Numerical Solution of Resonantly Forced ODE with Applications to Weakly Nonlinear Instability of Shallow Water Flows

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Abstract: - Numerical method for the solution of linear boundary value problems for resonantly forced ODE is presented in the present paper. The solution is obtained by the singular value decomposition method. The method is applied to numerical calculation of the coefficients of the complex Ginzburg-Landau equation which describes the development of instability of shallow water flow behind obstacles such as islands. The results of the present study can be applied in practice to control water quality on the sheltered side of islands.

Key-Words: - singular value decomposition, linear stability analysis, weakly nonlinear theory, shallow water flows

1 Introduction

Shallow water flows are flows where the transverse length scale of the flow is much larger than the water depth. Understanding the major mechanisms responsible for vortex formation is quite important in applications. In particular, weak water circulation in wake flows (i.e., flows behind obstacles such as islands or headlands) can lead to poor water quality. Experiments [1] – [3] confirmed that complex flows behind islands can trap sediments and pollutants. Decisions on where to locate outfall discharges, mud disposals and cooling intakes can be made on the basis of the knowledge of the structure of shallow water flows. Poor water quality induced by eddies behind obstacles such as islands can lead to fish deceases and mortality [4].

Methods of linear stability theory are widely used to analyze the structure of shallow flows [4] – [7]. The linear stability theory can be used to obtain critical values of the parameters separating regions of stability and instability in the parameter space. In order to "trace" the development of the most unstable mode above the threshold methods of weakly nonlinear theory can be used [4], [8] – [10]. Direct application of the method of multiple scales to equations of motion usually results in relatively simple amplitude evolution equations such as complex Ginzburg-Landau equation (CGLE). Examples include plane Poiseuille flow [8], generation of waves by wind [10], shallow water flows [4] and flows in the nearshore [9]. An excellent survey of the properties of CGLE can be found, for example, in [11] and [12]. In some cases (see, for example, [13] and [14]) the CGLE is used as a phenomenological model equation where the coefficients of the CGLE are determined from experimental data. In the examples mentioned above (see [4], [8] – [10]) the CGLE is derived from the equations of motion. In this case the coefficients of the CGLE are calculated from the characteristics of the corresponding linear stability problem. One of the steps of the procedure includes numerical solution of linear ODE which is resonantly forced.

In the present paper numerical method for the solution of resonantly forced ODE is presented. The method is applied to the calculation of the coefficients of the CGLE for shallow wake flows.

2 Description of the method

Consider a linear boundary value problem of the form

$$Lf(x,\lambda) = g, \tag{1}$$

$$f(a)=0, \quad f(b)=0 \tag{2}$$

where the solution f depends on the parameter λ. We assume that the corresponding homogeneous problem

$$Lf(x,\lambda_0)=0 \tag{3}$$

$$f(a)=0, \quad f(b)=0 \tag{4}$$

has a nontrivial solution if $\lambda=\lambda_0$. In accordance with the Fredholm's alternative, problem (1), (2) has a solution if and only if certain solvability conditions are satisfied. Our aim is to construct a numerical solution of (1), (2) for the case $\lambda=\lambda_0$.

Suppose that after discretization problem (1), (2) can be written as a system of linear equations of the form

$$Ax=b, \tag{5}$$

where the matrix A is singular (since the corresponding homogeneous equation $Ax=0$ has a nontrivial solution).

Numerical solution of (5) is sought by means of the singular value decomposition method [15].

The idea of the method is briefly described below. A complex-valued matrix $Q\in C^{n\times n}$ is said to be unitary if

$$Q^H Q = QQ^H = E, \tag{6}$$

where E is the $n\times n$ identity matrix and Q^H is the conjugate transpose of Q (also called the Hermitian adjoint of Q).

If $A\in C^{n\times n}$ is a complex-valued matrix then there exist unitary matrices $U\in C^{n\times n}$ and $V\in C^{n\times n}$ such that

$$U^H AV=\Sigma, \tag{7}$$

where $\Sigma = diag(\sigma_1,\sigma_2,...,\sigma_n)$ is the diagonal matrix whose diagonal entries are the singular values of the matrix A. Equation (7) is called the singular value decomposition of A.

