Linear and Weakly Nonlinear Instability of a Particle-Laden Shallow Wake Flows

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Abstract: - The linear and weakly nonlinear analysis of a particle-laden shallow wake flows is investigated in the present paper under the simplifying assumptions that the mean velocity profile of the particle-laden flow is identical to that of a single-phase flow. The particle concentration is assumed to be uniform. The coupling between the phases is simplified by assuming that the particles do not respond to changes in fluid motion. The linear stability analysis shows that the presence of particles enhances the stability of the flow. In addition, bottom friction also acts a stabilizing mechanism. Weakly nonlinear analysis of the flow in a convectively unstable regime shows that the evolution of the most unstable mode is governed by the complex Ginzburg-Landau equation. The coefficients of the equation are calculated numerically.

Key-Words: - linear stability analysis, weakly nonlinear theory, Ginzburg-Landau equation

1 Introduction

Water flow is considered to be shallow if the transverse length scale of the flow is much larger than the water depth. Linear stability of shallow water flows is investigated in [1]-[4]. Shallow wake flows (that is, flows behind obstacles such as islands) are of a particular interest since these flows often occur in nature and engineering. It is shown experimentally in [5] that three different flow regimes can be identified in shallow wake flows: steady bubble, unsteady bubble and vortex street. Theoretical analyses in [2]-[4] showed that these regimes are related to convective/absolute instabilities is shallow wakes.

Two-phase shear flows (gas-solid particle, liquid-gas bubble, gas-droplet) can be found in many engineering applications (see [6]-[7]). The stability of the two-phase flows is studied in [6]-[7] under several simplifying assumptions. Two major assumptions are as follows. First, small perturbations imposed on the flow have no effect on the particles during the initial moment. Second, the particle distribution is assumed to be uniform. Linear stability calculations presented in [6]-[7] showed that an increase in the particle concentration has a stabilizing effect on the flow. Later the case of differential particle loading is considered in [8]. Dynamic interaction of particles with the base flow is analyzed in [9].

The linear stability analyses of single-phase and two-phase flows can be used to predict when a particular flow becomes unstable. One can calculate the marginal stability curve and the critical values of the parameters at the threshold. However, the linear stability theory cannot describe the evolution of the most unstable mode above the threshold. Weakly nonlinear theories are used [10]-[12], [3] in order to study further development of instability.

Relatively simple amplitude evolution equations are often used in fluid mechanics to describe spatio-temporal dynamics of complex flows. One of the popular dynamical models is the complex Ginzburg-Landau equation (GLE) [13], [14]. The Ginzburg-Landau model is widely used in applications such as the onset of wave-pattern forming instabilities. The complex GLE model is relatively simple and allows one to include physical effects such as diffusion and nonlinearity. In addition it is relatively simple task to integrate the equation numerically which makes it an effective tool to study spatio-temporal characteristics of complex flows. The complex GLE can exhibit a rich variety of solutions depending on the values of its coefficients [15].

Note that in many cases the GLE (or the Landau equation) is used as a phenomenological model equation. In the present paper we derive the GLE from the Navier-Stokes equations for two-phase flows under some simplifying assumptions. Linear stability of shallow wake two-phase flows is investigated numerically. The coefficients of the complex GLE are calculated analytically (in terms of integrals depending on linear stability characteristics) and are evaluated numerically for different parameters of the model. It is shown that both bottom friction and particle concentration has a stabilizing influence on the flow in both linear and weakly nonlinear regime.

2 Linear stability analysis

Consider the two-dimensional shallow water equations of the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{c_f}{2h}u\sqrt{u^2+v^2}=B(u^p-u), \quad (2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}+\frac{c_f}{2h}v\sqrt{u^2+v^2}=B(v^p-v), \quad (3)$$

where u and v are the depth-averaged velocity components in the x and y directions, respectively, h is water depth, c_f is the friction coefficient, p is the pressure and B is the particle loading parameter (see [6], [7]). Introducing the stream function $\psi(x,y,t)$ by the relations

$$u=\frac{\partial\psi}{\partial y},\ v=-\frac{\partial\psi}{\partial x}$$

(4)

we rewrite system (1)-(3) in the form

$$\left(\Delta\psi\right)_t+\psi_y\left(\Delta\psi\right)_x-\psi_x\left(\Delta\psi\right)_y+\frac{c_f}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2}$$
$$+\frac{c_f}{2h\sqrt{\psi_x^2+\psi_y^2}}\left(\psi_y^2\psi_{yy}+2\psi_x\psi_y\psi_{xy}+\psi_x^2\psi_{xx}\right)+B\Delta\psi=0 \quad (5)$$

where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

Consider a perturbed solution to (5) of the form

$$\psi=\psi_0(y)+\varepsilon\psi_1(x,y,t)+\varepsilon^2\psi_2(x,y,t)+\ldots \quad (6)$$

