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Asymptotic Methods for Stability Analysis of Markov Impulse Dynamical Systems

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Abstract: The paper deals with n-dimensional dynamical system of impulse type whose dynamical characteristics are dependent on the step Markov process with rapid switchings. The phase motion has small jumps at the moments of switchings and satisfies the ordinary differential equation in the intervals of constancy of the Markov process. The intensity of switchings, the quantities of jumps and the vector field of the differential equation are dependent on the phase coordinates and Markov process. Under some assumptions the limit averaged ordinary differential equation, the limit differential equation switched by the merged Markov process, the diffusion approximation and the limit stochastic differential equation of Ornstein-Uhlenbeck type for normalized deviations are constructed. It is proved that one can use the limit equations for stability analysis of an initial impulse dynamical system.

Keywords: *Stability analysis; impulsive dynamical systems; Markov process.*

Mathematics Subject Classification (2000): 34D20, 34A37, 34F05.

1 Introduction

The problem of asymptotic analysis of dynamical systems under small random perturbations has been discussed in many mathematical and engineering papers. Apparently, R.Z. Khasminsky was the first mathematician to have proved that the probabilistic limit theorems may be successfully used for differential equations with random right parts. The approach proposed in [12] makes it possible to apply for asymptotic analysis of real stochastic structural dynamical systems not only the Krylov-Bogolyubov averaging procedure but also diffusion approximation (see, for example, [6] and review there). It should be mentioned that in spite of the fact that the above result has been developed in [12] for

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the analysis of differential equations on a finite time interval, the diffusion approximation procedure has been applied in many engineering papers for Lyapunov stability analysis, that is, for analysis of differential equations as $t \to \infty$. To prove the validity of this approach the authors of papers [3, 5, 14, 15] had to use not only a special type of limit theorem in Skorokhod space [16] but also martingale techniques and a stochastic version of the Second Lyapunov method developed for stochastic Ito differential equations in [13]. These asymptotic methods of stochastic stability analysis have been applied in the above-mentioned papers to differential equations with continuous trajectories. But some dynamical systems of the recent Economics (see, for example, [1, 4, 9, 10] and review there) require an extension of "smooth" models to allow the phase motion to have a jump type discontinuity. A possible approach to this problem developed in [11, 17−19, 21] is discussed in the present paper.

To formulate the problems one needs first of all to describe the switching step process $\{y(t),\, t \geq 0\}$ with values in the set $\mathbf{Y}$. We suppose for simplicity that $\mathbf{Y}$ is discrete at most countable space but all our results easily can be reformulated for any metric topological space. We will assume that the above switching process is a right continuous homogeneous Markov process [8] with a weak infinitesimal operator defined by the equality

$$Qv(y) := a(y) \sum_{z\in\mathbf{Y}} [v(z) - v(y)]p(y,\, z)$$

for any bounded mapping $v\colon \mathbf{Y} \to \mathbb{R}$, where $p(y,z)$ is the transition probability of the embedded Markov chain and $a(y)$ is the intensity of switchings which satisfies the inequality $0 < \hat{a}_1 \leq a(y) \leq \hat{a}_2 < \infty$ for any $y \in \mathbf{Y}$. It is well known [8, 16] that the above $\{y(t)\}$ is a piecewise constant process with the switching moments $\{\tau_j,\, j \in \mathbb{N}\}$ which have the conditional exponential distributions defined by the equalities: $\tau_0 = 0$,

$$\mathbb{P}\{\tau_j - \tau_{j-1} > t \mid y(\tau_{j-1}) = y\} = \exp\{-a(y)t\}, \qquad j \in \mathbb{N}.$$

Now one can describe the Impulse Dynamical System (IDS) in $\mathbb{R}^n$ with small parameter $\varepsilon \in (0,1)$ this paper deals with. The phase motion $x(t)$ of this system satisfies:

– *the initial condition*

$$x(0) = x; \tag{1}$$

– *the differential equation*

$$\frac{dx}{dt} = \varepsilon\, f(x,\, y(t), \varepsilon) \tag{2}$$

for all $t \in (\tau_{j-1}, \tau_j)$, $j \in \mathbb{N}$;

– *the condition of jump*

$$x(t) = x(t-0) + \varepsilon g(x(t-0),\, y(t-0), \varepsilon) \tag{3}$$

for all $t \in \{\tau_j,\ j \in \mathbb{N}\}$, where

$$f(x,\, y, \varepsilon) = f_1(x,\, y) + \varepsilon\, f_2(x,\, y), \qquad g(x,\, y, \varepsilon) = g_1(x,\, y) + \varepsilon\, g_2(x,\, y) \tag{4}$$

and $f_j(x,y)$, $g_j(x,y)$, $j = 1, 2$ are twice boundedly continuously differentiable on x functions.

Under the above assumptions it is easy to prove [21] that *the pair* $\{x(t),\, y(t)\}$ *is a homogeneous Markov process with the weak infinitesimal operator*

$$(\mathcal{L})v(x,\, y) := \varepsilon(f(x,\, y,\, \varepsilon),\, \bigtriangledown)\, v(x,\, y) + Qv(x,\, y) + \varepsilon G^{\varepsilon} v(x,\, y), \tag{5}$$

where

$$G^{\varepsilon} v(x,\, y) = \frac{a(y)}{\varepsilon} \sum_{z \in \mathbf{Y}} [v(x + \varepsilon g(x,\, y, \varepsilon),\, z) - v(x,\, z)]p(y,\, z), \tag{6}$$

$(.,.)$ is scalar product and $\bigtriangledown$ is an operator-gradient in $\mathbb{R}^n$.

