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ASYMPTOTIC METHODS FOR RETARDING QUASILINEAR DYNAMICAL SYSTEMS

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Abstract. An asymptotic method for qualitative analysis of quasi-linear functional differential equations with small or rapidly oscillating perturbations dependent on phase coordinates and an ergodic Markov process is presented. The proposed method is based on an averaging procedure with respect to time and the invariant measure of the Markov process along the critical solutions of the linear equation. We have proposed an algorithm for dynamical analysis of the initial random equation with delay that permits approximate its solutions (which are stochastic processes) by corresponding solutions of a specially constructed averaged deterministic ordinary differential equation (called "fully simplified"). It is proved that for linear systems with small perturbations an exponential stability of the resulting fully simplified deterministic equation is suffice for exponential stochastic stability of the initial random system. Moreover, we have proved that problem of stability analysis of linear retarding dynamical systems with rapidly oscillating functionals one may reduce to stability analysis of deterministic linear functional differential equation applying an averaging procedure to above mentioned functionals.

1 Quasilinear functional differential equations

This paper deals with the n-dimensional functional differential equation in a quasi-linear form with a small parameter $\varepsilon \in [0, 1)$

$$\frac{du^\varepsilon(t)}{dt} = g(u_t^\varepsilon) + \varepsilon F(t, u_t^\varepsilon, y(t), \varepsilon), \tag{1}$$

where

- u_t^ε is part of solution defined by the equality $u_t^\varepsilon := \{u^\varepsilon(t+\theta), \theta \in [-h, 0]\}$ with some positive number h;

- $g(\varphi)$ is the linear continuous mapping of the space of the continuous n-dimensional vector-functions $\mathbf{C}_n := \mathbf{C}_n([-h,\, 0])$ to $\mathbb{R}^n$, defined by the equality

$$g(\varphi) := \int_{-h}^{0} \{dG(\theta)\}\varphi(\theta)$$

with matrix $G(\theta)$ consisting of bounded variation functions;

- $\{y(t),\, t \geq 0\}$ is a homogeneous ergodic Markov process on the probability space $(\Omega, \mathcal{F}F, \mathbf{P})$ with values in the phase space $\mathbf{Y}$, with infinitesimal operator Q, transition probability $P(t,\, y,\, dz)$ and unique invariant measure $\mu(dy)$ satisfying the condition of exponential ergodicity [1], that is, there exist positive constants M and δ such that $||P(t,\, y,\, .) - \mu|| \leq M \exp\{-\delta t\}$ for any $t \geq 0$;

- the perturbing term $F(t,\, \varphi,\, y,\, \varepsilon)$ is a continuous mapping of the product space $\mathbb{R}_+ \times \mathbf{C}_n([-h,\, 0]) \times \mathbf{Y} \times [0,\, 1)$ to the space $\mathbb{R}^n$, satisfying $F(t,\, 0,\, y,\, \varepsilon) \equiv 0$ and the Lipschitz condition

$$|F(t,\, \varphi,\, y,\, \varepsilon) - F(t,\, \psi,\, y,\, \varepsilon)| \leq l \int_{-h}^{0} |\varphi(s) - \psi(s)|\, d\nu(s) \tag{2}$$

for any $y \in \mathbf{Y},\ \varepsilon \in [0,1),\ t \in \mathbb{R}_+,\ \varphi,\, \psi \in \mathbf{C}_n$, with some function $\nu(s)$ of unit variation.

Under these conditions the random equation (1) with initial problem $u^\varepsilon(s+\theta) = \varphi(\theta),\ -h \leq \theta \leq 0$ has [4] a unique solution $u^\varepsilon = \{u^\varepsilon(t),\, t \geq 0\}$ for any continuous function φ; this solution is a continuous stochastic process with probability one. We will refer to the linear equation

$$\frac{du(t)}{dt} = \int_{-h}^{0} \{dG(\theta)\}\, u(t+\theta) \tag{3}$$

as *the generative equation* corresponding to (1). It is well known [4] that equation (3) defines in the space $\mathbf{C}_n$ a strong continuous semigroup $T(t)$ with infinitesimal operator given for sufficiently smooth function φ by

$$(\mathbb{A}\varphi)(\theta) := \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & \text{if } \; -h \leq \theta < 0, \\ g(\varphi), & \text{if } \; \theta = 0. \end{cases}$$

The spectrum $\sigma(\mathbb{A})$ of this operator is given by $\sigma(U) := \{z : det\{U(z)\} = 0\}$ where $U(z) := Iz - \int_{-h}^{0} e^{z\theta} dG(\theta)$. As in the deterministic case [3], we will proceed in this paper with the assumption that *the generative equation is on the border of stability,* that is, $\sigma(U) \cap \{z : \Re z > 0\} = \emptyset,\quad \sigma_0 := \sigma(U) \cap \{z : \Re z = 0\} \neq \emptyset$. We will refer to the spectral subspace of the operator $\mathbb{A}$ corresponding to σ_0 as *the critical subspace* and to the solutions of (3) lying in the critical subspace as *the critical solutions.*

Using projection on the critical subspace we will construct a finite-dimensional differential equation with Markov parameters and rapid switching which has the same stability properties of the trivial solution as equation (1) for all sufficiently small $\varepsilon > 0$. It will be proven that for stability analysis under some additional assumptions one can perform averaging with respect to the invariant measure of the Markov process and with respect to time along the critical solutions of the generative equation as one can do for the deterministic delay equations [3]. Stability results can then be obtained applying the Second Lyapunov method using a specially constructed [1, 7] Lyapunov functional and recursive approximations of the solutions of (1) given by the solutions of the corresponding averaged equation.

2 Fully simplified equation for quasilinear systems with small perturbations

Some preliminary preparation is needed in order to obtain the resulting averaged equation. Firstly one needs to rewrite equation (1) in the operator form [4]

$$\frac{du_t^\varepsilon}{dt} = \mathbb{A}\, u_t^\varepsilon + \varepsilon \mathbf{1}\, F(t,\, u_t^\varepsilon,\, y(t),\, \varepsilon), \tag{4}$$

where the matrix-valued function $\{\mathbf{1}(\theta),\ -h \leq \theta \leq 0\}$ is defined by the equality

$$\mathbf{1}(\theta) := \begin{pmatrix} 0, & \text{if } \ -h \leq \theta < 0, \\ I, & \text{if } \ \theta = 0, \end{pmatrix}$$

and I is the $n \times n$ identity matrix. Next, one must define the spectral projective operator P_0 corresponding to $\sigma_0 \subset \sigma(\mathbb{A})$. For this we will use its integral representation [5] in the form

$$(P_0\psi)(\theta) := \frac{1}{2\pi i} \int\limits_{\mathcal{B}} \big((Jz - \mathbb{A})^{-1}\psi\big)(\theta) dz \tag{5}$$

where $\mathcal{B} := \cup_{j=1}^m \{z : \ |z - z_j| = \delta\}$ with sufficiently small $\delta > 0$. It can be easily seen that both the projective operator P_0 and $I - P_0$ are bounded.

One can apply the projective operator P_0 not only on any continuous vector-function $\psi(\theta)$ but also on any vector- or matrix-valued measurable function. Inserting into the integral representation (5) the above matrix-valued function $\mathbf{1}$ one can define the $n \times n$-matrix-function

$$\Gamma(\theta) := \frac{1}{2\pi i} \int\limits_{\mathcal{B}} \big((Jz - \mathbb{A})^{-1}\mathbf{1}\big)(\theta) dz = \sum_{j=1}^{m} res\{U^{-1}(z) e^{z\theta}\}|_{z=z_j}. \tag{6}$$

