathbf{A}_0$ is given by $\mathbf{A}_0\, q = A_0^T\, q + q\, A_0$ which can be viewed as an operator in $\mathbf{L}(\mathbb{R}^{\frac{n(n+1)}{2}})$. Hence, one can satisfy the equations (11) with any basis $\mathbf{B}^{\,0} \subset P(0)\, \hat{\mathbf{M}}_n(\mathbb{R}) \subset \hat{\mathbf{M}}_n(\mathbb{R})$ and the matrix Λ_0 with elements $\lambda_{kj}^{(0)}$ of the operator $\mathbf{A}_0$ in this basis.
On the second step we have to deal with the system of equations

$$\mathbf{A}_0 \star \mathbf{B}^{\,1} - \mathbf{B}^{\,1} \diamond \Lambda_0 = \mathbf{B}^{\,0} \diamond \Lambda_1 - F \star \mathbf{B}^{\,0} \tag{12}$$

involving the N components of the basis $\mathbf{B}^{\,1}$. This system has solution if and only if the right part is orthogonal to N linearly independent solutions of the adjoint equation. It can be easily seen that the adjoint homogeneous equation for (12) based on the inner product (4) and using the earlier notations has the form $\mathbf{A}_0^* \star H - \Lambda_0^T \diamond H = 0$, where the adjoint operator $\mathbf{A}_0^*$ is defined by $\mathbf{A}_0^*\, p := A_0\, p + p\, A_0^T$ and the column H consists of N elements of the space $\hat{\mathbf{M}}_n(\mathbb{R})$

$$H := \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix}.$$

It can be proven that we can choose matrices $h_j,\ j = \overline{1, N}$ satisfying the condition

$$H \bullet \mathbf{B}^{\,0} = I, \tag{13}$$

where $I \in \mathbf{M}_N(\mathbb{R})$ is the identity matrix. Next one must verify the condition of orthogonality of all elements of H with respect to the right hand part of (12), that is

$$H \bullet (\mathbf{B}^{\,0} \diamond \Lambda_1 - \mathbf{F} \star \mathbf{B}^{\,0}) = 0, \tag{14}$$

where the operator $\mathbf{F}$ is represented by $\mathbf{F}\, q(y) = y\,(A_1^T\, q(y) + q(y)\, A_1) := y\, \mathbf{A}_1\, q(y)$ Hence from (13) and (14) taking in the account that $\mathbf{E}\, \hat{y}(0) = 0$ one can write the equalities $\Lambda_1 = H \bullet \mathbf{F} \star \mathbf{B}^{\,0} = 0$ and equation (12) has the form

$$\mathbf{A}_0 \star \mathbf{B}^{\,1} - \mathbf{B}^{\,1} \diamond \Lambda_0 + Q \star \mathbf{B}^{\,1} = -y\, \mathbf{A}_1 \star \mathbf{B}^{\,0}, \tag{15}$$

that is the elements $\vec{b}_j^1,\ j = \overline{1, N}$ of $\mathbf{B}^{\,1}$ must satisfy the matrix ordinary differential equation

$$(\frac{d^2}{dy^2} - 2y\, \frac{d}{dy})\, \vec{b}_j^1 + A_0^T\, \vec{b}_j^1 + \vec{b}_j^1\, A_0 - \sum_{k=1}^{N} \vec{b}_k^1\, \lambda_{kj}^{(0)} = -y\,(A_1^T\, \vec{b}_j^0 + \vec{b}_j^0\, A_1).$$