In this paper we will discuss the problem of asymptotic analysis of the IDS (2) – (3) for sufficiently small positive ε. Under the condition of ergodicity of the Markov process $\{y(t)\}$ with limit distribution $\{\mu(y),\, y \in \mathbf{Y}\}$ we shall do this starting with the limit *averaged ordinary differential equation* for (2) – (3)

$$\frac{d\bar{x}}{dt} = \bar{F}_1\,(\bar{x}), \tag{7}$$

where

$$\bar{F}_1(x) := \sum_{y \in \mathbf{Y}} F_1(x,\, y)\mu(y), \qquad F_1(x,\, y) := f_1(x,y) + a(y)g_1(x,y). \tag{8}$$

It will be proven that this deterministic approximation may be successfully used not only on a finite interval but also for asymptotic stability analysis of the initial system. If $F_1(x) \equiv 0$ one will then be able to do the next step in asymptotic analysis of (2) – (3) using the limit theorem in Skorokhod space [16]. This approach leads us in the second section to the limit Ito stochastic differential equation which also can be successfully used for stability analysis of (2) – (3). The third section contains a derivation of a merger procedure and stability theorem based on the merged differential equation.

A word should be said about tools. To prove the limit theorems for (2) – (3) the methods and results of paper [2] can be successfully applied. But in the above paper the author uses specially constructed recurrent equations in the moments of switchings and does not use any infinitesimal characteristics of the Markov process $\{x(t),\, y(t)\}$. This approach is poorly consistent with the Second Lyapunov method, which is mainly used for stability analysis of stochastic dynamical systems [3, 5, 6, 13, 14] and is the main tool of our paper. To prove the classical averaging or merger theorems, unlike the martingale approach of [14], this paper applies the Lyapunov method with specially constructed Lyapunov functions, reflecting the distance between corresponding solutions of the system (2) – (3) and averaged or merged differential equations.

2 Averaging and Stability

Let us assume that the spectrum $\sigma(Q)$ of the weak infinitesimal operator Q has the simple spectrum point 0, $\sigma(Q)\diagdown\{0\} \subset \{z \in \mathbf{C} : \Re z < -\rho < 0\}$ and let the distribution $\{\mu(y)\}$ be the solution of the equation $Q^*\mu = 0$, where Q^* is a conjugate operator. Under these conditions one can extend [8] the potential of the above Markov process and define the linear continuous operator $\Pi\colon \mathbb{B}\,(\mathbf{Y}) \to \mathbb{B}\,(\mathbf{Y})$ by equality

$$(\Pi v)(y) := \int_0^{\infty} \sum_{z \in \mathbf{Y}} v(z)[P(t,y,z) - \mu(z)]\, dt, \tag{9}$$

where $\mathbb{B}(\mathbf{Y})$ is the space of bounded mappings $\{v(y),\, y\in\mathbf{Y}\}$ of $\mathbf{Y}$ to $\mathbb{R}$ and $P(t,y,z)$ is the transition probability.

It is easy to prove that any considerable variations of any solution of (2)−(3) can happen only on a sufficiently large time interval of order ε^{-1}. Therefore it is convenient to pass to the slow time $s=\varepsilon t$ and to analyze the process with rapid switchings defined by the equality $x^\varepsilon(s):=x(s/\varepsilon)$.

Theorem 2.1 (Averaging principle) *Under the above assumptions the processes* $\{x^\varepsilon(s)\}$ *for any* $r>0,\ T>0$ *uniformly on* $y\in\mathbf{Y},\ x\in U_r:=\{|x|\le r\},\ t\in[0,T]$ *converge on probability as* $\varepsilon\to 0$ *to the solution of (7) with initial condition* $\bar{x}(0)=x$, *that is, for any* $\delta>0$

$$\lim_{\varepsilon\to 0}\sup_{y,|x|<r}\mathbb{P}_{x,y}\Big(\sup_{0\le t\le T}|x^\varepsilon(t)-\bar{x}(t,\,x)|>\delta\Big)=0.$$

Proof Under the assumptions of twice continuously boundedly differentiability on x of the functions $f_1(x,y)$ and $g_1(x,y)$, the function $\bar{F}_1(x)$ from (8) also has the continuous bounded derivative $D\,\bar{F}_1(x)$ and therefore the Cauchy problem $\bar{x}(0)=\bar{x}$ for (7) has a unique solution $\bar{x}(s,\bar{x})$ for any $\bar{x}\in\mathbb{R}^n$. It is easy to prove that the joined process $\{x^\varepsilon(s),\, y(s/\varepsilon),\, \bar{x}(s)\}$ one can consider as the Markov process with the weak infinitesimal operator [8]

$$\mathbb{L}(\varepsilon):=(f(x,\,y,\,\varepsilon),\,\bigtriangledown^{(x)})+(\bar{F}_1(\bar{x}),\,\bigtriangledown^{(\bar{x})})+\frac{1}{\varepsilon}\,Q+G^\varepsilon,$$

where the gradients are acting by indicated indices. Let us choose constant c so large that for all $\varepsilon\in(0,1)$ and phase variables $x,\,y,\,\bar{x}$ the function

$$v_\varepsilon(x,\,y,\,\bar{x}):=|x-\bar{x}|^2+\varepsilon[2(x-\bar{x},\,(\Pi F_1)(x,\,y))+c\,(1+|x|^2+|\bar{x}|^2)]$$