Let us denote the critical subspace as $\mathbf{X}_0 := P_0 \mathbf{C}_n$, the matrix of a basis in this subspace consisting of m columns and n rows as $V(\theta)$, the restriction of the operator $\mathbb{A}$ on $\mathbf{X}_0$ as $\mathbb{A}_0$ and let A_0 be the matrix of this restriction, defined by the equation $\mathbb{A}_0 V(\theta) = V(\theta) A_0$. Furthermore, one can define the $m \times m$-matrix $\hat{\mathbf{\Psi}}$, writing the identity $\Gamma(\theta) = V(\theta)\hat{\mathbf{\Psi}}$. Let us use the above notations, along with the notation $\mathbf{V} := \{V(\theta),\ -h \leq \theta \leq 0\}$ and assume the existence of the m-dimensional vector function $\check{F}(x)$ of argument $x \in \mathbb{R}^m$ defined by

$$\check{F}(x) := \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \int\limits_{\mathbf{Y}} e^{-tA_0} \hat{\Psi} F(t,\, \mathbf{V} e^{tA_0} x,\, y,\, 0) \mu(dy) dt,$$

where $\mu(dy)$ is the invariant measure of the Markov process $y(t)$, $t \geq 0$. Thus we define *the averaged differential equation* (which is not random)

$$\frac{d\check{x}}{dt} = \check{F}(\check{x}). \tag{7}$$

We say that *the trivial solution of the equation* (7) *is exponentially stable in the large* if there exist positive constants $\check{a}_1$ and $\check{a}_2$ such that

$$|\check{x}(t+s,\ s,\ x)| \le \check{a}_1 e^{-\check{a}_2 t}|x| \tag{8}$$

for any $s, t \ge 0,\ x \in \mathbb{R}^m$. We say that the trivial solution of the random equation (1) is *exponentially p-stable in the large for all sufficiently small positive* ε if there exist positive constants ε_0, a_1 and for any $\varepsilon \in (0,\ \varepsilon_0)$ there exists positive number $a_2(\varepsilon)$ such that $\mathbf{E}^{(s)}_{y,\varphi}\{|u^\varepsilon(t+s)|^p\} \le a_1 e^{-a_2(\varepsilon)t}||\varphi||^p$ for any $s,\ t \ge 0,\ y \in \mathbf{Y},\ \varphi \in \mathbf{C}_n$. In this definition and everywhere further throughout this paper the above upper and lower indices of expectation (or probability) denote the conditions $y(s) = y,\ u^\varepsilon_s = \varphi$. All the relations below involving random variables and processes are understood as such.

One must note that the selection of the linear mapping $g(\varphi)$ in the right part of equation (1) can be done somewhat arbitrarily. One can add any arbitrary linear continuous mapping $\varepsilon g_1(\varphi)$ to the linear part of (1) and subtract it from the second term. Using this arbitrariness and because the set σ_0 consists of the finite number of points [4] $\sigma_0 = \{z_j,\ j = 1,\ 2,\ ...,\ m\}$ it may be assumed that the selection of the terms in the right part of (1) has been done in such manner so that $((\det U(z_j))' \ne 0,\ j = 1,\ 2,\ ...,\ m$.

Lemma 1. *Under the above assumptions one can find constant* c *such that the solution of the equation* (1) *with initial condition*

$$u^\varepsilon(s+\theta) = \varphi(\theta),\ \ -h \le \theta \le 0, \tag{9}$$

satisfies the inequality

$$\sup_{\substack{0 \leqslant t \leqslant T/\varepsilon \\ s \geqslant 0, y \in \mathbf{Y}}} |u^\varepsilon(t+s,\ s,\ \varphi)| \le c e^{clT}||\varphi|| \tag{10}$$

for any $T > 0$, $\varepsilon \in (0,\ 1)$, *where* l *is the Lipschitz constant from inequality* (2).

Proof. Let $H(t)$ denote the matrix-solution of the generative equation (3) satisfying the initial condition $H(\theta) = \mathbf{1}(\theta),\ -h \leqslant \theta \le 0$. Using this matrix-valued function, one can [4] rewrite equation (1) in the integral form $u^\varepsilon(t+s,\ s,\ \varphi) = u^0(t,\ 0,\ \varphi) + \varepsilon \int\limits_0^t H(t-\tau)F(s+\tau,\ u^\varepsilon_{s+\tau},\ y(s+\tau),\ \varepsilon)d\tau$, where $u^0(t,\ 0,\ \varphi)$ is the solution of the generative equation with the same initial condition. Due to our assumptions regarding the spectrum part σ_0 there exists [4] $c := \sup\limits_{t \ge 0}||T(t)||$, whence $||H(t)|| \le c \sup\limits_{-h \leqslant \theta \leqslant 0}||\mathbf{1}(\theta)|| \le c$, $|u^0(t,\ 0,\ \varphi)| = |(T(t)\varphi)(0)| \le c||\varphi||$ for any $t \ge 0$ and $\varphi \in \mathbf{C}_n$. Therefore, the proof follows from the integral inequality $\sup\limits_{t_1 \leqslant t}|u^\varepsilon(t_1+s,\ s,\ \varphi)| \le c||\varphi|| + \varepsilon lc \int\limits_0^t \sup\limits_{t_1 \leqslant \tau}|u^\varepsilon(t_1+s,\ s,\ \varphi)|d\tau$ after applying the Gronwall's lemma on the segment $0 \le t \le T/\varepsilon$.

Using the matrix $\Gamma(\theta)$ from (6) and the decomposition $u^\varepsilon_t(\theta) = (P_0 u^\varepsilon_t)(\theta) + ((I-P_0)u^\varepsilon_t)(\theta)$ one can rewrite equation (4) as a system of two equations for the vector-functions

$r_0(t,\,\theta) := (P_0 u_t^\varepsilon)(\theta)$ and $r_1(t,\,\theta) := ((I-P_0)u_t^\varepsilon)(\theta)$:

$$\frac{\partial r_0(t,\,\theta)}{\partial t} = (\mathbb{A}_0\, r_0)(t,\,\theta) + \varepsilon\Gamma(\theta)\, F(t,\, \mathbf{r_0}(t) + \mathbf{r_1}(t),\, y(t),\, \varepsilon), \tag{11}$$

$$\frac{\partial r_1(t,\,\theta)}{\partial t} = (\mathbb{A}_1\, r_1)(t,\,\theta) + \varepsilon(\mathbf{1}(\theta) - \Gamma(\theta))\, F(t,\, u_t^\varepsilon,\, y(t),\, \varepsilon), 2 \tag{12}$$

where $\mathbf{r_j}(t) = \{r_j(t,\,\theta),\ \theta \in [-h,\,0]\}$, $j = 0,\,1$, and the linear closed operator $\mathbb{A}_1 := (J-P_0)\mathbb{A}$ is acting on the same subspace $\mathcal{D}(\mathbb{A}) \subset \mathbf{C}_n$ as the operator $\mathbb{A}$ and has the spectrum $\sigma_1 := \sigma(U) \setminus \sigma_0 \subset \{\Re z \le -\rho < 0\}$.

Lemma 2. *Under the above conditions there exists constant c_1 such that the solution of equation* (12) *satisfies the inequality*

$$\sup_{\substack{s\geqslant 0,\, y\in\mathbf{Y}\\ -h\leqslant\theta\leqslant 0\\ 0\leqslant t\leqslant T/\varepsilon}} |r_1(t,\,\theta) - (T(t)(I-P_0)\varphi)(\theta)| \le \varepsilon c_1 ||\varphi|| e^{lcT}$$

for any $T>0$, $\varepsilon\in(0,\,1)$ with constant c from inequality (10).

Proof. The operator $\mathbb{A}_1$ can be considered [5] as the infinitesimal operator of the contractive semigroup $\{T_1(t),\ t\ge 0\}$, which satisfies the inequality $||T_1(t)|| \le \gamma e^{-\rho t}$ with the above defined positive ρ and some positive constant γ for any $t\ge 0$. Using this semigroup one can rewrite equation (12) in the integral form

$$r_1(s+t,\,\theta) = (T(t)(I-P_0)\varphi)(\theta) + \varepsilon\int_s^{s+t} T_1(t-\tau)(\mathbf{1}(\theta)-\Gamma(\theta))\, F(\tau,\, u_\tau^\varepsilon,\, y(\tau),\, \varepsilon)d\tau.$$

Due to Lemma 1, inequality (10), the Lipschitz condition and the exponential decay of the semigroup $T_1(t)$ the above integral equality allows us to complete the proof using the inequality $\sup\limits_{\substack{0\leqslant t\leqslant T/\varepsilon\\ s\geqslant 0,\, y\in\mathbf{Y}}} ||\mathbf{r_1}(t+s) - T(t)(I-P_0)\varphi|| \le \varepsilon(1+||P_0||)\frac{lc\gamma}{\rho}e^{lcT}||\varphi||$ or

$$\sup_{s\geqslant 0,\ y\in\mathbf{Y}} ||\mathbf{r_1}(T/\varepsilon + s)|| \le \varepsilon\gamma_1 e^{lcT}||\varphi|| + \gamma e^{-\frac{t\rho}{\varepsilon}} \tag{13}$$

for any $T>0$, $\varepsilon\in(0,\,1)$, $\varphi\in\mathbf{C}_n$ and with γ_1 being a positive constant.

Theorem 1. *Let in addition to the previous assumptions the function $F(t,\mathbf{V}x,y,\varepsilon)$ be uniformly continuous at zero as a function of ε, that is, assume that the quantity*

$$\alpha(\varepsilon) := \sup_{\substack{t\geqslant 0,\, y\in\mathbf{Y}\\ x\in\mathbb{R}^n}} \frac{|F(t,\,\mathbf{V}x,\,y,\,\varepsilon) - F(t,\,\mathbf{V}x,\,y,\,0)|}{|x|} \tag{14}$$

is infinitesimal as $\varepsilon\to 0$ and the limit function $F(t,\,\mathbf{V}x,\,y,\,0)$.