Using orthogonality condition (6) we transform (7) to the form

$$A=U\Sigma V^H. \tag{8}$$

Assume that $rank(A)=r=n-1$. In this case $\sigma_1\ge\sigma_2\ge...\ge\sigma_{n-1}>\sigma_n=0$. Thus, we delete the last column of U and the last row of V^H from the analysis. Using Z as the new U, W^H for the new V^H and T as the new Σ, equation (5) becomes

$$ZTW^H x=b. \tag{9}$$

The matrix T in (9) is invertible so that the solution x can be written in the form

$$x=\sum_{i=1}^{n-1}\frac{u_i^H b v_i}{\sigma_i}, \tag{10}$$

where u_i and v_i, $i=1,2,...,n-1$ are the columns of the matrices U and V, respectively. The singular value decomposition is computed by means of the IMSL routine LSVCR, which provides Σ as well as the matrices U and V

3 Application of the method to weakly nonlinear instability of shallow water flows

The development of instability of a particular flow above the threshold (where the flow is linearly unstable [16]) in some cases can be described by amplitude evolution equations for the most unstable linear mode. Such equations sometimes are derived from the equations of motion [4], [8]-[10], [17], [18]. In order to calculate the coefficients of the amplitude evolution equations numerically one needs to use the ideas described in the previous section.

The procedure includes several steps. These steps are briefly summarized below.

Consider the shallow water equations of the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{11}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{c_f}{2h}u\sqrt{u^2+v^2}=0, \tag{12}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}+\frac{c_f}{2h}v\sqrt{u^2+v^2}=0. \tag{13}$$

The following notations are used in (11) – (13): u and v are the velocity components in the x and y directions, respectively, h is water depth, c_f is the friction coefficient, and p is the pressure. Introducing the stream function $\psi(x,y,t)$ by the relations

$$u=\frac{\partial\psi}{\partial y},\ v=-\frac{\partial\psi}{\partial x} \tag{14}$$

we transform the system (11) – (13) to the single partial differential equation

$$(\Delta\psi)_t+\psi_y(\Delta\psi)_x-\psi_x(\Delta\psi)_y+\frac{c_f}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2} + \frac{c_f}{2h\sqrt{\psi_x^2+\psi_y^2}}\left(\psi_y^2\psi_{yy}+2\psi_x\psi_y\psi_{xy}+\psi_x^2\psi_{xx}\right)=0 \tag{15}$$

where $\Delta=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$.

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The base flow is defined as follows: $\psi_{0y}(y) = u_0(y)$. A perturbed solution to (15) is sought in the form

$$\psi = \psi_0(y) + \varepsilon\psi_1(x,y,t) + \varepsilon^2\psi_2(x,y,t) + ... \quad (16)$$

Substituting (16) into (15) we obtain the first-order approximation (in ε) in the form

$$L\psi_1 = 0, \quad (17)$$

where the linear operator L is

$$L\psi_1 = \psi_{1xxt} + \psi_{1yyt} + \psi_{0y}(\psi_{1xxx} + \psi_{1yyx}) - \psi_{0yyy}\psi_{1x}$$

$$+\frac{c_f}{2h}\left[(\psi_{1xx} + 2\psi_{1yy})\psi_{0y} + 2\psi_{1y}\psi_{0yy}\right]$$

Equation (17) can be used to study linear stability of the flow $u = u_0(y)$. The following one-parameter family of wake flow profiles is adopted in the present study:

$$u_0(y) = 1 + \frac{2R}{1-R}\frac{1}{\cosh^2(\alpha y)}, \quad (18)$$

where $R = (U_c - U_a)/(U_c + U_a)$ is the velocity ratio, $\alpha = \sinh^{-1}(1)$, U_c is the velocity on the centerline and U_a is the ambient velocity.

The method of normal modes is used to solve the linear stability problem. The perturbation $\psi_1(x,y,t)$ is assumed to be of the form

$$\psi_1(x,y,t) = \varphi_1(y)\exp[ik(x-ct)], \quad (19)$$

where $\varphi_1(y)$ is the amplitude of the normal perturbation (19), k is the wavenumber and c is the phase speed of the perturbation. Substituting (19) into (17) we obtain the modified Rayleigh equation for the function $\varphi_1(y)$ in the form

$$\varphi_1''(-ikc + iku_0 + 2Su_0) + Su_{0y}\varphi_1'$$
$$+\varphi_1\left(ik^3c - ik^3u_0 - iku_{0yy} - Sk^2u_0\right) = 0 \quad (20)$$

with the boundary conditions

$$\varphi_1(\pm\infty) = 0 \quad (21)$$

Equation (20) contains the stability parameter $S = \frac{c_f b}{2h}$, where b is the wake half-width. The eigenvalues, $c = c_r + ic_i$, of the problem (20) – (21) determine the linear stability of the flow (18). The flow (18) is said to be linearly stable if all c_i are negative and unstable if at least one $c_i > 0$.