satisfies the inequalities

$$|x-\bar{x}|^2+\varepsilon\,(1+|x|^2+|\bar{x}|^2)\le v_\varepsilon(x,\,y,\,\bar{x})\le|x-\bar{x}|^2+\varepsilon\,c_1\,(1+|x|^2+|\bar{x}|^2)$$

with some positive constant c_1. Applying the equality

$$Q\,(\Pi F_1)(x,\,y)=-\,F_1(x,\,y)+\bar{F}_1(x) \tag{10}$$

and well-known Dynkin's formula [8]

$$\mathbb{E}_{x,y}\,v_\varepsilon(x^\varepsilon(t),\,y(t/\varepsilon),\,\bar{x}(t,\bar{x}))=v_\varepsilon(x,\,y,\,\bar{x})$$
$$+\int_0^t\mathbb{E}_{x,y}\,\mathbb{L}(\varepsilon)\,v_\varepsilon(x^\varepsilon(s),\,y(s/\varepsilon),\,\bar{x}(s,\bar{x}))\,ds$$

one can obtain the inequality $\mathbb{L}(\varepsilon)v_\varepsilon(x,\,y,\,\bar{x})\le k\,v_\varepsilon(x,\,y,\,\bar{x})$ which guarantees the stochastic process

$$\zeta(t,x,y,\bar{x}):=v_\varepsilon(x^\varepsilon(t),\,y(t/\varepsilon),\,\bar{x}(t,\bar{x}))\,\exp\{-kt\}$$

be supermartingale [7, 13]. To complete the proof one can use the supermartingale properties and to write the inequalities

$$\mathbb{P}_{x,y}\Big(\sup_{0\le t\le T}|x^\varepsilon(t)-\bar{x}(t,x)|\ge\delta\Big)\le\mathbb{P}\Big(\sup_{0\le t\le T}\zeta(t,\,x,\,y,\,x)\ge\delta^2\,e^{-kT}\Big)$$
$$\le\delta^{-2}\,e^{kT}v_\varepsilon(x,\,y,\,x)\le\varepsilon\,c_1\,\delta^{-2}\,e^{kT}\,(1+2|x|^2)$$

for any $\delta>0,\ T>0$.

Let now $f(0,y,\varepsilon)\equiv g(0,y,\varepsilon)\equiv 0$. Then also $\bar{F}_1(0)=0$ and both systems (2)–(3) and (7) have the trivial solution. We will say that the trivial solution of (7) is exponentially stable if there exist positive constants $M,\ \gamma$ such that $|\bar{x}(t,\bar{x})|\le M|\bar{x}|\,\exp\{-\gamma t\}$ for any $t\ge 0$ and $\bar{x}\in\mathbb{R}^n$. For the IDS (2)–(3) we will use the following two definitions of stability [13]:

1) the trivial solution of (2)–(3) is called *asymptotically stochastic stable* if for any $\eta>0$ there exists a δ-neighborhood $B_\delta:=\{|x|<\delta\}$ such that any motion starting within B_δ remains within an η-neighborhood with probability not less than $1-\eta$ and tends to zero as $t\to\infty$;
2) the trivial solution of (2)–(3) is called *exponentially p-stable*, if there exist positive numbers K and β such that the inequality $\mathbb{E}|x(t)|^p\le K\,|x|^p\times\exp\{-\beta t\}$ is satisfied for all $t\ge 0$ and initial conditions $x\in\mathbb{R}^n,\ y\in\mathbf{Y}$.

Theorem 2.2 *Under the above assumptions if the trivial solution of (7) is exponentially stable then for any $p>0$ there exists $\varepsilon_p>0$ such that the trivial solution of IDS (2)–(3) is exponentially p-stable for any $\varepsilon\in(0,\varepsilon_p)$.*

Proof Owing to exponential decrease of the solutions of (7) and the boundedness of the derivative of $\bar{F}_1(x)$ one can define the Lyapunov function

$$v^{(p)}(x):=\int_0^T|\bar{x}(t,x)|^p\,dt,$$

where $T=\frac{\ln M+\ln p}{\gamma}$ and the constants $M,\ \gamma$ are taken from the above definition of exponential stability. It is easy to verify that this function satisfies the inequalities

$$m_1\,|x|^p\le v^{(p)}(x)\le m_2\,|x|^p \tag{11}$$

with some positive constants $m_1,\,m_2$. By definition of the gradient and due to exponential stability of (7) one can write the inequalities

$$(\bar{F}_1(x),\,\triangledown)\,v^{(p)}(x)=|\bar{x}(T,\,x)|^p-|x|^p\le-\frac{1}{2}\,|x|^p\le-\frac{1}{2m_2}\,v^{(p)}(x) \tag{12}$$

for any $x\in\mathbb{R}^n$. To prove the theorem we will use the Lyapunov function

$$v_\varepsilon^{(p)}(x,y):=v^{(p)}(x)+\varepsilon((\Pi F_1)(x,y),\,\triangledown)\,v^{(p)}(x).$$