- *has uniformly bounded continuous x-derivative $DF(t,\,\mathbf{V}x,\,y,\,0)$;*

- *belongs to the domain $\mathcal{D}(Q)$ of the operator Q;*
- *has continuous bounded t-derivative $\frac{\partial}{\partial t}F(t,\, \mathbf{V}x,\, y,\, 0)$;*
- *has the above defined average $\check{F}(x)$ along the solutions of the generative equation and there exists constant b such that*

$$\sup_{\substack{y\in\mathbf{Y},T>0\\ s\geqslant 0}} \left| \int\limits_{s}^{s+T}\int\limits_{\mathbf{Y}} e^{-tA_0}\hat{\Psi}F(t,\,\mathbf{V}e^{tA_0}x,\,y,\,0)\mu(dy)\,dt - T\,\bar{F}(x)\right| \leq b|x|, \tag{15}$$

for any $x \in \mathbb{R}^{>}$.

If the trivial solution of the averaged equation (7) *is exponentially stable in the large, then the trivial solution of the random equation* (1) *is exponentially p-stable in the large for all sufficiently small positive ε.*

Proof. According to the definition of the basis $V(\theta)$ the n-dimensional vector-function $r_0(t,\,\theta)$ in equation (11) can be decomposed as follows: $r_0(t,\,\theta) := V(\theta)\vec{u}^{\varepsilon}(t),\ \forall\theta \in [-h,\,0]$. After substitution of this decomposition in (11) one can conclude that the m-dimensional vector-function $\vec{u}^{\varepsilon}(t)$ satisfies the ordinary random differential equation in $\mathbb{R}^m$

$$\frac{d\vec{u}^{\varepsilon}(t)}{dt} = A_0\vec{u}^{\varepsilon}(t) + \varepsilon\hat{\Psi}F(t,\,\mathbf{V}\vec{u}^{\varepsilon}(t) + \mathbf{r_1}(t),\,y(t),\,\varepsilon). \tag{16}$$

One can consider the decomposition of equation (1) in the form (11)-(12) with the decomposed initial condition (9):

$$r_0(s,\,\theta) = P_0\varphi(\theta) := V(\theta)\vec{u},\ \ r_1(s,\,\theta) = (I-P_0)\varphi(\theta) \tag{17}$$

which uniquely defines the vector $\vec{u} \in \mathbb{R}^m$ for given basis $\mathbf{V}$. Consequently, equation (16) should be considered with initial condition

$$\vec{u}^{\varepsilon}(s) = \vec{u}. \tag{18}$$

Let $\tilde{u}^{\varepsilon}(t)$ be the solution of the random differential equation in $\mathbb{R}^m$

$$\frac{d\tilde{u}^{\varepsilon}(t)}{dt} = A_0\tilde{u}^{\varepsilon}(t) + \varepsilon\hat{\Psi}F(t,\,\mathbf{V}\tilde{u}^{\varepsilon}(t),\,y(t),\,\varepsilon) \tag{19}$$

with the above initial condition (18). Using the substitutions $\vec{u}^{\varepsilon}(t) = e^{tA_0}\vec{z}^{\varepsilon}(t)$, $\tilde{u}^{\varepsilon}(t) = e^{tA_0}\tilde{z}^{\varepsilon}(t)$ one can derive from (16) for the vector-valued functions $\vec{z}^{\varepsilon}(t)$ and $\tilde{z}^{\varepsilon}(t)$ the equations

$$\frac{d\vec{z}^{\varepsilon}(t)}{dt} = \varepsilon e^{-tA_0}\hat{\Psi}F(t,\,\mathbf{V}e^{tA_0}\vec{z}^{\varepsilon}(t) + \mathbf{r_1}(t),\,y(t),\,\varepsilon), \tag{20}$$

$$\frac{d\tilde{z}^{\varepsilon}(t)}{dt} = \varepsilon e^{-tA_0}\hat{\Psi}F(t,\,\mathbf{V}e^{tA_0}\tilde{z}^{\varepsilon}(t),\,y(t),\,\varepsilon). \tag{21}$$

Due to the Lipschitz condition (2) and the assumptions regarding the spectrum σ_0 of the matrix A_0 the difference of the solutions satisfies the integral inequality

$$\begin{aligned} |\vec{z}^{\varepsilon}(t+s) - \tilde{z}^{\varepsilon}(t+s)| &\leq \varepsilon cl||\hat{\Psi}|| \int\limits_0^t\int\limits_{-h}^0 ||V(\theta)|||r_1(\tau+s,\,\theta)|d\nu(\theta)d\tau \\ &+ \varepsilon cl||\hat{\Psi}|| \int\limits_{-h}^0 ||V(\theta)||d\nu(\theta)\int\limits_0^t |\vec{z}^{\varepsilon}(\tau+s) - \tilde{z}^{\varepsilon}(\tau+s)|d\tau. \end{aligned}$$

Using inequality (13) one can derive the inequality $||\mathbf{r_1}(\tau + s)|| \le \gamma e^{-\rho\tau}(1 + ||P_0||)||\varphi|| + \varepsilon c_1 e^{lcT}||\varphi||$ for any $T > 0,\ \tau \in [0,\, T/\varepsilon)$. Therefore, there exist a constant l_1 and a function $l_2(T)$ such that the difference between the solutions of equations (20) and (21) satisfies the inequality $|\bar{z}^\varepsilon(t+s) - \tilde{z}^\varepsilon(t+s)| \le \varepsilon l_2(T)||\varphi|| + \varepsilon l_l \int\limits_0^t |\bar{z}^\varepsilon(\tau+s) - \tilde{z}^\varepsilon(\tau+s)|d\tau$. Applying Gronwall's inequality one can find a function $c_3(T)$ such that

$$\sup_{\substack{0 \leqslant t \leqslant T/\varepsilon \\ s \geqslant 0,\, y \in \mathbf{Y}}} |\bar{z}^\varepsilon(t+s) - \tilde{z}^\varepsilon(t+s)| \le \varepsilon l_3(T)||\varphi|| \tag{22}$$

for any $T > 0,\ \varepsilon \in [0,\, 1),\ \varphi \in \mathbf{C}_n$.

To simplify the notations we denote $\tilde{x}^\varepsilon(t) := \tilde{z}^\varepsilon(t/\varepsilon),\ \ f(t,\, x,\, y,\, \varepsilon) := e^{-tA_0}\hat{\Psi}F(t,\, \mathbf{V}e^{tA_0}x,\, y,\, \varepsilon)$. Let $x^\varepsilon(t)$ be the solution of the random equation

$$\frac{dx^\varepsilon}{dt} = f(t/\varepsilon,\, x^\varepsilon,\, y(t/\varepsilon),\, 0). \tag{23}$$

It is easy to verify that $\tilde{x}^\varepsilon(t)$ satisfies the random differential equation

$$\frac{d\tilde{x}^\varepsilon}{dt} = f(t/\varepsilon,\, \tilde{x}^\varepsilon,\, y(t/\varepsilon),\, \varepsilon). \tag{24}$$

Consider this equation with initial conditions $\tilde{x}^\varepsilon(s) = x^\varepsilon(s) = \vec{u}$ with vector $\vec{u}$ from (17).
Due to the existence of uniformly bounded x-derivative $DF(t,\, \mathbf{V}x,\, y,\, 0)$ the right hand sides of equations (23) and (24) satisfy the Lipschitz condition with some constant L. Furthermore, it follows from (14) that the function $f(t,\, x,\, y,\, \varepsilon)$ is uniformly continuous at point zero as a function of ε, that is, the quantity $\beta(\varepsilon) := \sup\limits_{\substack{t \geqslant 0,\, y \in \mathbf{Y} \\ x \in \mathbb{R}^n}} \frac{|f(t,x,y,\varepsilon) - f(t,x,y,0)|}{|x|}$ is infinitesimal as $\varepsilon \to 0$. Using the latter property and the Lipschitz constant L one can write the inequalities

$$\begin{aligned} &|\tilde{x}^\varepsilon(t+s,\, s,\, \vec{u}) - x^\varepsilon(t+s,\, s,\, \vec{u})| \le \\ &\qquad \int\limits_s^{s+t} |f(\tau/\varepsilon,\, \tilde{x}^\varepsilon(\tau,\, s,\, \vec{u}),\, y(\tau/\varepsilon),\, \varepsilon) - f(\tau/\varepsilon,\, x^\varepsilon(\tau,\, s,\, \vec{u}),\, y(\tau/\varepsilon),\, 0)|d\tau \le \\ &L \int\limits_0^t |\tilde{x}^\varepsilon(\tau+s,\, s,\, \vec{u}) - x^\varepsilon(\tau+s,\, s,\, \vec{u})|d\tau + \beta(\varepsilon) \int\limits_s^{s+t} |x^\varepsilon(\tau,\, s,\, \vec{u})|d\tau \end{aligned} \tag{25}$$