Substituting (6) into (5) and neglecting the terms of order ε^2 we obtain

$$L\psi_1=0, \quad (7)$$

where

$$L\psi_1=\psi_{1xxt}+\psi_{1yyt}+\psi_{0y}(\psi_{1xxx}+\psi_{1yyx})-\psi_{0yyy}\psi_{1x}$$
$$+\frac{c_f}{2h}\left[(\psi_{1xx}+2\psi_{1yy})\psi_{0y}+2\psi_{1y}\psi_{0yy}\right]+B(\psi_{1xx}+\psi_{1yy})$$

Here $\psi_{0y}=u_0(y)$ is the base flow solution. The function $u_0(y)$ in the present study is chosen in the form

$$u_0(y)=1+\frac{2R}{1-R}\frac{1}{\cosh^2(\alpha y)}, \quad (8)$$

where $R=(U_c-U_a)/(U_c+U_a)$ is the velocity ratio, $\alpha=\sinh^{-1}(1)$, U_c is the velocity on the centerline and U_a is the ambient velocity. The profile (8) is suggested by Monkewitz [16] after careful analysis of experimental data for single-phase wake flows and is adopted in the present study.

We assume that the function ψ_1 in (7) has the form

$$\psi_1(x,y,t)=\varphi_1(y)\exp[ik(x-ct)] \quad (9)$$

where $\varphi_1(y)$ is the amplitude of the normal perturbation (9). Substituting (9) into (7) we obtain the modified Raykeigh equation for the function $\varphi_1(y)$ in the form

$$\varphi_1''(-ikc+iku_0+2Su_0+B)+Su_{0y}\varphi_1'$$
$$+\varphi_1\left(ik^3c-ik^3u_0-iku_{0yy}-Sk^2u_0-Bk^2\right)=0 \quad (10)$$

with the boundary conditions

$$\varphi_1(\pm\infty)=0 \quad (11)$$

Here $S=\frac{c_f b}{2h}$ is the bed friction number and b is the wake half-width. Note that (10)-(11) is an eigenvalue problem where complex eigenvalues are $c=c_r+ic_i$. The base flow (8) is stable if all c_i are negative and unstable if at least one $c_i>0$.

The linear stability problem (10) – (11) is solved by means of a pseudospectral collocation method based on Chebyshev polynomials. First, we transform the interval $-\infty<y<+\infty$ onto the interval $-1<r<1$ by means of the substitution $r=\frac{2}{\pi}\arctan y$. Second, the solution to (10) – (11) is sought in the form

$$\varphi_1(r)=\sum_{k=0}^{N}a_k(1-r^2)T_k(r), \quad (12)$$

where $T_k(r)$ is the Chebyshev polynomial of degree k. The form of (12) guarantees that the boundary conditions (11) in terms of the variable r are satisfied automatically. The collocation points r_j are

$$r_j = \cos\frac{\pi j}{N}, \quad j = 0,1,...,N. \tag{13}$$

Substituting (12) into (10)-(11) and evaluating the derivatives of the function $\varphi_1(r)$ at the collocation points (13) we obtain a generalized eigenvalue problem of the form

$$(A - \lambda B)a = 0 \tag{14}$$

where A and B are complex-valued matrices and

$a = (a_1 a_2 ... a_N)^T$,

the superscript T denotes transpose.

Note that solution of the form (12) is more convenient than one obtained by traditional collocation methods [17] for two reasons: first, the matrix B in (14) is not singular, and second, the fact that the function (12) satisfies the boundary conditions automatically reduces the condition number of the matrices in this method [18].

Problem (37) is solved numerically by means of IMSL routine DGVCCG.