By definition (9) and due to equality

$$Q((\Pi F_1)(x,\,y),\,\triangledown)\,v^{(p)}(x)+(F_1(x,y),\,\triangledown)\,v^{(p)}(x)=(\bar{F}_1(x),\,\triangledown)\,v^{(p)}(x)$$

and inequalities (11) – (12) one can choose such a constant $\varepsilon_p > 0$ that the above Lyapunov function satisfies the inequalities

$$\hat{m}_1 \, |x|^p \le v_\varepsilon^{(p)}(x,y) \le \hat{m}_2 \, |x|^p, \qquad \mathbb{L}(\varepsilon) \, v_\varepsilon^{(p)}(x,y) \le -\frac{1}{4m_2} \, v_\varepsilon^{(p)}(x,y)$$

with some positive constants $\hat{m}_1$, $\hat{m}_2$ for any $\varepsilon \in (0, \varepsilon_p)$. Using Dynkin's formula for the stochastic process

$$\xi(s) := v_\varepsilon^{(p)}(x^\varepsilon(s), \, y(s/\varepsilon)) \, e^{\frac{1}{4m_2} \, s}$$

one can get the inequalities

$$\hat{m}_1 e^{\frac{1}{4m_2} \, s} \, \mathbb{E}_{x,y} \, |x^\varepsilon(s)|^p \le v_\varepsilon^{(p)}(x, \, y) \le \hat{m}_2 \, |x|^p,$$

for any $s \ge 0$ and the proof is complete.

By using the supermartingale property of the above defined stochastic process $\xi(s)$ one can make sure that *under the conditions of the Theorem 2.2 the trivial solution of the IDS (2) – (3) is asymptotically stochastic stable for all sufficiently small positive* ε*.*

3 Diffusion Approximation and Stability

In this Section we will assume that in addition to the condition of ergodicity of the Markov process and twice bounded differentiability of the functions (4) on x the average function satisfies the condition $\bar{F}_1(x) \equiv 0$. Thus, any solution of the averaged equation (7) is constant and we have no information on the behavior of the solutions of the IDS (2) – (3). Then we can go to the "very slow" time $\theta = \varepsilon s = \varepsilon^2 t$, where t is the initial time of the IDS (2) – (3). Let us denote $x_\varepsilon(\theta) := x(\theta/\varepsilon^2)$. The infinitesimal operator of the Markov process $\{x_\varepsilon(\theta), \, y(\theta/\varepsilon^2)\}$ has the form

$$\mathcal{L}_\varepsilon := \frac{1}{\varepsilon}(f(x, \, y, \, \varepsilon), \, \bigtriangledown) + \frac{1}{\varepsilon^2} \, Q + \frac{1}{\varepsilon} \, G^\varepsilon. \tag{13}$$

In spite of the fact that the operator (13) has a singular type as $\varepsilon \to 0$ under the above condition one can prove the following assertions.

Lemma 3.1 [21] *For any positive* p *there exist positive constants* c_p, γ_p, ε_p *such that*

$$\mathbb{E}_{x,y} \, |x_\varepsilon(\theta)|^p \le c_p \, (1 + |x|)^p \, e^{\gamma_p \theta}$$

for all $x \in \mathbb{R}^n$, $y \in \mathbf{Y}$, $\varepsilon \in (0, \varepsilon_p)$, $\theta > 0$.

Corollary 3.1 [21] *For any* $T > 0$, $r > 0$ *there exists* $\varepsilon_T > 0$ *such that*

$$\lim_{\rho \to \infty} \sup_{0 \le \varepsilon \le \varepsilon_T} \mathbb{P}_{x,y} \Big(\sup_{0 \le \theta \le T} |x_\varepsilon(\theta)| \ge \rho \Big) = 0$$

uniformly on $y \in \mathbf{Y}$ *and* $x \in U_r$.

The family of the stochastic processes $\{\{x_\varepsilon(\theta), \, 0 \le \theta \le T\}, \, \varepsilon \in (0, \, \varepsilon_0)\}$ with initial condition $x_\varepsilon(0) = x$ we will consider as the family of random variables in Skorokhod space [16] $D([0,\,T],\,\mathbb{R}^n)$. The probability measures corresponding to these random variables we will denote $\mathbb{P}^\varepsilon$. Owing to Corollary 3.1 we may confirm that for any natural m and any moments of time $\theta_m > \theta_{m-1} > \dots > \theta_1 \geq 0$ the distribution family of the random vectors $\{x_\varepsilon(\theta_1),\, x_\varepsilon(\theta_2),\, \dots,\, x_\varepsilon(\theta_m)\}$ is weak compact. That is, the family $\{\mathbb{P}^\varepsilon\}$ is relatively compact (as $\varepsilon \to 0$) in the meaning of the weak convergence of finite-dimensional distributions. We will prove that there exist the weak limit of the family $\{\mathbb{P}^\varepsilon\}$ as $\varepsilon \to 0$. To describe the limit process let us introduce the vector

$$b(x) := \sum_{y\in\mathbf{Y}} [f_2(x,\,y) + a(y)\,g_2(x,\,y)]\mu(y) + \sum_{y\in\mathbf{Y}} [\Pi\,DF_1(x,\,y)]\,F_1(x,\,y)\,\mu(y) - \sum_{y\in\mathbf{Y}} [DF_1(x,\,y)]\,g_1(x,\,y)\,\mu(y)$$

and the positive symmetrical matrix $\sigma(x)$ defined by the equality

$$(\sigma(x)\,z,\,z) = 2\sum_{y\in\mathbf{Y}} \big[(F_1(x,\,y),\,z)\,(\Pi\,F_1(x,\,y),\,z) - (g_1(x,\,y),\,z)\,\big(f_1(x,\,y) + \frac{1}{2}\,a(y)\,g_1(x,\,y),\,z\big)\big]\,\mu(y)$$

with an arbitrary vector $z \in \mathbb{R}^n$.