It can be easily shown that the Lipschitz condition for the right hand side of equation (23) guarantees the existence of a constant B such that $|x^\varepsilon(t+s,\, s,\, \vec{u})| \le Be^{Lt}|\vec{u}|$ for any $\vec{u} \in \mathbb{R}^m,\ s \ge 0,\ t \ge 0,\ \varepsilon \in [0,\, 1)$. Thus, substituting this bound in the last term of (25) and applying Gronwall's inequality one can obtain the relation

$$\sup_{\substack{0 \leqslant t \leqslant T \\ s \geqslant 0,\, y \in \mathbf{Y}}} |\tilde{x}^\varepsilon(t+s,\, s,\, \vec{u}) - x^\varepsilon(t+s,\, s,\, \vec{u})| \le \beta(\varepsilon)\, BTe^{LT}|\vec{u}|$$

for any $T \geq 0$, $\vec{u} \in \mathbb{R}^m$ and sufficiently small $\varepsilon > 0$. For further analysis it is convenient to rewrite this inequality for the time $t\varepsilon$ and use the norm of the initial condition (9):

$$\sup_{\substack{0 \leqslant t \leqslant T/\varepsilon \\ s \geqslant 0, y \in \mathbf{Y}}} |\tilde{x}^\varepsilon(t\varepsilon + s,\, s,\, \vec{u}) - x^\varepsilon(t\varepsilon + s,\, s,\, \vec{u})| \leq \beta(\varepsilon)\, BTe^{LT}||\varphi|| \tag{26}$$

for any $T \geq 0$, $\varphi \in \mathbf{C}_n$.

3 Stability analysis by fully simplified equation

It is known [1, 9] that under the above assumptions the solutions of equation (24) tend to the corresponding solutions of equation (7) and that the stability of the trivial solution of equation (7) guarantees [7] the stability with probability one of the trivial solution of equation (24). However, in order to prove our theorem we need stronger evaluation of the rate of convergence to zero of the p-moments of the solutions of (24) as $t \to \infty$. For this purpose we will apply the second Lyapunov method to a specially constructed functional $v(t,\, x,\, y,\, \varepsilon)$ using the ideas of [1]. Since for any random variable ξ the quantity $(\mathbf{E}(|\xi|^p))^{1/p}$ is monotonically nondecreasing function of $p > 0$ we can assume in our proof without loss of generality that $p \geq 2$.

One can consider the pair $\{x^\varepsilon,\, y(t/\varepsilon)\}$ as a Markov process in the phase space $\mathbb{R}^m \times \mathbf{Y}$ [1, 8, 9] with weak infinitesimal operator $\mathcal{Q}$ defined on sufficiently smooth continuous function $v(x, y)$ by the equality $(\mathcal{Q}v)(x,\, y) := (\bigtriangledown v)(x,\, y) + \frac{1}{\varepsilon}(Qv)(x,\, y)$ where $(.\,,\,.)$ and $\bigtriangledown$ are the scalar product and the gradient-operator in $\mathbb{R}^m$, respectively. Since the right hand side of equation (23) depends on the time coordinate t one needs to extend the phase space by adding a new phase coordinate $t \in \mathbb{R}_+$ and consider the above nonhomogeneous Markov process with infinitesimal operator

$$(\mathcal{L}v)(t,\, x,\, y) := \frac{1}{\varepsilon}\big((\frac{\partial}{\partial t}v)(t,\, y,\, x) + (Qv)(t,\, x,\, y)\big) + \big((\bigtriangledown v)(t,\, x,\, y),\, f(t/\varepsilon,\, x,\, y,\, 0)\big). \tag{27}$$

For a given function $w(t,\, y)$ let $\hat{w}(t)$ denote the function obtained by averaging $w(t,\, y)$ with respect to the invariant measure $\mu(dy)$, that is, $\hat{w}(t) := \int\limits_{\mathbf{Y}} w(t,\, y)\, \mu(dy)$, and $\bar{w}(y)$ denote the function obtained by averaging $w(t,\, y)$ with respect to the time t, that is, $\bar{w}(y) := \lim\limits_{T\to\infty} \frac{1}{T} \int\limits_0^T w(t,\, y)dt$. The projection operator $\mathbf{P}_\mu$ on the subspace $\mathbf{C}_\mu := \{g \in \mathbf{C}(\mathbf{Y}) : \hat{g} = 0\}$ can be defined by the equality $\mathbf{P}_\mu g(y) := g(y) - \hat{g}$. Due to our assumption of exponential ergodicity we can use the inequality

$$\sup_{y\in\mathbf{Y}} |\mathbf{E}_y(\mathbf{P}_\mu g)(y(t)| \leq Me^{-\rho t} \sup_{y\in\mathbf{Y}} |g(y)| \tag{28}$$

for any $t \geq 0$ and $g \in \mathbf{C}(\mathbf{Y})$ and, therefore, the potential Π of the Markov process $\{y(t)\}$ can be defined as the improper integral $(\Pi g)(y) := \int\limits_0^\infty \mathbf{E}_y\, g(y(t))dt$, which satisfies,

$$\sup_{y\in\mathbf{Y}} |(\Pi g)(y)| \leq \frac{M}{\delta} \sup_{y\in\mathbf{Y}} |g(y)| \tag{29}$$

for any $g \in \mathbf{C}_\mu$. According to the definition of the weak infinitesimal operator [2] one can write the equality $\frac{\partial}{\partial t}\mathbf{E}_y h(s,\, y(s-t)) = -\mathbf{E}_y(Qh)(s,\, y(s-t))$ for any $s > t \geq 0$ and continuous bounded function $h(s,\, y)$. If in addition $\hat{h}(s) \equiv 0$ then the inequality $|\mathbf{E}_y h(s,\, y(s-t))| \leq Me^{-\delta(s-t)} \sup\limits_{y\in\mathbf{Y}} |h(s,\, y)|$ follows from (28). Therefore, there exists the improper integral $\int\limits_t^\infty \mathbf{E}_y h(s,\, y(s-t))ds := G(h)(t,\, y)$ for any $y \in \mathbf{Y}$ and

$$\sup_{\substack{y\,\in\,\mathbf{Y}\\ s\,\geqslant\,0}} |(Gh)(t,\, y)| \leq \frac{M}{\delta} \sup_{\substack{y\,\in\,\mathbf{Y}\\ s\,\geqslant\,0}} |h(s,\, y)|. \tag{30}$$

In view of (29) one can easily verify that the function $r(t,\, y) := G(h)(t,\, y)$ satisfies the ordinary differential equation

$$\frac{d}{dt}r(t,\, y) + Qr(t,\, y) = -h(t,\, y). \tag{31}$$

Using this result and the representation $h(t,\, y) = (h(t,\, y) - \hat{h}(t)) + \hat{h}(t)$ one can find a solution of equation (31) for arbitrary bounded function $h(t,\, y)$ in the form:

$$R(h)(t,\, y) := G(h - \hat{h})(t) + \int\limits_0^t \hat{h}(s)ds \tag{32}$$

and from inequality (30) obtain the inequality

$$|R(h)| \leq 2\frac{M}{\delta} \sup_{t\geqslant 0,y\in\mathbf{Y}} |h(t,\, y)| + \sup_{t\geqslant 0} \left| \int\limits_0^t \hat{h}(s)ds \right|. \tag{33}$$

To prove the exponential p-stability of the trivial solution of equation (24) we will use the Lyapunov functional

$$w(t,\, x,\, y,\, \varepsilon) := v(x) + \varepsilon v_1(t/\varepsilon,\, x,\, y) \tag{34}$$

where $v_1(t,\, x,\, y) = \big(\nabla v(x),\, R(f - \check{F})(t,\, x,\, y,\, 0)\big)$ and the operator R acts on the function $f(t,\, x,\, y,\, 0) - \check{F}(x)$ according to arguments t and y as defined in (32). The inequalities (15) and (33) allow us to estimate the second term in the latter scalar product as follows:

$$|R(f - \check{F})(t,\, x,\, y,\, 0)| \leq 2\sup_{t\geqslant 0} \left| \int\limits_0^t \hat{h}(s)ds \right| + 2\frac{M}{\delta} \sup_{\substack{y\,\in\,\mathbf{Y}\\ t\,\geqslant\,0}} |f(t,\, x,\, y,\, 0)|.$$