Next we must solve these equations, taking into account the condition of orthogonality of the required solution to all elements of H. We consider (15) as an equation with respect to $\mathbf{B}^1$ in the space defined by the following tensor product of $N+1$ spaces $\mathbf{H} := \hat{\mathbf{M}}_n(\mathbb{R}) \otimes \hat{\mathbf{M}}_n(\mathbb{R}) \otimes \cdots \otimes \hat{\mathbf{M}}_n(\mathbb{R}) \otimes \mathbf{L}_2$. It is clear that the left part of equality (15) is the tensor sum of operator Q and operator $\hat{\mathbf{A}}$ defined by $\hat{\mathbf{A}}\, G := \mathbf{A}_0 \star G - G \diamond \Lambda_0$. Therefore, the solution of (15) can be written in the form

$$\mathbf{B}^1 = \int_0^\infty e^{\hat{\mathbf{A}}\, t} \mathbf{E}_y\, y(t)\, dt,$$

where $y(t)$ is the solution of the Ito equation (2) with initial condition $y(0) = y$. Hence, all elements of $\mathbf{B}^1$ are some constant matrices multiplied by y and we can put
$\mathbf{B}^1 = y\, G := \{y\vec{g}_1,\, y\vec{g}_2,\, \cdots,\, y\vec{g}_N\}$. Next we can rewrite the equation (15) in the form

$$\mathbf{A}_0 \star G - G \diamond \Lambda_0 - 2G = -\mathbf{A}_1 \star \mathbf{B}^0. \tag{16}$$

as the equation for unknown matrix $G := \{\vec{g}_1,\, \vec{g}_2,\, \cdots,\, \vec{g}_N\}$. By construction the above described operator $\hat{\mathbf{A}}$ has the space $P(0)\, \hat{\mathbf{M}}_n(\mathbb{R})$ as its root space and then the operator $\hat{\mathbf{A}} - 2J$ is invertible.
Let $\mathbf{B}^1 = y\, G$, with G from (16), be the solution of equation (15). The next step is analyzing of the equation

$$\mathbf{A}_0 \star \mathbf{B}^2 - \mathbf{B}^2 \diamond \Lambda_0 = \mathbf{B}^0 \diamond \Lambda_2 - \mathbf{F} \star \mathbf{B}^1. \tag{17}$$

The condition of solvability of this equation has the form $H \bullet (\mathbf{B}^0 \diamond \Lambda_2 - F \star \mathbf{B}^1) = 0$ or $\Lambda_2 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} y^2\, e^{-y^2}\, \mathbf{A}_1 \star G = \frac{1}{2}\, \mathbf{A}_1 \star G$ with above defined G. If we substitute this Λ_2 in (17) we obtain the following equation for $\mathbf{B}^2$ $\mathbf{A}_0 \star \mathbf{B}^2 - \mathbf{B}^2 \diamond \Lambda_0 = \mathbf{B}^0 \diamond \Lambda_2 - y^2\, \mathbf{A}_1 \star G$. We can solve this equation and continue on to the equation for $\mathbf{B}^3$ and Λ_3 and so on. This algorithm permits us to construct the decomposition $\Lambda(\varepsilon) = \Lambda_0 + \varepsilon^2\, \Lambda_1 + \cdots$ up to any desired accuracy. The same algorithm can be used for equation

$$\frac{dx}{dt} = \left(A_0 + \sum_{j=1}^{\infty} \varepsilon^j\, (A_j + \sum_{k=1}^{m} h_k\, y_k(t)\, B_j) \right)\, x,$$

where $y_j\,(t)$, $j = \overline{1,m}$ are the coordinates of uniformly ergodic diffusion vector process $y(t)$ which satisfies the Ito equation $dy = C\, y\, dt + D\, dw(t)$, where $C \in \mathbf{M}_m(\mathbb{R})$, $D \in \mathbf{M}_m(\mathbb{R})$ and $w(t)$ is the standard Wiener process in $\mathbb{R}^m$. In this case one must consider the matrix valued function space $\mathbf{Q}_2$ with inner product defined by the invariant measure μ of the Markov process $y(t)$. The next Section contains some examples using the above described algorithm.