Theorem 3.1 (Diffusion approximation) *Under the above assumptions the family $\{\mathbb{P}^\varepsilon\}$ weak converges as $\varepsilon \to 0$ to the diffusion Markov process with weak infinitesimal operator*

$$\mathbb{L}_0 := (b(x),\,\triangledown) + \frac{1}{2}\,(\sigma(x)\,\triangledown,\,\triangledown). \tag{14}$$

Proof To prove this theorem it is sufficient to verify [16] that for any twice continuously differentiable function $v(x)$ with bounded support the equality

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \sup_{0<h<\varepsilon} \left| \mathbb{E}_{x,y} \left\{ v(x_\varepsilon(s+h)) - v(x_\varepsilon(s)) - \int_s^{s+h} \mathbb{L}_0\, v(x_\varepsilon(\tau))\,d\tau \right\} \right| = 0$$

can be written for all $0 \leq s < T$, $x \in \mathbb{R}^n$, $y \in \mathbf{Y}$. This equality one can get using Dynkin's formula for $\mathbb{E}_{x,y}\, w(x_\varepsilon(s),\, y(s/\varepsilon^2),\, \varepsilon)$, where

$$w(x,y,\varepsilon) = v(x) + \varepsilon\,(\Pi\,F_1(x,y),\,\triangledown)\,v(x) + \varepsilon^2\,u(x,y),$$

and $u(x,\,y)$ is a solution of the equation

$$Q\,u(x,\,y) = -\Big\{ (f_2(x,\,y) + a(y)\,g_2(x,\,y) + [\Pi\,D\,F_1(x,\,y)]\,F_1(x,\,y) - [D\,F_1(x,\,y)]\,g_1(x,\,y) - b(x),\,\triangledown)\,v(x) + ([D\,\triangledown\,v(x)]\,F_1(x,\,y),\,\Pi\,F_1(x,\,y)) - \Big([D\,\triangledown\,v(x)]\,g_1(x,\,y),\,f_1(x,\,y) + \frac{1}{2}\,a(y)\,g_1(x,\,y)\Big) - \frac{1}{2}\,(\sigma(x)\,\triangledown,\,\triangledown)\,v(x) \Big\}.$$

Owing to the Fredholm alternative [11] and by construction the vector $b(x)$ and the matrix $\sigma(x)$ the above equation has solution and the proof is complete.

There exists [8] a Markov process $X(t)$ with infinitesimal operator (14) which satisfies the stochastic Ito differential equation

$$dX = b(X)\,dt + \sum_{k=1}^{n} \sigma_k(X)\,d\,w_k(t), \tag{15}$$

where $w_k(t),\ k = 1, 2, \dots, n$ are the coordinates of the standard Wiener process in $\mathbb{R}^n$ and the matrices $\sigma_k(X),\ k = 1, 2, \dots, n$ are defined such that the process $X(t)$ has the infinitesimal operator (14). Equation (15) is called [2] *the diffusion approximation of the process* $\{x_\varepsilon(t)\}$.

If $\bar{F}_1(x)$ is not identically equal to zero one can use the diffusion approximation for *the normalized deviations* $\xi_\varepsilon(t) := [x(t/\varepsilon) - \bar{x}(t)]/\sqrt{\varepsilon}$ as $\varepsilon \to 0$ applying Theorem 3.1 to the $2n$ dimensional process $\{\xi_\varepsilon(t),\, \bar{x}(t)\}$ with small parameter $\sqrt{\varepsilon}$.

Theorem 3.2 [19, 21] *Under the assumptions of this Section the probability measures* $\{\hat{\mathbb{P}}^\varepsilon\}$ *corresponding to the normalized deviations* $\{\xi_\varepsilon(t),\, 0 \le t \le T\}$ *weak converge as* $\varepsilon \to 0$ *to the measure* $\hat{\mathbb{P}}$ *corresponding to the solution* $\{\hat{X}(t),\, 0 \le t \le T\}$ *of the stochastic Ito equation*

$$d\hat{X} = D\bar{F}_1(\bar{x}(t))\hat{X}\,dt + \sum_{k=1}^{n} \sigma_k(\bar{x}(t))\,dw_k(t) \tag{16}$$

with initial condition $\hat{X}(0) = 0$*, where* $\bar{x}(t)$ *is the solution of (7) with the initial condition* $\bar{x}(0) = x$*.*

The diffusion approximation (15) in just the same way as for the Markov dynamical systems without jumps [3, 5] can be successfully used for stability analysis of the IDS (2) – (3).