It is obvious that under the assumption of exponential stability condition in the large of equation (7), the pth power of the absolute value of any solution of (7) decreases also exponentially when $t \to \infty$ for any $p > 0$. Therefore, one can consider the function $v(x) = \int\limits_0^S |x(t,\, 0,\, x)|^p dt$ with sufficiently large positive S as the Lyapunov function for the averaged system. Using the smoothness with respect to $\check{x}$ of the right hand side of equation (7) one can prove that $v(x)$ has continuous derivative $\nabla v(x)$ and that the following inequalities are satisfied:

$$\begin{aligned} v_1|x|^p &\leq v(x) \leq v_2|x|^p, \quad (\nabla v(x),\, \check{F}(x)) \leq -v_3|x|^p, \\ \sup_{t\geqslant 0, y\in \mathbf{Y}} & |v_1(t,\, x,\, y)| \leq v_4|x|^p, \quad \sup_{t\geqslant 0, y\in \mathbf{Y}} |\nabla v_1(t,\, x,\, y)| \leq v_4|x|^{p-1} \end{aligned} \tag{35}$$

for any $x \in \mathbb{R}^m$ with some positive numbers $v_1,\, v_2,\, v_3,\, v_4,\, v_5$. Therefore, for sufficiently small positive ε_1 one can write the inequalities

$$w_1|x|^p \leq w(t,\, x,\, y,\, \varepsilon) \leq w_2|x|^p \tag{36}$$

with some positive number w_1 for all $\varepsilon \in [0,\, \varepsilon_1)$ and arbitrary values of the remaining variables involved in (36). Furthermore, using the definitions (27) and (32) of the operators $\mathcal{L}$ and R respectively one can obtain for the quantity $\mathcal{L}w$ the following inequality:

$$(\mathcal{L}w)(t,\, x,\, y, \varepsilon) \;=\; \big((\nabla v)(x),\, \check{F}(x)\big) + \varepsilon(\nabla v_1)(t/\varepsilon,\, x,\, y),\, f(t/\varepsilon,\, x,\, y,\, 0)\big) \leq -w_3|x|^p \tag{37}$$

for sufficiently small values of ε. Let us assume that ε_1 has been chosen small enough so that both inequalities (36) and (37) are fulfilled simultaneously. Then, using the well known Dynkin formula [2] and inequalities (36)-(37) one can obtain the inequality

$$\begin{aligned} \mathbf{E}\Big\{|x^\varepsilon(t)|^p|x^\varepsilon(s) = \vec{u},\, y(s/\varepsilon) \;&=\; y)\Big\} \leq= \frac{1}{w_1}\Big\{w(t,\, x,\, y,\, \varepsilon) + \\ &\int_s^t \mathbf{E}\Big\{\mathcal{L}w(\tau,\, x^\varepsilon(\tau),\, y(\tau/\varepsilon,\, \varepsilon)|x^\varepsilon(s) = \vec{u},\, y(s/\varepsilon)\Big\}d\tau\Big\} \end{aligned}$$

Therefore, the conditional p-moment of the solution of (23) satisfies the inequality

$$\mathbf{E}\{|x^\varepsilon(t)|^p|x^\varepsilon(s) = \vec{u},\, y(s/\varepsilon) = y)\} \leq \frac{w_2}{w_1}|x|^p - \frac{w_3}{w_1}\int_s^t \mathbf{E}\Big\{|x^\varepsilon(\tau)|^p|x^\varepsilon(s) = \vec{u},\, y(s/\varepsilon)\Big\}d\tau$$

whence one can conclude that $\mathbf{E}\{|x^\varepsilon(t)|^p|x^\varepsilon(s) = \vec{u},\, y(s/\varepsilon) = y)\} \leq \beta_1|\vec{u}|^p e^{-\beta_1(t-s)}$ for any $t \geq s \geq 0,\ \vec{u} \in \mathbb{R}^m$ and $\varepsilon \in (0,\ \varepsilon_1)$ with some positive constant β_1. Using this inequality one can evaluate the rate of decay of the second moment of the supremum of the solution of (23) in the time-interval $[-h\varepsilon,\, 0]$ from:

$$\mathbf{E}\Big\{\sup_{t-h\varepsilon\leqslant\tau\leqslant t} |x^\varepsilon(\tau+s)|^2|x^\varepsilon(s) = \vec{u},\, y(s/\varepsilon) = y)\Big\} \leq |\vec{u}|^p + h\varepsilon \int_{t-h\varepsilon}^t \beta_1|\vec{u}|^p e^{-\beta_1\tau}d\tau \leq \beta_2|\vec{u}|^p$$

for any $t \geq h\varepsilon,\ x \in \mathbb{R}^m,\ y \in \mathbf{Y}$ and $\varepsilon \in (0,\ \varepsilon_1)$. It can be easily seen that using this formula the previous inequality can be rewritten in the form

$$\sup_{\substack{s\,\geqslant\,0,\\ y\,\in\,\mathbf{Y}}} \mathbf{E}^s_{y,\vec{u}}\Big\{\sup_{t-h\varepsilon\leqslant\tau\leqslant t} |x^\varepsilon(\tau+s)|^p|x^\varepsilon(s) = \vec{u},\, y(s/\varepsilon) = y)\Big\} \leq \beta_2|\vec{u}|^p e^{-\beta_1 t}.$$

Since the initial condition $\vec{u}$ of equation (23) is the projection of the initial condition (9), it follows from the above inequality that:

$$\sup_{\substack{s \geqslant 0,\\ y \in \mathbf{Y}}} \mathbf{E}^s_{y,\varphi}\Big\{ \sup_{t-h\varepsilon \leqslant \tau \leq t} |x^\varepsilon(\tau+s)|^p \Big\} = \sup_{\substack{s \geqslant 0,\\ y \in \mathbf{Y}}} \mathbf{E}\Big\{ \sup_{t-h\varepsilon \leqslant \tau \leq t} |x^\varepsilon(\tau+s)|^p | u^\varepsilon(s) = \varphi,\, y(s/\varepsilon) = y) \Big\}$$
$$\leq \beta_3 ||\varphi||^p e^{-\beta_1 t} \tag{38}$$

for any $\varphi \in \mathbf{C}_n$.

By construction the solution of (1)-(9) satisfies the inequalities

$$\sup_{\substack{s \geqslant 0,\\ y \in \mathbf{Y}}} ||u^\varepsilon_{t+s}(s,\,\varphi)|| \leq \sup_{\substack{s \geqslant 0,\\ y \in \mathbf{Y}}} ||\mathbf{r_0}(t+s)|| + \sup_{\substack{s \geqslant 0,\\ y \in \mathbf{Y}}} ||\mathbf{r_1}(t+s)|| \leq$$
$$\leq h_1 \sup_{\substack{s \geqslant 0,\\ y \in \mathbf{Y}}} \big(|\bar{z}^\varepsilon(t+s) - \tilde{z}^\varepsilon(t+s)| + |\tilde{z}^\varepsilon(t+s)| \big) + \sup_{\substack{s \geqslant 0,\\ y \in \mathbf{Y}}} ||\mathbf{r_1}(t+s)||$$

with some positive constant h_1. Taking into account the definition of the initial condition $\vec{u}$ given by (17) and the formulas (13),(26) and (38), one can find sufficiently large $A(T)$ as $T \to \infty$ and infinitesimal $\alpha(\varepsilon)$ as $\varepsilon \to 0$ such that

$$\sup_{\substack{s \geqslant 0,\\ y \in \mathbf{Y}}} \mathbf{E}^{(s)}_{\varphi,y}\Big\{ ||u^\varepsilon_{s+T/\varepsilon}||^p \Big\} \leq \big(\alpha(\varepsilon) A(T) + \beta e^{-\beta_1 T}\big) ||\varphi||^p$$

for any $\varphi \in \mathbf{C}_n$ with some positive constant β. Choosing the numbers $T = \frac{ln 4\beta}{\beta_1}$ and ε_0 such that
$\alpha(\varepsilon_0) A(\frac{ln 4\beta}{\beta}) = \frac{1}{4}$ this inequality can be rewritten in the form

$$\sup_{\substack{s \geqslant 0,\\ y \in \mathbf{Y}}} \mathbf{E}^{(s)}_{\varphi,y}\Big\{ ||u^\varepsilon_{s+T/\varepsilon}||^p \Big\} \leq \frac{1}{2} ||\varphi||^p. \tag{39}$$