Numerical solution of (20) – (21) is obtained by means of a collocation method based on Chebyshev polynomials. Using the substitution $r = \frac{2}{\pi}\arctan y$ we transform the interval $-\infty < y < +\infty$ onto the interval $-1 < r < 1$. Next, it is assumed that the function $\varphi_1(r)$ can be represented by the following expansion

$$\varphi_1(r) = \sum_{k=0}^{N} a_k(1-r^2)T_k(r), \quad (22)$$

where $T_k(r)$ is the Chebyshev polynomial of degree k. The collocation points r_j are

$$r_j = \cos\frac{\pi j}{N}, \; j = 0,1,...,N. \quad (23)$$

Using (20) – (23) we obtain a generalized eigenvalue problem of the form

$$(A - \lambda B)a = 0 \quad (24)$$

where A and B are complex-valued matrices and $a = (a_1 a_2 ... a_N)^T$.

Problem (24) can be solved numerically by means of IMSL routine DGVCCG.

Nonlinear development of the most unstable mode can be analyzed by means of weakly nonlinear theory. In this case the application of the method of multiple scales allows one to derive an amplitude evolution equation. It is shown in [4], [17], [18] that for different shallow flows the evolution equation is the complex Ginzburg-Landau equation (CGLE). Explicit formulas for the calculation of the coefficients of the CGLE are derived in [4], [17], [18]. For the sake of completeness, the basic steps of the derivation are reproduced below.

First, we assume that the stability parameter is slightly smaller than the critical value S_c, that is,

$$S = S_c(1-\varepsilon^2). \quad (25)$$

Then, following [8], the slow time and longitudinal variables are introduced:

$$\tau = \varepsilon^2 t, \quad \xi = \varepsilon(x - c_g t), \quad (26)$$

where c_g is the group velocity.

Thus, the differential operators $\partial/\partial t$ and $\partial/\partial x$ are then replaced by

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \varepsilon c_g\frac{\partial}{\partial \xi} + \varepsilon^2\frac{\partial}{\partial \tau},$$

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial \xi}.$$

Introducing a slowly varying amplitude $A(\xi,\tau)$, we represent the unknown function ψ_1 in (17) in the form

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$$\psi_1(x,y,t) = A(\xi,\tau)\varphi_1(y)\exp[ik(x-ct)] + c.c. \tag{27}$$

Here $\varphi_1(y)$ is the eigenfunction of the linear stability problem (20)-(21) where the values of k and c correspond to the critical state, and c.c. means the complex conjugate.

The following equations are derived in [4] for the functions ψ_2 and ψ_3:

$$\begin{aligned} L\psi_2 &= c_g(\psi_{1xx\xi} + \psi_{1yy\xi}) - 2\psi_{1x\xi t} - \psi_{0y}(3\psi_{1xx\xi} + \psi_{1yy\xi}) \\ &-\psi_{1y}(\psi_{1xxx} + \psi_{1yyx}) + \psi_{1x}(\psi_{1xxy} + \psi_{1yyy}) + \psi_{1\xi}\psi_{0yyy} \\ &-S[(\psi_{1xx} + \psi_{1yy})\psi_{1y} + 2\psi_{1x\xi}\psi_{0y} + \psi_{1yy}\psi_{1y} - 2\psi_{0y}\psi_{0yy} \\ &+2\psi_{1x}\psi_{1xy}] \end{aligned} \tag{28}$$