Figure 1 plots neutral stability curves for the parameter S versus k for $R = -0.5$ and different values of the particle loading parameter B. As can be seen from the figure the increase in B leads to more stable flow (the region of instability is below the curves).

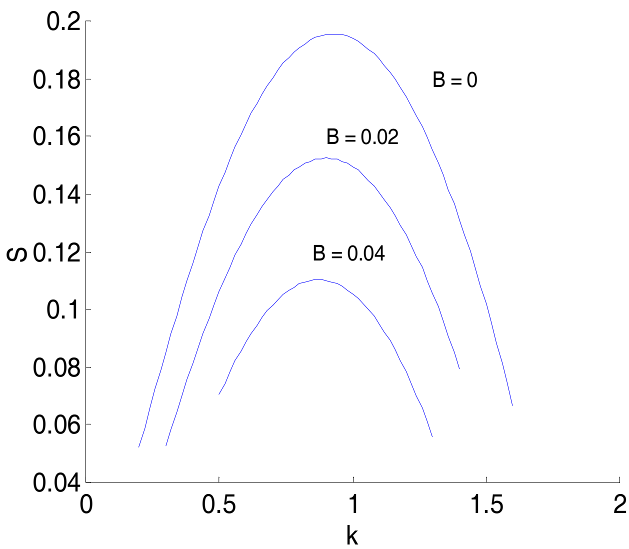

Fig.1. Neutral stability curves for different values of B at $R = -0.5$.

The critical values of the bed friction number S^{cr} for different values of B at $R = -0.5$ are shown in Fig.2. Stabilizing effect of the particle loading parameter B is clearly seen in the figure since the critical values of S are decreasing almost linearly as the parameter B increases.

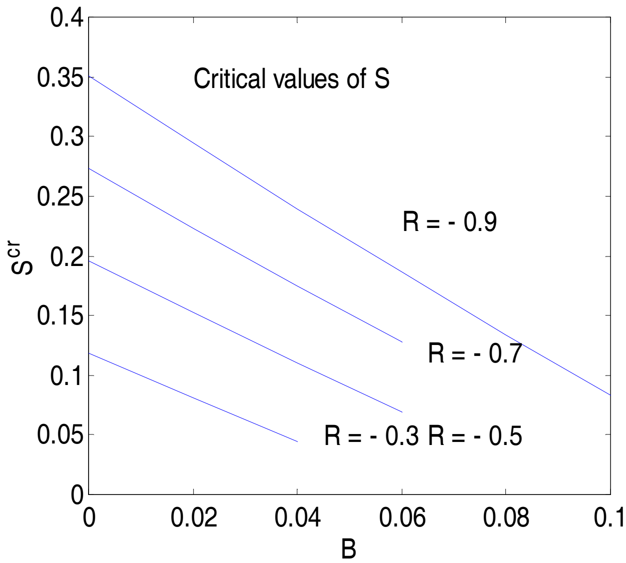

Fig. 2. Critical values of S versus B for different values of R.

Figure 3 plots the growth rates for the most unstable mode in unstable regime for different values of B. As the particle loading parameter increases, the growth rates decrease.

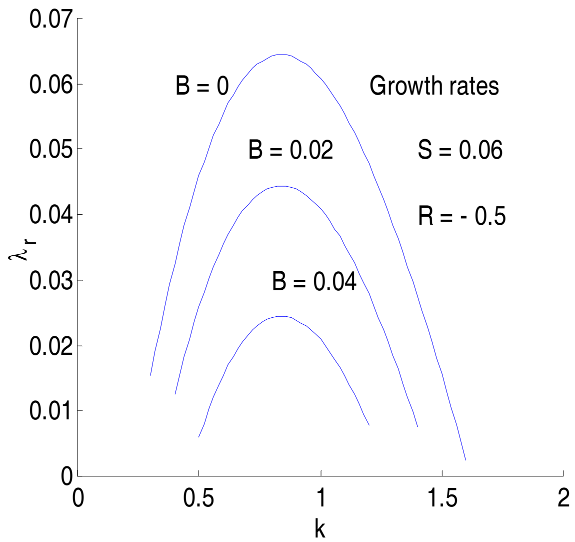

Fig. 3. Growth rates for the most unstable mode for different values of B.