Next, we apply the Markov property for conditional expectation in the form

$$\mathbf{E}^{(s)}_{\varphi,y}\Big\{ ||u^\varepsilon_{s+t_1+t_2}||^p \Big\} = \mathbf{E}^{(s)}_{\varphi,y}\Big\{ \mathbf{E}^{(\tau)}_{\psi,z}\Big\{ ||u^\varepsilon_{\tau+t_2}||^p \Big\}\Big|_{\tau = t_1+s, \psi = u^\varepsilon_{s+t_1}, z = y_{s+t_1}} \Big\}$$

This allows us to use the inequality (39) and to evaluate the second moment of the norm of the solution (1)-(9) in the recursive form

$$\sup_{y \in \mathbf{Y}} \mathbf{E}^{(s)}_{\varphi,y}\Big\{ ||u^\varepsilon_{s_{k+1}}||^p \Big\} = \sup_{y \in \mathbf{Y}} \mathbf{E}^{(s)}_{\varphi,y}\Big\{ \mathbf{E}^{(\tau)}_{\psi,z} ||u^\varepsilon_{\tau+T/\varepsilon}||^p \Big|_{\tau = s_k, \psi = u^\varepsilon_{s_k}, z = y_{s_k}} \Big\} \leq \frac{1}{2} \sup_{y \in \mathbf{Y}} \mathbf{E}^{(s)}_{\varphi,y}\Big\{ ||u^\varepsilon_{s_k}||^p \Big\}$$

for any given $s > 0$, $k \in \mathbb{N}$ and $\varphi \in \mathbf{C}_n$. Therefore, for $t \in [s_k,\, s_{k+1}),\ k \in \mathbb{N}$ the reiteration of the above inequality allows us to write the following inequalities for the second moment of the solution of (1)-(9)

$$\mathbf{E}^{(s)}_{y,\varphi}\Big\{ |u^\varepsilon(s+t/\varepsilon)|^p \Big\} \leq \sup_{\substack{s_k \leqslant t < s_{k+1},\\ y \in \mathbf{Y}}} \mathbf{E}^{(s)}_{y,\varphi}\Big\{ |u^\varepsilon(s+t/\varepsilon)|^p \Big\} \leq c^p e^{pclT} \mathbf{E}^{(s)}_{y,\varphi}\Big\{ ||u^\varepsilon_{s_k}||^p \Big\} \leq \frac{c^p}{2^k} e^{pclT} ||\varphi||^p$$

or $\mathbf{E}^{(s)}_{y,\varphi}\Big\{ |u^\varepsilon(s+t)|^p \Big\} \leq a_1 e^{-a_2 t \varepsilon} ||\varphi||^p$ with some positive constants a_1, a_2. This completes the proof of our theorem.

4 Stability of deterministic linear functional differential equations with rapid switching

This Section deals with the linear equation of given in an operator form

$$\frac{dx(t)}{dt} = A\left(\frac{t}{\varepsilon}\right) x_t \tag{40}$$

where the linear operators $A(t) : \mathbf{C}_n \to \mathbb{R}^n$ are continuous and uniformly bounded by t, i.e. there exists a such number c that for any $t \in \mathbb{R}$ one may write inequality $||A(t)|| \leq C$. Let us denote $\{X_s^t, t \geq s, s \in \mathbb{R}\}$ the family of shift operators defined by equality $X_s^t \varphi := x(t, s, \varphi$ for a given initial $\varphi \in \mathbf{C}_n$. The main assumption this Section needs is *uniform averaging assumption:*

(Av) there exists such a linear continuous operator $\bar{A} : \mathbf{C}_n \to \mathbf{C}_n$ that $\lim_{t\to\infty} \frac{1}{t} \int\limits_{t_0}^{t_0+t} A(s)\varphi ds = \bar{A}\varphi$ for any $c > 0$ uniformly on $t_0 \in \mathbb{R}$ and

$$\varphi \in \mathbf{K}_c := \{\varphi \in \mathbf{C}_n^1 : ||\varphi|| \leq 1, \max_{-h\leq\theta\leq 0} \left|\frac{d\varphi(\theta)}{d\theta}\right| \leq c\}$$

Side by side with equation (40) we will consider so called *an averaged equation*

$$\frac{dz(t)}{dt} = \bar{A} z_t \tag{41}$$

with resolving semigroup $\{Z^t, t \geq 0\}$ with generator $\mathbb{A}_0$ defined by an equality

$$(\mathbb{A}_0\varphi)(\theta) := \begin{pmatrix} \frac{d\varphi(\theta)}{d\theta}, & \text{if } -h \leq \theta < 0, \\ \bar{A}\varphi, & \text{if } \theta = 0. \end{pmatrix}$$

It is not so difficult to make sure of inequalities $||X_s^t|| \leq e^{C(t-s)}$, $||Z_s^t|| \leq e^{C(t-s)}$ for any $t \geq s$ and $\varepsilon > 0$.

Lemma 3. *If $\sigma(\mathbb{A}_0) \in \{\Re\lambda < 0\}$ than under condition* **(Av)** *for any $T > 0$ and $\delta > 0$ there exists such a constant $\varepsilon_0 > 0$ that $t_0 \in \mathbb{R}, \varepsilon \in (0, \varepsilon_0)$, and$\varphi \in \mathbf{K}_c$ one may write an inequality*

$$g(T, \varepsilon) := \max_{0\leq t\leq T} \left| \int\limits_{t_0}^{t_0+t} \left[A(\left(\frac{\tau}{\varepsilon}\right) - \bar{A}\right] z_\tau(s, \varphi) d\tau \right| \leq \delta.$$

Proof. Let $\{[t_k, t_{k+1}) := t_0 + \frac{kT}{m},\ k = 0, 1, ..., m-1\}$ be a partition of segment $[t_0, t_0 + T]$. Because under condition of Lemma there exists such a constant M that $sup_{t\geq 0}||Z^t|| = M < \infty$ the analyzing function $g(T, \varepsilon)$ may be estimated in a following way

$$\begin{aligned} g(T, \varepsilon) &= \max_{0\leq k\leq m-1} \max_{t_k\leq t_{k+1}} \left| \int\limits_{t_0}^{t_0+t} \left[A(\left(\frac{\tau}{\varepsilon}\right) - \bar{A}\right] z_\tau(s, \varphi)| d\tau \right| \\ &\leq \max_{0\leq k\leq m-1} \left| \int\limits_{t_0}^{t_k} \left[A(\left(\frac{\tau}{\varepsilon}\right) - \bar{A}\right] z_\tau(s, \varphi)| d\tau \right| + \frac{2MTc}{m}. \end{aligned}$$

Let us choose $m \geq \frac{4MTc}{\delta}$. Than

$$\max_{0\leq k\leq m-1}\left|\int\limits_{t_0}^{t_k}\left[A\left(\frac{\tau}{\varepsilon}\right)-\bar{A}\right]z_\tau(s,\varphi)|d\tau\right| = \max_{0\leq k\leq m-1}\left|\sum_{i=0}^{k-1}\int\limits_{t_i}^{t_{i+1}}\left[A(\left(\frac{\tau}{\varepsilon}\right)-\bar{A}\right]z_\tau(s,\varphi)|d\tau\right| \leq$$

$$2cT\max_{0\leq t\leq T/m}||z_{t+\tau+t_0}(s,\varphi)-z_{t+t_0}(s,\varphi)||+TM\max_{\psi\in\mathbf{K}_c,s\in\mathbb{R}}\left|\frac{\varepsilon m}{T}\int\limits_{s}^{s+\frac{T}{m}\varepsilon}(A(\tau)-\bar{A})\psi d\tau\right| \quad (42)$$

Taking number m sufficiently large and applying inequality $\left|\frac{d\varphi(\theta)}{d\theta}\right| \leq c$ one can make the first item in the right part of the inequality (42) less than $\delta/2$ for all $\varphi \in \mathbf{K}_c$. Then due to inequality **(Av)** the second item in the right part of the above inequality (42) may be done less than $\delta/2$ at the expense of smallness of ε.