6 Stability of stochastic oscillator with colored noise

The linear differential equation of second order

$$\frac{d^2 z}{dt^2} + 2\delta(y(t))\, \frac{dz}{dt} + (\omega^2 + h(y(t)))\, z = 0, \tag{18}$$

where $y(t)$ is stochastic process is called [1,2] *stochastic or random oscillator*. First let us assume that the parameters of the oscillator (18) have decompositions $\delta(y) = \varepsilon\, (\delta_1 y + \overline{\delta_1}) + \varepsilon^2\, (\delta_2 y + \overline{\delta_2})$, $h(y) = \varepsilon\, (h_1 y + \overline{h_1}) + \varepsilon^2\, (h_2 y + \overline{h_2})$, where ε is small positive parameter and $y(t)$

is a uniformly ergodic Markov process with Gaussian invariant measure $\mu(dy) \sim N(0,\, 1/2)$ described by equation (2). To use the above method equation (18) must be rewritten as an equation in $\mathbb{R}^2$ as follows

$$\frac{dx}{dt} = (A + 2\delta(y(t))\, B + h(y(t))\, C)\, x,$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Let $\mathbf{Q}_2$ be the space of symmetric 2×2-matrix-valued functions and $\mathbf{A}(\varepsilon)$ be the infinitesimal operator of the corresponding semigroup (3) in $\mathbf{Q}_2$. This operator can be decomposed in the form $\mathbf{A}(\varepsilon) = \mathbf{A}_0 + \varepsilon\, \mathbf{A}_1 + \varepsilon^2\, \mathbf{A}_2$, where the operator-coefficients are defined by

$$\begin{aligned} \mathbf{A}_0\, q(y) &= A^T q(y) + q(y)\, A + Q\, q(y), \\ \mathbf{A}_j\, q(y) &= (2(\delta_j y + \overline{\delta_j})\, B^T + (h_j y + +\overline{h_j})\, C^T)\, q(y) + \\ &+ q(y)\, (2(\delta_j y + +\overline{\delta_j})\, B + (h_j y + +\overline{h_j})\, C), \quad j = 1, 2 \end{aligned}$$

We will extract only one point $\lambda(\varepsilon) = \varepsilon\, \lambda_1 + \varepsilon^2\, \lambda_2 + \cdots$ of the spectrum of the operator $\mathbf{A}(\varepsilon)$ which corresponds to the spectrum point 0 of the operator $\mathbf{A}_0$. This spectrum point, accordingly the formulae (5) and (6), has multiplicity one. The corresponding eigen-element q is a symmetric matrix satisfying the matrix equation $A^T q + q\, A = 0$ that is one can put $q = \begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix}$. The solution of the adjoint equation $A\, p + p\, A^T = 0$ also is the matrix $p = \begin{pmatrix} \frac{1}{2\omega^2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ which satisfies the condition $Tr\{pq\} = 1$. Next we can write the equation $\mathbf{A}(\varepsilon)\, q^\varepsilon = \lambda(\varepsilon)\, q^\varepsilon$ in the form

$$(\mathbf{A}_0 + \varepsilon\, \mathbf{A}_1 + \varepsilon^2\, \mathbf{A}_2)\, (q + \varepsilon\, q_1(y) + \varepsilon^2\, q_2(y) + \cdots) = (\varepsilon\, \lambda_1 + \varepsilon^2\, \lambda_2 + \cdots)\, (q + \varepsilon\, q_1(y) + \varepsilon^2\, q_2(y) + \cdots)$$

and by equating the coefficients corresponding to the same powers of ε we obtain:

$$A^T q_1(y) + q_1(y)\, A + Q q_1(y) = -\mathbf{A}_1\, q + \lambda_1\, q, \tag{19}$$

$$A^T q_2(y) + q_1(y)\, A + Q q_1(y) = -\mathbf{A}_1\, q_1(y) - \mathbf{A}_2\, q + \lambda_2\, q + \lambda_1\, q_1(y), \tag{20}$$

and so on. In the next step of the algorithm we look at equation (19) using the previously found q. We obtain:

$$\begin{aligned} A^T q_1(y) + q_1(y)\, A + Q\, q_1(y) = \lambda_1 \begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix} + \\ +4(\delta_1 y + \overline{\delta_1}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + (h_1 y + \overline{h_1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{21}$$