Theorem 3.3 *Under the assumptions of this Section if the trivial solution of (15) is exponentially* p*-stable then the trivial solution of the IDS (2) – (3) is also exponentially* p*-stable for all sufficiently small* ε*.*

Proof It is shown in [13] that trivial solution of equation (12) is exponentially p-stable if and only if there exists such a sufficiently smooth Lyapunov function $V(x)$ that

$$h_1|x|^p \le V(x) \le h_2|x|^p, \quad \mathbb{L}_0 V(x) \le -h_3|x|^p, \quad \|D^l \bigtriangledown V(x)\| \le h_4|x|^{p-l-1}$$

for any $x \in R^n,\ l = 1, 2, 3$ and some positive constants $h_j,\ j = 1, 2, 3, 4$. To prove the theorem we will use the Lyapunov function

$$V_\varepsilon(x, y) = V(x) + \varepsilon(\Pi F_1(x, y),\, \bigtriangledown)V(x) + \varepsilon^2\, U_2(x, y)$$

where $U_2(x, y)$ satisfies the equation

$$Q\,U_2(x, y) = -\Big\{(F_1(x, y),\, \bigtriangledown)\,(\Pi F_1(x, y),\, \bigtriangledown)V(x) + (f_2(x, y) + a(y)g_2(x, y),\, \bigtriangledown)V(x) + \frac{1}{2}\,(g_1(x, y),\, \bigtriangledown)V(x) + \mathbb{L}_0\, V(x)\Big\}.$$

One can apply the infinitesimal operator $\mathcal{L}_\varepsilon$ to the function $V_\varepsilon(x,y)$ and to obtain the equality $\mathcal{L}_\varepsilon V_\varepsilon(x,y) = \mathbb{L}_0 V(x) + r(x,y,\varepsilon)$, where the last term satisfies the inequality $|r(x,y,\varepsilon)| \le \alpha(\varepsilon)|x|^p$ with some infinitesimal $\alpha(\varepsilon)$ as $\varepsilon \to 0$. It is easy to verify that there exist the positive constants ε_0, r_1, r_2, r_3 such that the function $V_\varepsilon(x,y)$ satisfies the inequalities

$$r_1\,|x|^p \le V_\varepsilon(x,y) \le r_2\,|x|^p, \qquad \mathcal{L}_\varepsilon V_\varepsilon(x,y) \le -r_3\,|x|^p \le -\frac{r_3}{r_2}\,V_\varepsilon(x,y)$$

for all $x \in \mathbb{R}^n$, $y \in \mathbf{Y}$, $\varepsilon \in (0,\varepsilon_0)$. To complete the proof one can use the same calculations as in the end of the proof of Theorem 2.2.

4 Merger and Stability

To illustrate the asymptotic merger method of stability analysis proposed in [14] we will suppose that the infinitesimal operator of the step Markov process has the form $Q_\varepsilon = Q_0 + \varepsilon Q_1$, where

$$Q_j\,v(y) := \sum_{y\in\mathbf{Y}} [v(z) - v(y)]\,p_j(y,\,z), \quad j = 0,1$$

and $p_j(y,z)$ as functions of $z \in \mathbf{Y}$ are positive uniformly bounded on $y \in \mathbf{Y}$ discrete measures. Let $\{y_\varepsilon(t),\, t \ge 0\}$ be the Markov process corresponding to this infinitesimal operator. It is easy to see that this process is a step Markov process [8]. We will assume that the operator Q_0 has 0 as an isolated simple eigenvalue of multiplicity h, the eigenfunctions of this operator are defined by equalities

$$q_j\,(y) = \begin{cases} 1, & \text{for} \quad y \in \mathbf{Y}_j \\ 0, & \text{for} \quad y \in \mathbf{Y}_k,\ k \ne j \end{cases}$$

with nonintersecting supports $\mathbf{Y}_j$, $j = \overline{1,h}$ and the remaining part of the spectrum is situated in the half-plane $\{\lambda \in \mathbf{C}\colon \Re\lambda < -\rho\}$ for some positive ρ. The conjugate operator Q_0^* also [8, 11] has 0 as an isolated eigenvalue of multiplicity h and h invariant probabilistic measures $\mu_k(y)$ with the same supports $\mathbf{Y}_k$, $k = \overline{1,h}$.

In this section we will deal with stochastic process $\{x_\varepsilon(t),\, t \ge 0\}$ which satisfies the differential equation

$$\frac{dx_\varepsilon}{dt} = \varepsilon f(x_\varepsilon,\, y_\varepsilon(t),\, \varepsilon), \tag{17}$$

for all $t \in (\tau^\varepsilon_{j-1},\, \tau^\varepsilon_j)$, $j \in \mathbb{N}$, and the conditions of jump

$$x_\varepsilon(t) = x_\varepsilon(t-) + \varepsilon g(x_\varepsilon(t-),\, y_\varepsilon(t-),\, \varepsilon), \tag{18}$$

for all $t \in \{\tau^\varepsilon_j,\ j \in \mathbb{N}\}$, where $\{\tau^\varepsilon_j,\ j \in \mathbb{N}\}$ are switching time moments of the process $\{y_\varepsilon(t), t \ge 0\}$ and functions $f(x,y,\varepsilon)$, $g(x,y,\varepsilon)$ were defined in Section 2. The system (17) – (18) we will consider in slow time $s = \varepsilon t$ denoting $\tilde{x}_\varepsilon(s) = x_\varepsilon(s/\varepsilon)$.