Theorem 2. *If trivial solution of the equation (41) is asymptotically stable than under condition* **(Av)** *there exists such a positive number* ε_0 *that for any* $\varepsilon \in (0,\varepsilon_0)$ *the trivial solution of equation (40) is exponentially stable, i.e.*

$$\exists M>0,\exists\gamma>0,\forall s\in\mathbb{R},\forall t\geq s,\forall\varphi\in\mathbf{C}_n: \qquad |x(t,s,\varphi)|\leq Me^{-\gamma(t-s)}||\varphi|| \quad (43)$$

Proof. It is well known [4] than under condition of Theorem there exist such positive numbers M_0 and γ_0 that

$$\forall s\in\mathbb{R},\forall t\geq s,\forall\varphi\in\mathbf{C}_n: \qquad |x(t,s,\varphi)|\leq M_0e^{-\gamma_0(t-s)}||\varphi|| \quad (44)$$

where $x(t,s,\varphi)$ is the solution of (41) satisfying initial condition $x_s(s,\varphi)=\varphi$. Let us define continuous function $\psi := x_{s+h}(s,\varphi)e^{-ch}/||\varphi||$ belonging to the compact $\mathbf{K}_c$ for any nontrivial $\varphi \in \mathbf{C}_n$ and any positive number c. Taking this function as initial conditions for equations (40) and (41) we can derive for any $s \in \mathbb{R}, t \geq h$ inequality

$$|x(t+s,s+h,\psi)-z(t+s,s+h,\psi)| \leq c\int\limits_{h}^{T}|x(\tau+s,s+h,\psi)-z(\tau+s,s+h,\psi)|d\tau +$$

$$+\max_{h\leq t\leq T}\left|\int\limits_{s+h}^{s+t}\left(A\left(\frac{\tau}{\varepsilon}\right)z_\tau(s+h,\psi)-Az_\tau(s+h,\psi)\right)d\tau\right|$$

Therefore the left item in the above inequality by Gronwall's lemma [4]and Lemma 3 may be estimated uniformly by $\psi \in \mathbf{K}_c$ in a following form

$$\max_{h\leq t\leq T}|x(t+s,s+h,\psi)-z(t+s,s+h,\psi)| \leq g(T,\varepsilon)e^{cT} \quad (45)$$

This inequality permits for any $t \in [h,T]$ to write

$$\begin{aligned}|x(t+s,s,\varphi)| &\leq (|x(t+s,s+h,\psi)-z(t+s,s+h,\psi)|+|z(t+s,s+h,\psi)|)\,||\varphi||e^{ch} \leq \\ &\leq ||\varphi||e^{c(h+T)}\,(g(T,\varepsilon)+|z(t+s,s+h,\psi)|)\end{aligned}$$

Choosing $T = \frac{ln(4\hat{M})+(c+\hat{\gamma})}{\hat{\gamma}}$ and ε such a small that $g(T,\varepsilon) \le \frac{1}{4}e^{c(T+h)}$ for any $\varepsilon \in (0,\varepsilon_0)$ we can uniformly by $s \in \mathbb{R}$ and $\varphi \in \mathbf{C}_n$ get a formulae $||x_T(s,\varphi)|| \le \frac{1}{2}||\varphi||$, $\max\limits_{0\le t\le T}|x(t+s,s,\varphi)| \le ||\varphi||e^{cT}$ This permits to write for the segments of solutions (40) recurrent inequalities $|x_{kT}(s,\varphi)|| \le \frac{1}{2}||x_{(k-1)T}(s,\varphi)||$, $k = 1,2,...$ and

$$\max_{kT\le t\le (k+1)T}|x(t+s,s,\varphi)| = e^{cT}||x_{kT}(s,x_{(k-1)T+s}(s,\varphi))|| \le e^{cT}\left(\frac{1}{2}\right)^k ||\varphi||, k = 1,2,...$$

that is equivalent [4] to exponential stability of trivial solution of (40).

Example 1. Given scalar functional differential equation with fast oscillated delay

$$\frac{dx(t)}{dt} = ax\left(t-1+h\sin\frac{t}{\varepsilon}\right) \tag{46}$$

where $h \in [-1,1]$. The right part side of (46) may be rewritten in a functional form

$$A(t)\varphi := a\varphi(-1+h\sin t),\ \varphi \in \mathbf{C}_1\,.$$

It is not so difficult to make sure of equality

$$\lim_{\varphi\in\mathbf{K}_c,s\in\mathbb{R}}\int\limits_s^{s+t}\varphi(-1+h\sin\tau)d\tau = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\varphi(-1+h\sin\tau)d\tau$$

for any $c>0$. Now for asymptotic stability analysis of trivial solution of equation (46) with sufficiently small positive ε we may use more simple equation

$$\frac{dz(t)}{dt} = \frac{a}{2\pi}\int\limits_{-\pi}^{\pi}z(t-1+h\sin\tau)d\tau \tag{47}$$

Applying to this equation well known D-partition method [4], one can find for any $h\in[-1,1]$ the region of exponential stability in a form of inequality:

$$\frac{\pi^2}{\int\limits_{-\pi}^{\pi}\cos\left(\frac{\pi h}{2}\sin\theta\right)d\theta} < a < 0\,.$$

5 Limit theorems and stability of functional differential equations with rapid random switching

Let $(\Omega,\mathfrak{F},\mathbb{P})$ be the probability space and $\{\mathfrak{N}_s^t \subset \mathfrak{F}, -\infty < s \le t < \infty\}$ be family of σ-algebras, satisfying assumption $(s,t)\subset(s_1,t_1):\ \mathfrak{N}_s^t \subset \mathfrak{N}_{s_1}^{t_1}$ and strong mixing condition in a form

$$\xi \in \mathfrak{N}_{-\infty}^t, \eta \in \mathfrak{N}_{t+\tau}^{\infty}, |\xi|<1, |\eta|<1: \qquad \sup_{t\in\mathbb{R}}\sup_{\xi,\eta}|\mathbb{E}\xi\eta - \mathbf{E}\xi\mathbf{E}\eta| := \alpha(\tau)$$

with $\lim\limits_{\tau\to 0}\alpha(\tau)=0$. This section deals with functional differential equation with random right part side

$$\frac{x^\varepsilon(t)}{dt}=F\left(\frac{t}{\varepsilon},x_t^\varepsilon,\omega\right) \tag{48}$$

where $F(.,.,.)$ is $\{\mathfrak{N}_t^t, t\in\mathbb{R}\}$-adopted mapping $\mathbb{R}\times\mathbf{C}_n\times\Omega$ to $\mathbb{R}^n$ and ε – small positive parameter. We will assume that with probability one there exists unique solution $\{x_t^\varepsilon(s,\varphi),\ t\geq s\}$ of initial problem

$$\theta\in[-h,0]: \qquad x^\varepsilon(s+\theta,s,\varphi)=\varphi(\theta) \tag{49}$$

for any $s\in\mathbb{R},\varphi\in\mathbf{C}_n$ and this equation satisfies condition of *integral continuity* in a following form: there exists such continuous mapping $\bar{F}:\mathbf{C}_n\to\mathbb{R}^n$ that for any $r>0$ and $T>0$

$$\lim_{\varepsilon\to 0}\left|\int\limits_{t_0}^{t_0+T}\mathbf{E}\left\{F\left(\frac{t}{\varepsilon},x_t^\varepsilon,\omega\right)\right\}-\bar{F}(\varphi)dt\right|=0 \tag{IC}$$

uniformly on $t_0\in\mathbb{R}$ and $\varphi\in S_r:=\{\varphi\in\mathbf{C}_n:||\varphi||\geq r\}$. Side by side with equation (48) we will consider so called *an averaged equation*

$$\frac{x^0(t)}{dt}=\bar{F}(x_t^0) \tag{50}$$

with the same initial condition

$$\theta\in[-h,0]: \qquad x^0(s+\theta,s,\varphi)=\varphi(\theta) \tag{51}$$

The monograph [10] discussed a performance of normalized deviations $z^\varepsilon(t):=[x^\varepsilon(s+t,s,\varphi)-x^0(t,0,\varphi)]/\sqrt{\varepsilon}$ for sufficiently small positive ε. It has been proved that under boundedness condition of mapping F with its two Frechet derivatives and condition

$$\int\limits_0^\infty\tau\alpha(\tau)d\tau<\infty \tag{52}$$

the above normalized deviations are uniformly bounded for all sufficiently small ε, i.e.

$$\exists\varepsilon_0>0,\forall\varepsilon\in(0,\varepsilon_0): \quad \sup_{0\leq t\leq T}\mathbf{E}\{|z^\varepsilon(t)|^2\}\leq c(T,||\varphi||)$$

for any $T>0,\varphi\in\mathbf{C}_n$. It permits to approximate the solution of (48)-(49) on any finite interval by solution of averaged equation (50)-(51) with sufficiently small (of order $\sqrt{\varepsilon}$) uniformly bounded residual. This result is very helpful for discovery of bounded attracting manifolds [4]. But for further qualitative analysis one needs also to know a performance of linearized equation (48)in some neighborhood of possible attractors.

Let the linearized equation (48) has a form

$$\frac{dx(t)}{dt}=A\left(\frac{t}{\varepsilon},\omega\right) \tag{53}$$

where ε is small positive parameter and $A(t,\omega) : \mathbf{C}_n \to \mathbb{R}^n$ is $\mathfrak{N}_t^t$-adopted operators family satisfying boundedness condition $\mathbb{P}\{\sup_{t\in\mathbb{R}} ||A(t,\omega)|| = L < \infty) = 1$ and law of large numbers in a form

$$\lim_{T\to\infty} \mathbb{E}\left\{\sup_{s\geq t} \frac{1}{T}\left|\int_s^{s+T} [A(\tau,\omega)\varphi - \bar{A}\varphi]d\tau\right| \Big/ \mathfrak{N}_{-\infty}^t\right\} = 0 \tag{54}$$

uniformly on $t \in \mathbb{R}, \varphi \in \mathbf{C}_n$.