The right part of (21) is orthogonal to the above matrix p by means of the inner product $[p,\, q] := Tr\{\overline{pq}\}$. Therefore if $\overline{\delta_1} \neq 0$, then from the above equality one can by using one step of the proposed algorithm obtain the following necessary and sufficient condition of stability: *if $\overline{\delta}_1 > 0$ then for all sufficiently small positive ε the stochastic oscillator (18) is exponentially mean square stable. If $\overline{\delta}_1 < 0$ then there exist solutions of (18) with exponentially growing second moment.*

Next we consider the case where $\overline{\delta}_1 = 0$. In this case one should put $\lambda_1 = 0$ in (20) and (21). Then equation (21) has solution

$$
\begin{aligned}
q_1(y) \;&= -2\int\limits_0^\infty \begin{pmatrix} \omega^2-\omega^2\, cos2\omega t & -\omega sin2\omega t \\ -\omega sin2\omega t & 1+cos2\omega t \end{pmatrix} \mathbf{E}_y\, \delta_1 y(t)\, dt - \\
&- \int\limits_0^\infty \begin{pmatrix} -\omega sin2\omega t & cos2\omega t \\ cos2\omega t & \frac{sin2\omega t}{\omega} \end{pmatrix} \mathbf{E}_y\, (h_1 y(t)+\overline{h_1})\, dt.
\end{aligned}
$$

Next, we must satisfy the condition of orthogonality of the right side of (20) to the matrix p. This condition gives the second term in the decomposition of $\lambda(\varepsilon)$ in the form $\lambda_2 = Tr\{\overline{p\,(\mathbf{A}_1\, q_1 + \mathbf{A}_2\, q_1}\}$, where

$$
\begin{aligned}
\mathbf{A}_2\, q(y) \;&=\; -4(\delta_2 y+\overline{\delta_2}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - (h_1 y+\overline{h_1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\mathbf{A}_1\, q_1(y) \;&=\; -2(h_1 y+\overline{h_1})\,\delta_1 \int\limits_0^\infty \begin{pmatrix} 2\omega sin2\omega t & -1-cos2\omega t \\ -1-cos2\omega t & 0 \end{pmatrix} \mathbf{E}_y\, y(t)\, dt + \\
&+\; (h_1 y+\overline{h_1}) \int\limits_0^\infty \begin{pmatrix} 2cos2\omega t & \frac{sin2\omega t}{\omega} \\ \frac{sin2\omega t}{\omega} & 0 \end{pmatrix} \mathbf{E}_y\, (h_1 y(t)+\overline{h_1})\, dt + \\
&+\; 4\delta_1^2 y \int\limits_0^\infty \begin{pmatrix} 0 & \omega sin2\omega t \\ \omega sin2\omega t & -2-2cos2\omega t \end{pmatrix} \mathbf{E}_y\, y(t)\, dt + \\
&+\; 2\delta_1 y \int\limits_0^\infty \begin{pmatrix} 0 & cos2\omega t \\ cos2\omega t & 2\frac{sin2\omega t}{\omega} \end{pmatrix} \mathbf{E}_y\, (h_1 y(t)+\overline{h_1})\, dt.
\end{aligned}
$$