To define the limit merged Markov process for the family $\{\tilde{x}_\varepsilon(s)\}$ as $\varepsilon \to 0$ [14] one needs the merged state space $\hat{\mathbf{Y}} := \{\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_h\}$ which for simplicity we will denote $\hat{\mathbf{Y}} := \{1, 2, \ldots, h\}$ and the infinitesimal matrix $\Gamma = \{\gamma_j^k\}$, where

$$\gamma_j^k := \begin{cases} \sum\limits_{y\in\mathbf{Y}_k} p_1(y, \mathbf{Y}_j)\, \mu_k(y), & \text{if} \quad j \neq k, \\ -\sum\limits_{\substack{l=1 \\ l\neq k}}^{h} \gamma_k^l, & \text{if} \quad j = k, \end{cases} \tag{19}$$

$k,\, j = \overline{1,\, h}$. Corresponding to this infinitesimal matrix process $\{\hat{y}(t),\ t \geq 0\}$ is called *a merged Markov process.* To use the merger method of [14] first of all one has to define the function

$$\tilde{F}_1(x,\, y) \equiv \sum_{z\in\mathbf{Y}_k} (f_1(x,\, z) + g_1(x,\, z) p_0(z, \mathbf{Y}))\mu_k(z), \quad y \in \mathbf{Y}_k$$

for each $k = \overline{1, h}$ and differential equation

$$\frac{d\tilde{x}_\varepsilon}{ds} = \tilde{F}_1(\tilde{x}_\varepsilon(s),\, y_\varepsilon(s/\varepsilon)). \tag{20}$$

Substituting the above defined merged step Markov process $\{\hat{y}(s),\ s \geq 0\}$ instead of the initial Markov process $\{y_\varepsilon(s/\varepsilon),\ s \geq 0\}$ in (20) we will obtain *the limit merged differential equation for the system* (17) – (18)

$$\frac{d\hat{x}}{ds} = \hat{F}_1(\hat{x}(s), \hat{y}(s)), \tag{21}$$

where $\hat{F}_1(x, k) := \tilde{F}_1(x, y)$ for any $k = \overline{1, h}$ and $y \in \mathbf{Y}_k$.

Owing to assumption on spectrum structure of the operator Q_0 one can define [11] the projective operator $\mathcal{P}$ by the equalities

$$\forall\, y \in \mathbf{Y}_k,\ \ v \in \mathbb{B}(\mathbf{Y}):\ (\mathcal{P}v)(y) \equiv \sum_{z\in\mathbf{Y}_k} v(z)\mu_k(z)$$

for each $k = \overline{1, h}$ and the linear continuous operator $\hat{\Pi}\colon \mathbb{B}\,(\mathbf{Y}) \to \mathbb{B}\,(\mathbf{Y})$ by equality

$$(\hat{\Pi}v)(y) := \int_0^\infty \sum_{z\in\mathbf{Y}} P_0(t, y, z)(v - \mathcal{P}v)(z)\, dt, \tag{22}$$

where $P_0(t, y, z)$ is the transition probability corresponding to infinitesimal operator Q_0. The operator (22) we will use in the same way as the potential Π in Section 2.

Theorem 4.1 (Merger principle) *Under the above assumptions the family of processes* $\{x^\varepsilon(s)\}$ *weak converges as* $\varepsilon \to 0$ *to the solution of (21) with corresponding initial condition.*

Proof It is easy to prove that the processes $\{x_\varepsilon(s),\, y_\varepsilon(s/\varepsilon),\, \tilde{x}_\varepsilon(s)\}$ one can consider jointly as the Markov process with the weak infinitesimal operator [8]

$$\tilde{\mathbb{L}}(\varepsilon) := (f(x, y, \varepsilon),\, \nabla^{(x)}) + (\tilde{F}_1(\tilde{x}, y),\, \nabla^{(\tilde{x})}) + \frac{1}{\varepsilon}\,(Q_0 + \varepsilon Q_1) + \tilde{G}^\varepsilon,$$

where the operator $\tilde{G}^\varepsilon$ is defined by the equality

$$\tilde{G}^\varepsilon v(x,y) = \frac{1}{\varepsilon}\sum_{y\in\mathbf{Y}}[v(x+\varepsilon g(x,y,\varepsilon),\, z) - v(x,z)](p_0(y,z)+\varepsilon p_1(y,z))$$

and the gradients are acting by indicated indices. As in Section 2 we will use the function

$$v_\varepsilon(x,y,\tilde{x}) := |x-\tilde{x}|^2 + \varepsilon[2(x-\tilde{x},\, (\tilde{\Pi}F_1)(x,y)) + c\,(1+|x|^2+|\tilde{x}|^2)],$$

which for sufficiently large c satisfies the inequalities

$$|x-\tilde{x}|^2 + \varepsilon\,(1+|x|^2+|\tilde{x}|^2) \le v_\varepsilon(x,y,\tilde{x}) \le |x-\tilde{x}|^2 + \varepsilon c_1\,(1+|x|^2+|\tilde{x}|^2)$$

with some positive constant c_1 for all $\varepsilon \in (0,1)$ and $x\in\mathbb{R}^n,\ y\in\mathbf{Y},\ \tilde{x}\in\mathbb{R}^n$. Applying the equality

$$Q_0\,(\tilde{\Pi}F_1)(x,y) = -F_1(x,y) + \tilde{F}_1(x,y)$$

one can obtain the inequality $\tilde{\mathbb{L}}(\varepsilon)v_\varepsilon(x,y,\tilde{x}) \le kv_\varepsilon(x,y,\tilde{x})$ with some positive constant k. Hence, for any $x\in\mathbb{R}^n,\ y\in\mathbf{Y},\ \tilde{x}\in\mathbb{R}^n$ the stochastic process $v_\varepsilon(x_\varepsilon(t),\, y_\varepsilon(t/\varepsilon),\, \tilde{x}_\varepsilon(t))$ $\times \exp\{-kt\}$ is a positive supermartingale [7]. Therefore under the initial conditions $x_\varepsilon(0)=x,\ \tilde{x}_\varepsilon(0)=x$ one can write the inequality

$$\mathbb{P}_{x,y}\Big(\sup_{0\le t\le T}|x_\varepsilon(t)-\tilde{x}_\varepsilon(t)|\ge\delta\Big) \le \varepsilon c_1\,\delta^{-2}\,e^{k_2T}\,(1+2|x|^2) \tag{23}$$

for any $\delta>0,\ T>0$.