3 Derivation of the Ginzburg-Landau equation

The effect of nonlinearity on the structure of the most unstable mode can be investigated if the perturbation expansion for the stream function is assumed to have the form (6). In this case one can use the methods of weakly nonlinear theory (see, for example, [3]) in order to derive the amplitude evolution equation for the case where S is slightly smaller than the critical value S_c. Following [3] we assume that

$$S = S_c(1-\varepsilon^2) \qquad (15)$$

and introduce the slow time τ and the stretched longitudinal variable ξ such that

$$\tau = \varepsilon^2 t, \qquad \xi = \varepsilon(x - c_g t), \qquad (16)$$

where c_g is the group velocity.

Weakly nonlinear theory is therefore applied in the vicinity of the critical point (see Fig. 4):

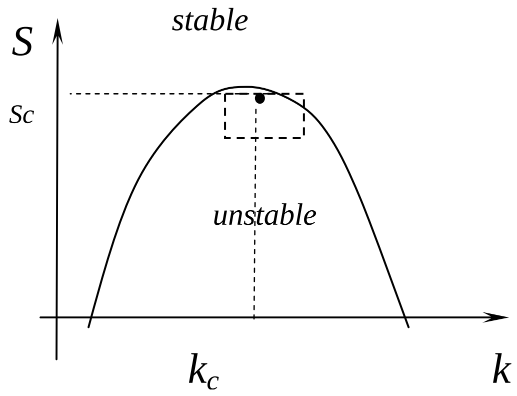

Fig. 4 Schematic diagram of the critical values of the bed friction number versus k. The dashed rectangle shows the region where weakly nonlinear theory is applicable.

The differential operators $\partial/\partial t$ and $\partial/\partial x$ are then replaced by

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \varepsilon c_g \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \xi}. \qquad (17)$$

The function ψ_1 in (6) is represented in the form

$$\psi_1(x,y,t) = A(\xi,\tau)\varphi_1(y)\exp[ik(x-ct)] + c.c. \qquad (18)$$

where A is a slowly varying amplitude, $\varphi_1(y)$ is the eigenfunction of the linear stability problem (10)-(11), the values of k and c correspond to the critical state, and c.c. means the complex conjugate.

In order to find an amplitude evolution equation for A we need to consider higher terms of the perturbation expansion (6). Substituting (6) into (5) and collecting the terms of order ε^2 we obtain

$$\begin{aligned} L\psi_2 &= c_g(\psi_{1xx\xi} + \psi_{1yy\xi}) - 2\psi_{1x\xi t} - \psi_{0y}(3\psi_{1xx\xi} + \psi_{1yy\xi}) \\ &-\psi_{1y}(\psi_{1xxx} + \psi_{1yyx}) + \psi_{1x}(\psi_{1xxy} + \psi_{1yyy}) + \psi_{1\xi}\psi_{0yyy} \\ &-S[(\psi_{1xx} + \psi_{1yy})\psi_{1y} + 2\psi_{1x\xi}\psi_{0y} + \psi_{1yy}\psi_{1y} - 2\psi_{0y}\psi_{0yy} \\ &+2\psi_{1x}\psi_{1xy}] - 2B\psi_{1x\xi} \end{aligned} \qquad (19)$$

Substituting (6) into (5) and collecting terms of order ε^3 we obtain

$$\begin{aligned} L\psi_3 &= c_g(\psi_{2xx\xi} + 2\psi_{1x\xi\xi} + \psi_{2yy\xi}) - \psi_{1xx\tau} - \psi_{1yy\tau} - 2\psi_{2x\xi t} \\ &-\psi_{1\xi\xi t} - 3\psi_{0y}(\psi_{2xx\xi} + \psi_{1x\xi\xi}) - \psi_{1y}(\psi_{2xxx} + 3\psi_{1xx\xi}) \\ &-\psi_{2y}(\psi_{1xxx} + \psi_{1yyx}) - \psi_{1y}(\psi_{2yyx} - \psi_{1\xi yy}) - \psi_{0y}\psi_{2\xi yy} \\ &+\psi_{2x}\psi_{1xxy} + \psi_{1\xi}\psi_{1xxy} + \psi_{1x}\psi_{2xxy} + 2\psi_{1x}\psi_{1xy\xi} + \psi_{1x}\psi_{2yyy} \\ &+\psi_{2x}\psi_{1yyy} + \psi_{1\xi}\psi_{1yyy} + \psi_{2\xi}\psi_{0yyy} \\ &-S[\psi_{2y}(\psi_{1xx} + \psi_{1yy}) + 2\psi_{2yy}\psi_{1y} + 1.5\psi_{1xx}\psi_{1x}^2/\psi_{0y} \\ &+\psi_{2xx}\psi_{1y} + 2\psi_{1x\xi}\psi_{1y} + 2\psi_{0y}\psi_{2x\xi} + \psi_{1\xi\xi}\psi_{0y} - \psi_{1xx}\psi_{0y} \\ &-2\psi_{0yy}\psi_{1y} - 2\psi_{0y}\psi_{1yy} + \psi_{1yy}\psi_{2y} - \psi_{1y}\psi_{2yy} + 2\psi_{1x}\psi_{2xy} \\ &+2\psi_{1x}\psi_{1yy} + 2\psi_{2x}\psi_{1xy} + 2\psi_{1\xi}\psi_{1xy}] - B(2\psi_{2x\xi} + \psi_{1\xi\xi}) \end{aligned} \qquad (20)$$