$$\begin{aligned} L\psi_3 &= c_g(\psi_{2xx\xi} + 2\psi_{1x\xi\xi} + \psi_{2yy\xi}) - \psi_{1xx\tau} - \psi_{1yy\tau} - 2\psi_{2x\xi t} \\ &-\psi_{1\xi\xi t} - 3\psi_{0y}(\psi_{2xx\xi} + \psi_{1x\xi\xi}) - \psi_{1y}(\psi_{2xxx} + 3\psi_{1xx\xi}) \\ &-\psi_{2y}(\psi_{1xxx} + \psi_{1yyx}) - \psi_{1y}(\psi_{2yyx} - \psi_{1\xi yy}) - \psi_{0y}\psi_{2\xi yy} \\ &+\psi_{2x}\psi_{1xxy} + \psi_{1\xi}\psi_{1xxy} + \psi_{1x}\psi_{2xxy} + 2\psi_{1x}\psi_{1xy\xi} + \psi_{1x}\psi_{2yyy} \\ &+\psi_{2x}\psi_{1yyy} + \psi_{1\xi}\psi_{1yyy} + \psi_{2\xi}\psi_{0yyy} \\ &-S[\psi_{2y}(\psi_{1xx} + \psi_{1yy}) + 2\psi_{2yy}\psi_{1y} + 1.5\psi_{1xx}\psi_{1x}^2/\psi_{0y} \\ &+\psi_{2xx}\psi_{1y} + 2\psi_{1x\xi}\psi_{1y} + 2\psi_{0y}\psi_{2x\xi} + \psi_{1\xi\xi}\psi_{0y} - \psi_{1xx}\psi_{0y} \\ &-2\psi_{0yy}\psi_{1y} - 2\psi_{0y}\psi_{1yy} + \psi_{1yy}\psi_{2y} - \psi_{1y}\psi_{2yy} + 2\psi_{1x}\psi_{2xy} \\ &+2\psi_{1x}\psi_{1yy} + 2\psi_{2x}\psi_{1xy} + 2\psi_{1\xi}\psi_{1xy}] \end{aligned} \tag{29}$$

The function ψ_2 is represented in the form

$$\begin{aligned} \psi_2 &= AA^*\varphi_2^{(0)}(y) + A_\xi\varphi_2^{(1)}(y)\exp[ik(x-ct)] \\ &+A^2\varphi_2^{(2)}(y)\exp[2ik(x-ct)] + c.c. \end{aligned} \tag{30}$$

where A^* is the complex conjugate of A and the functions $\varphi_2^{(0)}(y)$, $\varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$ are determined as the solutions of the following boundary value problems (for details see [4], [17], [18]):

$$\begin{aligned} 2S[u_{0y}\varphi_{2y}^{(0)} + u_0\varphi_{2yy}^{(0)}] &= ik[\varphi_{1y}\varphi_{1yy}^* - \varphi_{1y}^*\varphi_{1yy} \\ &+\varphi_1\varphi_{1yyy}^* - \varphi_1^*\varphi_{1yyy}] - S[k^2\varphi_1\varphi_{1y}^* + k^2\varphi_1^*\varphi_{1y} \\ &+2\varphi_{1y}^*\varphi_{1yy} + 2\varphi_{1yy}^*\varphi_{1y}], \end{aligned} \tag{31}$$

$$\varphi_2^{(0)}(\pm\infty) = 0, \tag{32}$$

$$\begin{aligned} &(iku_0 - ikc)\varphi_{2yy}^{(1)} + (ik^3c - ik^3u_0 - iku_{0yy})\varphi_2^{(1)} \\ &+S[2u_0\varphi_{2yy}^{(1)} + 2u_{0y}\varphi_{2y}^{(1)} - k^2u_0\varphi_2^{(1)}] \\ &= (c_g - u_0)\varphi_{1yy} + [-2k^2c + 3k^2u_0 + u_{0yy} - k^2c_g \\ &-iku_0S]\varphi_1, \end{aligned} \tag{33}$$

$$\varphi_2^{(1)}(\pm\infty) = 0, \tag{34}$$

$$\begin{aligned} &2(iku_0 - ikc)\varphi_{2yy}^{(2)} + (8ik^3c - 8ik^3u_0 - 2iku_{0yy})\varphi_2^{(2)} \\ &+ S[2u_0\varphi_{2yy}^{(2)} + 2u_{0y}\varphi_{2y}^{(2)} - 4k^2u_0\varphi_2^{(2)} \\ &= ik(\varphi_1\varphi_{1yyy} - \varphi_{1y}\varphi_{1yy}) - S(2\varphi_{1y}\varphi_{1yy} - 3k^2\varphi_1\varphi_{1y}), \end{aligned} \tag{35}$$