Theorem 3. *Under the above assumptions an asymptotic stability of the trivial solution of averaged equation*

$$\frac{dy(t)}{dt} = \bar{A}y_t \tag{55}$$

for sufficiently small ε guarantees exponential decreasing of any any solution of (53) in the mean.

Proof. The solutions of linear equation (5) satisfy the stationarity condition [4] in a form $x(t,s,\varphi) = x(t+u, s+u, 0, \varphi)$ for any $t \geq s \in \mathbb{R}, u \in \mathbb{R}$ and $\varphi \in \mathbf{C}_n$. Therefore one may derive for any $T > 0$ an inequality

$$\begin{aligned}\max_{s\leq t\leq T} |x(t,s,\varphi) - y(t,s,\varphi)| \;\;&\leq\;\; L\int_s^T \max_{s\leq\tau\leq\theta} |x(\tau,s,\varphi) - y(\tau+z, s+z, \varphi)|d\theta + \\ &\max_{s\leq t\leq T}\left|\int_s^t \left[A\left(\frac{u}{\varepsilon},\omega\right) y_{u+z}(s+z,\varphi) - \bar{A}y_{u+z}(s+z,\varphi)\right] du\right|\end{aligned} \tag{56}$$

with an arbitrary $z \in \mathbb{R}$. To estimate the second item of the above inequality we break the segment $[s, s+T]$ to pieces $[t_k, t_{k+1}), k = 0, 1, ..., m$ by points $t_k := s + \frac{kT}{m}$. At any interval one can write inequalities

$$\begin{aligned}\max_{t_k\leq t\leq t_{k+1}} &\left|\int_{t_k}^t \left[A\left(\frac{u}{\varepsilon},\omega\right) y_{t_k+z}(s+z,\varphi) - \bar{A}y_{t_k+z}(s+z,\varphi)\right] du\right| \leq \\ &\leq 2\frac{TL}{m}\max_{0\leq u\leq T} ||y_u(0,\varphi) \leq 2\frac{TL}{m}e^{LT},\end{aligned} \tag{57}$$

$$\begin{aligned}\max_{s\leq t\leq s+T} &\left|\int_s^t \left[A\left(\frac{u}{\varepsilon},\omega\right) y_{u+z}(s+z,\varphi) - \bar{A}y_{u+z}(s+z,\varphi)\right] du\right| \leq \\ \leq \max_{s\leq t\leq s+T} &\left|\int_s^t \left[A\left(\frac{u}{\varepsilon},\omega\right) \hat{y}_{u+z}(s+z,\varphi) - \bar{A}\hat{y}_{u+z}(s+z,\varphi)\right] du\right| + \\ &+ \int_0^T ||y_t(0,\varphi) - \hat{y}_t(0,\varphi)||du\end{aligned} \tag{58}$$

where $\hat{y}(t,s,\varphi) := y(t_k,s,\varphi)$ for any $t \in [t_k, t_{k+1}), k = 0,1,2,...,m$. Owing the inequality (57) and tractable inequality $\sup_{t_k\leqslant\tau\leqslant t_{k+1}} |y(\tau,s,\varphi) - y(t_k,s,\varphi)| \leq TLe^{LT}/m$ the inequality (58) may be rewritten in a following form

$$\max_{s\le t\le s+T}\left|\int\limits_s^t \left[A\left(\frac{u}{\varepsilon},\omega\right)y_{u+z}(s+z,\varphi)-\bar{A}y_{u+z}(s+z,\varphi)\right]du\right|\le$$

$$\le \max_{0\le k\le m-1}\left|\int\limits_{t_k}^{t_{k+1}}\left[A\left(\frac{u}{\varepsilon},\omega\right)\hat{y}_{t_k}(s,\varphi)-\bar{A}\hat{y}_{t_k}(s,\varphi)\right]du\right|+2\frac{TL}{m}e^{TL}(1+TL)$$

Under Theorem assumption there exist such positive numbers c_1 and ρ that $||y_t(s,\varphi)||\le c_1E^{-\rho(t-s)}$ for any $\varphi\in\mathbf{C}_n$ and $t\ge s$. Therefore choosing $m=\left[\varepsilon^{-1/2}\right]$ and applying inequality (54) we can make sure of infinite smallness as $\varepsilon\to 0$ for expression

$$g(\varepsilon,T):=T\sup_{s\ge 0}\mathbb{E}\left\{\sup_{r\ge s/\varepsilon}\left|\int\limits_r^{r+\frac{T}{\varepsilon m}}[A(t,\omega)y_r(s,\varphi)-\bar{A}y_r(s,\varphi)]dt\right|\Big/\,\mathfrak{N}_{-\infty}^{\frac{s}{\varepsilon}}\right\}+2\frac{TL}{m}e^{TL}(1+TL)$$

Now for any $T>0$ and $\varphi\in\mathbf{C}_n$ taking into account linearity of initial equation (53) we may estimate an expectation of the left part side of inequality (58)

$$\mathbf{E}\left\{\max_{s\le t\le s+T}\left|\int\limits_s^t\left[A\left(\frac{u}{\varepsilon},\omega\right)y_u(s,\varphi)-\bar{A}y_u(s,\varphi)\right]du\right|\Big/\,\mathfrak{N}_{-\infty}^{\frac{s}{\varepsilon}}\right\}\le g(\varepsilon,T)$$

Therefore by well known Gronwall's inequality the above formula permits estimate the conditional expectation of the difference of solutions in form

$$\mathbb{E}\left\{\max_{s\le t\le s+T}|x(t,s,\varphi)-y(t,s,\varphi)|\Big/\,\mathfrak{N}_{-\infty}^{\frac{s}{\varepsilon}}\right\}\le g(\varepsilon,T)e^{LT}||\varphi|| \qquad (59)$$

Having noted inequality $\max\limits_{0\le t\le T}|x(s+t,s,\varphi)|\le||\varphi||e^{LT}$ to prove Theorem one has to establish exponential decreasing of the sequence $\mathbb{E}\left\{||x_{s+kT}(s,\varphi)||\,/\,\mathfrak{N}_{-\infty}^{\frac{s}{\varepsilon}}\right\}$ as $k\to\infty$. For that we can chose such a large T that $||y_{s+T}(s,\varphi)||\le||\varphi||/4$ and ε_0 such a small that for the above T and any $\varepsilon\in(0,\varepsilon_)$ one may use inequality $g(\varepsilon,T)\le e^{-LT}/4$. Then

$$\begin{aligned}
&\mathbf{E}\left\{||x_{s+kT}(s,\varphi)||\,/\,\mathfrak{N}_{-\infty}^{\frac{s}{\varepsilon}}\right\}=\\
=\;&\mathbf{E}\left\{\mathbf{E}\left\{||x_{s+kT}(s+(k-1)T,x_{s+(k-1)T}(s,\varphi))||\,/\,\mathfrak{N}_{-\infty}^{[s+(k-1)T]/\varepsilon}\right\}/\,\mathfrak{N}_{-\infty}^{\frac{s}{\varepsilon}}\right\}\le\\
\le\;&\mathbf{E}\left\{||y_{s+kT}(s+(k-1)T,x_{s+(k-1)T}(s,\varphi))||\,/\,\mathfrak{N}_{-\infty}^{s/\varepsilon}\right\}+\\
&\qquad+\mathbf{E}\left\{\mathbf{E}\left\{||x_{s+kT}(s+(k-1)T,x_{s+(k-1)T}(s,\varphi))-\right.\right.\\
&\qquad\left.\left.-y_{s+kT}(s+(k-1)T,x_{s+(k-1)T}(s,\varphi))||\;/\,\mathfrak{N}_{-\infty}^{[s+(k-1)T]/\varepsilon}\right\}/\,\mathfrak{N}_{-\infty}^{\frac{s}{\varepsilon}}\right\}\le\\
\le\;&\frac{1}{2}\mathbf{E}\left\{||x_{s+(k-1)T}(s,\varphi)||\,/\,\mathfrak{N}_{-\infty}^{\frac{s}{\varepsilon}}\right\}
\end{aligned}$$

i.e.

$$\mathbf{E}\left\{||x_{s+kT}(s,\varphi)||\,/\,\mathfrak{N}_{-\infty}^{\frac{s}{\varepsilon}}\right\}\le||\varphi||2^{-k}$$

and proof is completed.

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