The form of the right-hand side of (19) and formula (18) suggest that the function ψ_2 should be sought in the form

$$\begin{aligned} \psi_2 &= AA^*\varphi_2^{(0)}(y) + A_\xi\varphi_2^{(1)}(y)\exp[ik(x-ct)] \\ &+A^2\varphi_2^{(2)}(y)\exp[2ik(x-ct)] + c.c. \end{aligned} \qquad (21)$$

where A^* denotes the complex conjugate of A and the functions $\varphi_2^{(0)}(y)$, $\varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$ have to be determined.

Substituting (18) for ψ_1 and (21) for ψ_2 into (19) and collecting terms proportional to AA^* gives

$$\begin{aligned} &2S[u_{0y}\varphi_{2y}^{(0)} + u_0\varphi_{2yy}^{(0)}] + 2B\varphi_{2yy}^{(0)} = ik[\varphi_{1y}\varphi_{1yy}^* - \varphi_{1y}^*\varphi_{1yy} \\ &+\varphi_1\varphi_{1yyy}^* - \varphi_1^*\varphi_{1yyy}] - S[k^2\varphi_1\varphi_{1y}^* + k^2\varphi_1^*\varphi_{1y} \\ &+2\varphi_{1y}^*\varphi_{1yy} + 2\varphi_{1yy}^*\varphi_{1y}] \end{aligned} \qquad (22)$$

with the boundary conditions

$$\varphi_2^{(0)}(\pm\infty) = 0. \quad (23)$$

Similarly, collecting the terms that are proportional to $A_\xi \exp[ik(x-ct)]$ we obtain

$$(iku_0 - ikc)\varphi_{2yy}^{(1)} + (ik^3c - ik^3u_0 - iku_{0yy})\varphi_2^{(1)}$$
$$+S[2u_0\varphi_{2yy}^{(1)} + 2u_{0y}\varphi_{2y}^{(1)} - k^2u_0\varphi_2^{(1)}] + B[\varphi_{2yy}^{(1)} - k^2u_0\varphi_2^{(1)}]$$
$$= (c_g - u_0)\varphi_{1yy} + [-2k^2c + 3k^2u_0 + u_{0yy} - k^2c_g$$
$$-iku_0S - 2ikB]\varphi_1 \quad (24)$$

The boundary conditions are

$$\varphi_2^{(1)}(\pm\infty) = 0. \quad (25)$$

Comparing (10) – (11) and (24) – (25) we see that the solution to (24), (25), namely, the function $\varphi_2^{(1)}$, is resonantly forced since the homogeneous equation which corresponds to (24) is satisfied at $S = S_c$, $k = k_c$ and $c = c_c$. Thus, (24) – (25) has a solution if and only if the right-hand side of (24) is orthogonal to all the eigenfunctions of the corresponding adjoint problem. The adjoint operator, L^a, and the adjoint eigenfunction, φ_1^a, are defined as follows:

$$\int_{-\infty}^{+\infty} \varphi_1^a L(\varphi_1)dy = \int_{-\infty}^{+\infty} \varphi_1 L^a(\varphi_1^a)dy = 0 \quad (26)$$