It is easy to obtain the following equalities $Tr(\mathbf{A}_2\, q)(y) = -4(\delta_2 y+\overline{\delta_2})$ and

$$
\begin{aligned}
Tr\,\{p\,(\mathbf{A}_1\, q_1(y))\} \;&=\; 4\delta_1^2 y \int\limits_0^\infty (1+cos2\omega t)\,\mathbf{E}_y y(t)\, dt + \\
&+\; \frac{(h_1 y+\overline{h_1})}{\omega^2} \int\limits_0^\infty cos2\omega t\,\mathbf{E}_y\,(h_1 y(t)+\overline{h_1})\, dt + \\
&+\; \frac{2\delta_1 y}{\omega} \int\limits_0^\infty sin2\omega t\,\mathbf{E}_y\,(h_1 y(t)+\overline{h_1})\, dt - \\
&-\; \frac{2(h_1 y+\overline{h_1}\delta_1)}{\omega} \int\limits_0^\infty sin2\omega t\,\mathbf{E}_y\, y(t)\, dt.
\end{aligned} \tag{22}
$$

Let us denote as $r(t)$ the correlation function of the stationary solution $\hat{y}(t)$ of the equation (2). The spectral density corresponding to this correlation function can be obtained by using cosine-transformation as follow

$$
|\phi(\lambda)|^2 := \frac{1}{\pi}\int\limits_0^\infty r(t)\, cos\lambda t\, dt. \tag{23}
$$

Then, using the definition of correlation function, one can write the equality

$$r(t) = 2\int\limits_0^\infty cos\lambda t|\phi(\lambda)|^2 d\lambda.$$

Substitution of (22) and (23) in equation for λ_2 yields the second coefficient of the decomposition of $\lambda(\varepsilon)$ in the form

$$\lambda_2 = -4\overline{\delta_2} + 4\pi\,\delta_1^2\,(|\phi(0)|^2 + |\phi(2\omega)|^2) + \frac{\pi\,h_1^2}{\omega^2}\,|\phi(2\omega)|^2. \tag{24}$$

It is easy to find the spectral density of the stationary solution of equation (2) $|\phi(\lambda)|^2 = (2\pi(\lambda^2+4)^{-1}$ and to substitute it in equality (24). Therefore, *if* $\overline{\delta_1} = 0$ *a sufficient condition for exponential mean square stability of the stochastic oscillator (18) is*

$$\overline{\delta_2} > \frac{1}{8(\omega^2+1)}\left[\delta_1^2(\omega^2+2) + \frac{h_1^2}{4\omega^2}\right]$$

and a sufficient condition for instability is

$$\overline{\delta_2} < \frac{1}{8(\omega^2+1)}\left[\delta_1^2(\omega^2+2) + \frac{h_1^2}{4\omega^2}\right].$$

The third step of the above algorithm should be necessary iff the right part of equality (24) is equal to zero. But, from a practical standpoint the equality

$$\overline{\delta_2} = \frac{1}{8(\omega^2+1)}\left[\delta_1^2(\omega^2+2) + \frac{h_1^2}{4\omega^2}\right]$$

is of no interest since it is extremely unlikely to achieve such perfect equality. Therefore, the proposed algorithm will be completed by the second step.
In the more general case the stationary diffusion Markov process satisfies [9] the stochastic differential equation

$$L_m(\frac{d}{dt})\,y(t) = \frac{dw(t)}{dt}, \tag{25}$$

where $L_m(\lambda)$ is a polynomial of degree m with roots in the halfplane $\{\mathbf{C}:\ \Re\lambda < 0\}$ and $\frac{dw(t)}{dt}$ is "white noise" process or generalized derivative of a standard Wiener process. The coefficients of oscillators also can be functions of the stationary solution $\hat{y}(t)$ of (25) and its derivatives. It may be proven that our proposed algorithm is able to successfully handle this more general case. In this case the algorithm has still the same steps and overlines denotes an averaging according to the corresponding invariant measure $\mu(dy)$. To illustrate how this could be done let us for example consider the coefficients of oscillator (18) in the form $\delta(\hat{y}(t)) = \varepsilon\,\delta_1\,(\hat{y}(t)) + \varepsilon^2\delta_2\,(\hat{y}(t))$, $h(\hat{y}(t)) = \varepsilon\,h_1\,(\hat{y}(t)) + \varepsilon^2\,h_2\,(\hat{y}(t))$, where