$$\varphi_2^{(2)}(\pm\infty) = 0. \tag{36}$$

There is no problem to calculate the functions $\varphi_2^{(0)}(y)$ and $\varphi_2^{(2)}(y)$ using the collocation method based on expansion (22) for $\varphi_2^{(0)}(y)$ and $\varphi_2^{(2)}(y)$ since in both cases the corresponding linear system after discretization has the form (5) with a nonsingular matrix A. However, the function $\varphi_2^{(1)}$ is resonantly forced since the corresponding homogeneous problem has a nontrivial solution at $S = S_c$, $k = k_c$ and $c = c_c$. As a result, problem (33) – (34) has a solution if and only if the right-hand side of (33) is orthogonal to all the eigenfunctions of the corresponding adjoint problem. The adjoint operator, L^a, and the adjoint eigenfunction, φ_1^a, are defined as follows:

$$\int_{-\infty}^{+\infty} \varphi_1^a L(\varphi_1)dy = \int_{-\infty}^{+\infty} \varphi_1 L^a(\varphi_1^a)dy = 0 \tag{37}$$

The adjoint eigenfunction φ_1^a is the solution of the problem

$$\begin{aligned} &(iku_0 + 2Su_0)(\varphi_1^a)'' + (2iku_{0y} + 2Su_{0y})(\varphi_1^a)' \\ &-(ik^3u_0 + u_0k^2S)\varphi_1^a + ikc[(\varphi_1^a)'' - k^2\varphi_1^a] = 0 \end{aligned} \tag{38}$$

$$\varphi_1^a(\pm\infty) = 0 \tag{39}$$

The solvability condition for equation (28) can be written in the form

$$c_g = \frac{I_1}{I_2}, \tag{40}$$

where c_g is the group velocity and

$$I_1 = \int_{-\infty}^{+\infty} [u_0\varphi_{1yy} - \varphi_1(3k^2u_0 + u_{0yy}$$
$$-2k^2c - 2iku_0S)]\varphi_1^a dy,$$
$$I_2 = \int_{-\infty}^{+\infty} \varphi_1^a(\varphi_{1yy} - k^2\varphi_1)dy.$$

Applying the solvability condition to (29) we obtain the amplitude evolution equation for A in the form of the complex Ginzburg-Landau equation

$$\frac{\partial A}{\partial \tau} = \sigma A + \delta \frac{\partial^2 A}{\partial \xi^2} - \mu |A|^2 A \quad (41)$$

where

$$\sigma = \frac{\sigma_1}{\gamma_1},\ \delta = \frac{\delta_1}{\gamma_1},\ \mu = \frac{\mu_1}{\gamma_1}. \quad (42)$$

The coefficients $\sigma = \sigma_r + i\sigma_i, \delta = \delta_r + i\delta_i, \mu = \mu_r + i\mu_i$ are defined by the formulas:

$$\gamma_1 = \int_{-\infty}^{+\infty} \varphi_1^a(\varphi_{1yy} - k^2\varphi_1)dy \quad (43)$$

$$\sigma_1 = S\int_{-\infty}^{+\infty} \varphi_1^a(2u_0\varphi_{1yy} + 2u_{0y}\varphi_{1y} - k^2u_0\varphi_1)dy \quad (44)$$

$$\delta_1 = \int_{-\infty}^{+\infty} \varphi_1^a[\varphi_{2yy}^{(1)}(c_g - u_0) + \varphi_2^{(1)}(-k^2c_g - 2k^2c$$
$$+3k^2u_0 + u_{0yy} - 2iku_0S) + \varphi_1(2ikc_g + ikc \quad (45)$$
$$-3iku_0 - US)]dy$$