The adjoint eigenfunction φ_1^a satisfies the equation

$$(iku_0 + 2Su_0 + B)(\varphi_1^a)'' + (2iku_{0y} + 2Su_{0y})(\varphi_1^a)'$$
$$-(ik^3u_0 + u_0k^2S + Bk^2)\varphi_1^a + ikc[(\varphi_1^a)'' - k^2\varphi_1^a] = 0 \quad (27)$$

with the boundary conditions

$$\varphi_1^a(\pm\infty) = 0 \quad (28)$$

Applying he solvability condition for equation (24) we obtain the group velocity, c_g, in the form

$$c_g = \frac{I_1}{I_2}, \quad (29)$$

where

$$I_1 = \int_{-\infty}^{+\infty} [u_0\varphi_{1yy} - \varphi_1(3k^2u_0 + u_{0yy}$$
$$-2k^2c - 2iku_0S - 2Bik)]\varphi_1^a dy$$

and

$$I_2 = \int_{-\infty}^{+\infty} \varphi_1^a(\varphi_{1yy} - k^2\varphi_1)dy$$

Finally, collecting the terms that are proportional to A^2 we obtain the following equation for the function $\varphi_2^{(2)}$

$$2(iku_0 - ikc)\varphi_{2yy}^{(2)} + (8ik^3c - 8ik^3u_0 - 2iku_{0yy})\varphi_2^{(2)}$$
$$+S[2u_0\varphi_{2yy}^{(2)} + 2u_{0y}\varphi_{2y}^{(2)} - 4k^2u_0\varphi_2^{(2)}] + B[\varphi_{2yy}^{(2)} - 4k^2\varphi_2^{(2)}]$$
$$= ik(\varphi_1\varphi_{1yyy} - \varphi_{1y}\varphi_{1yy}) - S(2\varphi_{1y}\varphi_{1yy} - 3k^2\varphi_1\varphi_{1y}) \quad (30)$$

with the boundary conditions

$$\varphi_2^{(2)}(\pm\infty) = 0. \quad (31)$$

The amplitude evolution equation for A is obtained from the solvability condition for equation (20) and has the form

$$\frac{\partial A}{\partial \tau} = \sigma A + \delta \frac{\partial^2 A}{\partial \xi^2} - \mu |A|^2 A \quad (32)$$

where

$$\sigma = \frac{\sigma_1}{\gamma_1}, \quad \delta = \frac{\delta_1}{\gamma_1}, \quad \mu = \frac{\mu_1}{\gamma_1}. \quad (33)$$

The coefficients

$$\sigma = \sigma_r + i\sigma_i, \delta = \delta_r + i\delta_i, \mu = \mu_r + i\mu_i$$

are complex. Equation (32) is the Ginzburg-Landau equation.

The coefficients $\gamma_1, \sigma_1, \delta_1$ and μ_1 are given by

$$\gamma_1 = \int_{-\infty}^{+\infty} \varphi_1^a(\varphi_{1yy} - k^2\varphi_1)dy \quad (34)$$

$$\sigma_1 = S\int_{-\infty}^{+\infty} \varphi_1^a(2u_0\varphi_{1yy} + 2u_{0y}\varphi_{1y} - k^2u_0\varphi_1)dy \quad (35)$$

$$\delta_1 = \int_{-\infty}^{+\infty} \varphi_1^a[\varphi_{2yy}^{(1)}(c_g - u_0) + \varphi_2^{(1)}(-k^2c_g - 2k^2c$$
$$+3k^2u_0 + u_{0yy} - 2iku_0S - 2ikB) + \varphi_1(2ikc_g + ikc) \quad (36)$$
$$-3iku_0 - US - B)]dy$$