$$\delta_j(\hat{y}(t)) = Q_{m-1}^{(j)}(\frac{d}{dt})\,\hat{y}(t),\ \ h_j(\hat{y}(t)) = R_{m-1}^{(j)}(\frac{d}{dt})\,\hat{y}(t),$$

and $Q_{m-1}^{(j)}(\lambda)$, $R_{m-1}^{(j)}(\lambda)$, $j = 1,2$ are polynomials of degrees less a m. As before $\lambda_1 = \overline{\delta_1}$ and if $\overline{\delta_1}$ does not equal zero this algorithm will contain only one step as before. If $\overline{\delta_1} = 0$ one can find the spectral densities of the corresponding auto- and cross-correlation functions. For this purpose the corresponding stationary processes can be represented in the form

$$\delta_j(\hat{y}(t)) = \frac{1}{\sqrt{2}\pi}\int_{-\infty}^{\infty} e^{i\lambda t}\,\frac{Q^{(j)}_{m-1}(-i\lambda)}{L_m(-i\lambda)}\,\nu\,(d\lambda),$$

$$h_j(\hat{y}(t)) = \frac{1}{\sqrt{2}\pi}\int_{-\infty}^{\infty} e^{i\lambda t}\,\frac{R^{(j)}_{m-1}(-i\lambda)}{L_m(-i\lambda)}\,\nu\,(d\lambda), \quad j=1,2,$$

where $\nu\,(d\lambda)$ is Fourier transformation [9] of $dw(t)$. During the second step of the above algorithm when stating the condition of solvability of equation (20) one can find equalities analogous to equality (24)

$$\lambda_2 = -4\overline{\delta}_2 + 4\rho_{11}(0) + 4\rho_{11}(2\omega) + \frac{1}{\omega^2}\,\rho_{22}(2\omega) + \frac{2}{\omega}\,(\rho_{12}(2\omega) - \rho_{21}(2\omega)),$$

where $\rho_{kj}(\lambda)$ are cosine-transformations of auto- and cross-correlations functions of the processes $\delta_1(\hat{y}(t))$ and $h_1(\hat{y}(t))$, that is

$$\rho_{11}(\lambda) = \frac{1}{2}\left|\frac{R^{(1)}_{m-1}(-i\lambda)}{L_m(-i\lambda)}\right|^2, \quad \rho_{22}(\lambda) = \frac{1}{2\omega^2}\left|\frac{Q^{(1)}_{m-1}(-i\lambda)}{L_m(-i\lambda)}\right|^2,$$

$$\rho_{12}(\lambda) - \rho_{21}(\lambda) = \frac{\mathbf{Im}\{Q^{(1)}_{m-1}(-i\lambda)\,R^{(1)}_{m-1}(-i\lambda)\}}{2|L_m(-i\lambda)|^2}.$$

After substitution of these expressions in (26) one can rewrite the coefficient λ_2 near ε^2 in the decomposition of $\lambda(\varepsilon)$

$$\lambda_2 = -4\overline{\delta_2} + 2\left(\left|\frac{R^{(1)}_{m-1}(2i\omega)}{L_m(2i\omega)}\right|^2 + \left|\frac{R^{(1)}_{m-1}(0)}{L_m(0)}\right|^2\right) +$$

$$+\frac{1}{2\omega^2}\left|\frac{Q^{(1)}_{m-1}(2i\omega)}{L_m(2i\omega)}\right|^2 + \frac{\mathbf{Im}\{Q^{(1)}_{m-1}(2i\omega)\,R^{(1)}_{m-1}(-2i\omega)\}}{\omega\,|L_m(i\omega)|^2}. \tag{26}$$

If this quantity does not equal to zero one can obtain the necessary and sufficient condition for exponential mean square stability of oscillator (18) as negativity of the right hand side of equality (26). The Markov process which satisfies equation (25) is often appearing in engineering applications by the name *narrow-band random process or colored noise.*

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