Under the assumptions of twice continuously boundedly differentiability on x of the functions $f_1(x,y)$ and $g_1(x,y)$, the function $\tilde{F}_1(x,y)$ also has two continuous bounded derivatives and therefore one can use the merger method and results of paper [14]. According to the above paper for any $T>0$ and $\tilde{x}\in\mathbb{R}^n$ the solution $\{\tilde{x}_\varepsilon(t),\, t\in[0,\,T]\}$ of Cauchy problem $\tilde{x}_\varepsilon(0)=x$ for (20) defines on Skorokhod space $D([0,T],\mathbb{R}^n)$ the family of probability measures $\{\mathbb{P}_\varepsilon,\, \varepsilon\in(0,1)\}$ which weak converges as $\varepsilon\to0$ to the probability measure corresponding to the solution of the Cauchy problem $\hat{x}(0)=x$ for (21). This assertion and inequality (23) complete the proof.

Let now $f(0,y,\varepsilon)\equiv g(0,y,\varepsilon)\equiv 0$. Then also $\tilde{F}_1(0,y)\equiv 0$ and both systems (17)–(18) and (21) have the trivial solution.

Theorem 4.2 *Under the above assumptions if the trivial solution of (21) is exponentially p-stable for all sufficiently small ε and some $p>0$ then there exists $\varepsilon_p>0$ such that the trivial solution of IDS (17)–(18) is exponentially p-stable for any $\varepsilon\in(0,\varepsilon_p)$.*

Proof Owing to exponential decrease of the p-moments of the solutions of (21) and a boundedness of the x-derivative of $\tilde{F}_1(x,y)$ one can define function

$$y\in\mathbb{Y}_k\colon v^{(p)}(x,y)\equiv\hat{v}^{(p)}(x,k) := \int_0^T \mathbb{E}_{x,k}|\hat{x}(t)|^p\,dt,\quad k=\overline{1,h},$$

with so large a constant T that the above function satisfies the inequalities $m_1|x|^p \le v^{(p)}(x,y) \le m_2|x|^p$ with some positive constants $m_1,\ m_2$. Owing to exponential p-stability of (21), the inequality $(\hat{F}_1(x,k),\,\bigtriangledown)\hat{v}^{(p)}(x,k) + \Gamma\hat{v}^{(p)}(x,k) \le -m_3\hat{v}^{(p)}(x,k)$ is held with some positive constant m_3 for any $k = \overline{1,h}$ and $x \in \mathbb{R}^n$. To prove the theorem we will use the Lyapunov function

$$v_\varepsilon^{(p)}(x,y) := v^{(p)}(x,y) + \varepsilon\tilde{\Pi}\{(F_1(x,y),\, \bigtriangledown)\, v^{(p)}(x,y) + Q_1 v^{(p)}(x,y)\},$$

which satisfies the inequalities $\hat{m}_1|x|^p \le v_\varepsilon^{(p)}(x,y) \le \hat{m}_2|x|^p$ with some positive constants $\hat{m}_1,\, \hat{m}_2$ for any $\varepsilon \in (0,1)$. By definition of the operator $\tilde{\Pi}$ one can write the equality

$$\begin{aligned}
&(F_1(x,y),\, \bigtriangledown)\, v^{(p)}(x,y) + Q_1 v^{(p)}(x,y) + Q_0 v_1^{(p)}(x,y)\\
&= (\tilde{F}_1(x,y),\, \bigtriangledown)\, v^{(p)}(x,y) + \mathcal{P}Q_1 v^{(p)}(x,y) + \varepsilon r(x,y,\varepsilon)\\
&= (\hat{F}_1(x,k),\, \bigtriangledown)\, \hat{v}^{(p)}(x,k) + \Gamma\hat{v}^{(p)}(x,k) + \varepsilon r(x,y,\varepsilon)\\
&\le -m_3\hat{v}^{(p)}(x,k) + \varepsilon\alpha(\varepsilon)|x|^p
\end{aligned}$$

and therefore

$$\begin{aligned}
&(F_1(x,y),\, \bigtriangledown)\, v^{(p)}(x,y) + Q_1 v^{(p)}(x,y) + Q_0 v_1^{(p)}(x,y)\\
&\le -m_3\hat{v}^{(p)}(x,k) + \varepsilon\alpha(\varepsilon)|x|^p
\end{aligned}$$

for any $y \in \mathbf{Y}_k$ and $k = \overline{1,h}$, where $\alpha(\varepsilon)$ is infinitesimal as $\varepsilon \to 0$. Owing to the above inequalities there exist such positive constants ε_p that for any $\varepsilon \in (0,\varepsilon_p)$

$$\mathbb{L}(\varepsilon)v_\varepsilon^{(p)}(x,y) \le -\frac{m_3}{2}\, v_\varepsilon^{(p)}(x,y).$$

Now we can use Dynkin's formula for $\mathbb{E}_{x,y}\Big\{v_\varepsilon^{(p)}(x_\varepsilon(s),\, y_\varepsilon(s/\varepsilon))\exp(sm_3/2)\Big\}$ and complete the proof as it has been done in the proof of Theorem 2.2.

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