$$\mu_1 = \int_{-\infty}^{+\infty} \varphi_1^a \{6ik^3\varphi_2^{(2)}\varphi_{1y}^* - 2ik\varphi_{1y}^*\varphi_{2yy}^{(2)} + 3ik^3\varphi_1^*\varphi_{2y}^{(2)}$$
$$+ik^3\varphi_1(\varphi_{2y}^{(0)} + \varphi^{*(0)}_{2y}) - ik\varphi_{1yy}(\varphi_{2y}^{(0)} + \varphi^{*(0)}_{2y})$$
$$+ik\varphi_{2y}^{(2)}\varphi_{1yy}^* - ik\varphi_1^*\varphi_{2yyy}^{(2)} + ik\varphi_1(\varphi_{2yyy}^{(0)} + \varphi_{2yyy}^{*(0)})$$
$$+2ik\varphi_{1yyy}^*\varphi_2^{(2)} - 2S[-k^2\varphi_1(\varphi_{2y}^{(0)} + \varphi^{*(0)}_{2y})$$
$$+3k^2\varphi_1^*\varphi_{2y}^{(2)} - 1.5k^4\varphi_1^2\varphi_1^*/u_0 + 2\varphi_{1yy}(\varphi_{2y}^{(0)} + \varphi^{*(0)}_{2y})$$
$$+2\varphi_{1yy}^*\varphi_{2y}^{(2)} + 2\varphi_{1y}(\varphi_{2yy}^{(0)} + \varphi^{*(0)}_{2yy}) + 2\varphi_{2yy}^{(2)}\varphi_{1y}^*]\}dy \quad (37)$$

4 Weakly nonlinear calculations

In order to evaluate the coefficients of the Ginzburg-Landau equation numerically, one needs to find the critical values of k, S and c from the linear stability problem (10)-(11). Then the corresponding eigenfunction φ_1^a of the adjoint problem (27), (28) is calculated. Next, three boundary value problems (22)-(25), (30)-(31) are solved and the functions $\varphi_2^{(0)}$, $\varphi_2^{(1)}$ and $\varphi_2^{(2)}$ are calculated. Finally, the group velocity c_g is computed from the solvability condition (29). In all cases pseudospectral method based on Chebyshev polynomials is used.

The coefficients of the complex GLE (32) are then evaluated numerically by means of (33)-(37). The results are shown in Table 1 for $R = -0.5$.

B	σ	δ	μ
0.0	0.0899+0.0004i	0.1150-0.1834i	4.5212+11.6033i
0.02	0.0716+0.0001i	0.1116-0.2131i	4.8302+11.7427i
0.04	0.0529-0.0000i	0.1062-0.2438i	5.3386+11.6620i
0.06	0.0300-0.0002i	0.0986-0.2819i	6.6213+11.8045i

Table 1. The coefficients of the Ginzburg-Landau equation (32) for different values of B.

As can be seen from Table 1, the real part of μ (known as the Landau constant in the literature) is positive, therefore, finite amplitude equilibrium is possible and the instability is supercritical. Thus, the Ginzburg-Landau equation may be used for the analysis of shallow wake two-phase flows in convectively unstable regime. Note that in cases where the real part of μ is negative, the higher powers of A (which are neglected in (32)) become also important, and the Ginzburg-Landau model cannot be used for the analysis. In such cases a finite equilibrium state is not possible. This means that the disturbances are linearly unstable and grow unbounded; that is, the instability is subcritical. An example of such case is given in [10] for plane Poiseuille flow.

Consider unmodulated (independent on ξ) equilibrium amplitude solution of (32) of the form

$$A = \sqrt{\frac{\sigma_r}{\mu_r}} \exp\left[i\tau \left(\sigma_i - \frac{\mu_i}{\mu_r}\sigma_r \right) \right]$$

where the amplitude A_0 and the frequency ω are given by

$$A_0 = \sqrt{\frac{\sigma_r}{\mu_r}}, \omega = \sigma_i - \frac{\mu_i}{\mu_r}\sigma_r \quad (38)$$

It is seen from (38) that both the amplitude and the frequency of the most unstable mode are modified by nonlinear effects. The values of A_0 and ω for different values of B are calculated for the case $R = -0.5$ and are shown in Table 2.

B	ω	A_0
0.0	0.230	0.141
0.02	0.174	0.122
0.04	0.116	0.100

Table 2. The amplitude A_0 and the frequency ω for different values of B at $R = -0.5$.

As can be seen from Table 2, the stabilizing effect of the particle loading parameter B is obvious also in weakly nonlinear regime: the finite amplitude is getting smaller as B increases.

5 Conclusion

Linear and weakly nonlinear analysis of two-phase shallow wake flow is performed in the present paper. Linear stability calculations show that both bottom friction and particle loading stabilize the flow. The amplitude evolution equation for the most unstable mode is derived. It is shown that the development of the most unstable mode in weakly nonlinear regime is governed by the complex Ginzburg-Landau equation. Results of numerical computation are presented.

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