$$\mu_1 = \int_{-\infty}^{+\infty} \varphi_1^a\{6ik^3\varphi_2^{(2)}\varphi_{1y}^* - 2ik\varphi_{1y}^*\varphi_{2yy}^{(2)} + 3ik^3\varphi_1^*\varphi_{2y}^{(2)}$$
$$+ik^3\varphi_1(\varphi_{2y}^{(0)} + \varphi^{*(0)}_{2y}) - ik\varphi_{1yy}(\varphi_{2y}^{(0)} + \varphi^{*(0)}_{2y})$$
$$+ik\varphi_{2y}^{(2)}\varphi_{1yy}^* - ik\varphi_1^*\varphi_{2yyy}^{(2)} + ik\varphi_1(\varphi_{2yyy}^{(0)} + \varphi_{2yyy}^{*(0)})$$
$$+2ik\varphi_{1yyy}^*\varphi_2^{(2)} - 2S[-k^2\varphi_1(\varphi_{2y}^{(0)} + \varphi^{*(0)}_{2y})$$
$$+3k^2\varphi_1^*\varphi_{2y}^{(2)} - 1.5k^4\varphi_1^2\varphi_1^*/u_0 + 2\varphi_{1yy}(\varphi_{2y}^{(0)} + \varphi^{*(0)}_{2y})$$
$$+2\varphi_{1yy}^*\varphi_{2y}^{(2)} + 2\varphi_{1y}(\varphi_{2yy}^{(0)} + \varphi^{*(0)}_{2yy}) + 2\varphi_{2yy}^{(2)}\varphi_{1y}^*]\}dy$$

(46)

4 Numerical results

In this section the coefficients of (41) are evaluated numerically. This is done in a few steps. First, the critical values of k, S and c from the linear stability problem (20)-(21) are calculated. The base flow is assumed to be of the form (18). The results of linear stability calculations for two different values of R are presented in Table 1.

R	k_c	S_c	c_c
– 0.2	0.875	0.079	0.740
– 0.9	0.998	0.351	0.499

Table 1. The critical values of linear stability problem (20) – (21) for different values of R.

Second, the eigenfunction φ_1^a of the adjoint problem (38), (39) is computed. Third, boundary value problems (31) – (32) and (35) – (36) are solved numerically. Finally, problem (33) – (34) is solved by the method described in Section 2 (the problem is resonantly forced, therefore, the singular value decomposition method is used to solve the problem numerically). As the last step of the procedure, we evaluate the group velocity c_g and the coefficients of the Ginzburg-Landau equation (41). The results for two values of R are presented in Table 2.

R	σ	δ	μ
– 0.2	0.046 + 0.004i	0.027 – 0.213i	9.945 + 17.268i
– 0.9	0.122 – 0.018i	0.178 – 0.142i	8.065 + 12.427i

Table 2. The coefficients of the Ginzburg-Landau equation (41) for different values of R.

The real part of μ (known as the Landau constant in the hydrodynamic stability literature) is positive for both values of R, therefore, the instability is supercritical and finite amplitude equilibrium is possible. Since the Landau constant has the "right" sign, the Ginzburg-Landau equation may be used for the analysis of shallow wake flows above the threshold.

Using the substitutions (see [11])

$$\tilde{\tau} = \tau\sigma_r,\ \tilde{\xi} = \xi\sqrt{\frac{\sigma_r}{\delta_r}},\ \tilde{A} = A\sqrt{\frac{\mu_r}{\sigma_r}}\exp(-ic_0\sigma_r\tau),$$

one can transform (41) to the form

$$\tilde{A}_{\tilde{\tau}} = \tilde{A} + (1 + ic_1)\tilde{A}_{\tilde{\xi}\tilde{\xi}} - (1 + ic_2)|\tilde{A}|^2\tilde{A} \quad (47)$$

where

$$c_0 = \frac{\sigma_i}{\sigma_r},\ c_1 = \frac{\delta_i}{\delta_r},\ c_2 = \frac{\mu_i}{\mu_r}. \quad (48)$$

Equation (47) has a plane wave solution of the form (see [11])

$$\tilde{A} = C\exp[i(K\tilde{\xi} - \Omega\tilde{\tau}] \quad (49)$$

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Stability of plane periodic solutions (49) is analyzed in [19] where it is shown that a sufficient condition for instability is

$$1 + c_1 c_2 < 0 \qquad (50)$$

Numerical values of the coefficients c_1 and c_2 are presented in Table 3.

R	c_1	c_2
– 0.2	– 0.798	1.541
– 0.9	– 7.889	1.736

Table 3. The coefficients of the Ginzburg-Landau equation (47) for different values of R.

It follows from Table 3 that for both values of R the instability condition (50) is satisfied. Hence, pure periodic waves (49) are unstable. This theoretical result is confirmed by experimental data in [1] where it is shown that pure periodic wake structures are not observable in the convectively unstable regime ("unsteady bubble").

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