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ASYMPTOTIC ANALYSIS OF SOME NONLINEAR REGRESSION MODEL WITH MARKOV PROCESS OF POISSON TYPE

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Abstract. In this paper the diffusion approximation is applied for some GARCH-M type impulse dynamical system with Markov switching. Some additional conditions on the processes were introduced.

Key words. dynamical impulse system, GARCH-M model, averaging principle, diffusion approximation.

1 Introduction

Let *x(t), y(t)* be piece-wise processes with jumps in random time moments $\{\tau_j, j \in \mathrm{N}\}$, the same for both processes. In switching moments the process *x(t)* is described by the auto-regressive equation of the first order with mean which is dependent on conditional variance error, i.e. *Generalized Auto-Regressive Conditional Heteroskedastic* model GARCH-M (Engle, Lilien, Robins, 1987) with some additional conditions. It differs from usual ARMA and GARCH processes because switching moments are random, not deterministic and it is the Poisson process.

As in ARMA and GARCH models, let take that in time intervals between switching $t \in [\tau_{j-1}, \tau_j),\ j \in \mathrm{N}$ both processes (*x(t)* and *y(t)*) are constant.
y(t) is the conditional variance of *x(t)*. I.e. the model can be described by the following dynamical system:

$$
\begin{cases}
\dfrac{dx(t)}{dt} = 0, & t \in [\tau_{j-1}, \tau_j),\ j \in \mathrm{N} \quad (1)\\
x(t) = A(y_t)x(t-) + \sigma_t & t \in \{\tau_j,\ j \in \mathrm{N}\} \quad (2)\\
\sigma_t = v_t\sqrt{y_t} & \quad (3)\\
\dfrac{dy(t)}{dt} = 0, & t \in [\tau_{j-1}, \tau_j),\ j \in \mathrm{N} \quad (4)\\
y(t) = a_0 + a_1\sigma_{t-}^2 + by(t-) & t \in \{\tau_j,\ j \in \mathrm{N}\} \quad (5)
\end{cases}
$$

In usual GARCH-M model ν_t is the white noise with mean equal to zero and variance equal to one. We will take $E\nu_t = 0$, $E\nu_t^2 = 1$ too. Then the unconditional and conditional mean of error is equal to zero: $E\sigma_t = E_{t-}\sigma_t = 0$.

The conditional variance $D_{t-}x(t) = E_{t-}\sigma_t^2 = y_t$ is not a constant, but it is GARCH(1,1) process, given by the formula (5). Due to the condition of the convergence to the stationary process all the coefficients of equation (5) need to be positive and $a_1 + b < 1$.

Let introduce the additional restriction, that the ν_t can obtain only two values.

Let ν_t is Markov process and its infinitesimal matrix is $Q = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}$. One can obtain the invariant measure of ν_t: $\mu = \{\mu_1, \mu_2\} = \left\{ \frac{q_2}{q_1 + q_2}; \frac{q_1}{q_1 + q_2} \right\}$. Then ν_t can be given as:

$$\nu_t = \begin{cases} \sqrt{q_1 / q_2} & \text{with prob. } \frac{q_2}{q_1 + q_2} \\ -\sqrt{q_2 / q_1} & \text{with prob. } \frac{q_1}{q_1 + q_2} \end{cases}$$

2 Materials and Methods

Let introduce some small positive parameter in the system (1)-(5) and rewrite the system for the process $z(t) = \varepsilon x(t)$. Let A has the form $A = 1 + \varepsilon A_1$. Let assume, that the coefficients a_0, a_1 are small, but b is near to 1 (but less than 1), i.e. $a_0 = \varepsilon\alpha_0$, $a_1 = \varepsilon\alpha_1$, $b - 1 = \varepsilon\beta$ (β <0).

After these notations our system can be rewritten in the form:

$$\begin{cases} \frac{dz(t)}{dt} = 0 & t \in [\tau_{j-1}, \tau_j),\ j \in \mathrm{N} \quad (6) \\ z(t) = z(t-) + \varepsilon g_1(z(t-), y(t-), \nu(t)) & t \in \{\tau_j,\ j \in \mathrm{N}\} \quad (7) \\ \frac{dy(t)}{dt} = 0 & t \in [\tau_{j-1}, \tau_j),\ j \in \mathrm{N} \quad (8) \\ y(t) = y(t-) + \varepsilon g_2(y(t-), \nu_t) & t \in \{\tau_j,\ j \in \mathrm{N}\} \quad (9) \end{cases}$$

where

$$g_1(z(t-), y(t), \nu(t)) = A_1(y(t)) z(t-) + \nu(t)\sqrt{\alpha_0 + (\alpha_1\nu_{t-}^2 + \beta) y(t-)},$$
$$g_2(y(t-), \nu(t)) = \alpha_0 + (\alpha_1\nu_{t-}^2 + \beta) y(t-)$$

Let assume, that the system (8)-(9) doesn't dependent on z. So we can analyze the solution of (8)-(9) at first, and then (6)-(7).

One can use the averaging principle [1] for (8)-(9), i.e. write the averaging equation and the solution of the system (8)-(9) tends to the solution of this averaging equation, as $\varepsilon \to 0$. The averaging equation has the form:

$$\frac{d\bar{y}(t)}{dt} = \bar{F}_2(\bar{y})$$

$$\bar{F}_2(y) = \int_Y a(v) \int_Y g_2(y, v, \tilde{v}) P(v, d\tilde{v}) \mu(dv)$$

where $a(v) = \{q_1, q_2\}$ are transition probabilities of Markov process v_t, Y is the range of definition of v_t: $Y = \left\{ \sqrt{\frac{q_1}{q_2}}, -\sqrt{\frac{q_2}{q_1}} \right\}$.

After integration we obtain:

$$\bar{F}_2(y) = \frac{q_1 q_2}{q_1 + q_2} \left(2\alpha_0 + 2\beta y + \alpha_1 \frac{q_1^2 + q_2^2}{q_1 q_2} \right)$$

The averaging equation for the system (6)-(7) can be obtained in the similar way:

$$\frac{d\bar{z}(t)}{dt} = \bar{F}_1(\bar{z})$$

$$\bar{F}_1(z) = \int_Y a(v) \int_Y g_1(y, v, \tilde{v}) P(v, d\tilde{v}) \mu(dv)$$

After integration we have:

$$\bar{F}_1(y) = \frac{q_1 q_2}{q_1 + q_2} \left(2A(y)z - \sqrt{\frac{q_2}{q_1}} \sqrt{\alpha_0 + \left(\alpha_1 \frac{q_1}{q_2} + \beta \right) y} + \sqrt{\frac{q_1}{q_2}} \sqrt{\alpha_0 + \left(\alpha_1 \frac{q_2}{q_1} + \beta \right) y} \right)$$

It is important the case, when $F_1(z) \equiv 0$, $F_2(y) \equiv 0$.

The diffusion approximation for systems in form (8)-(9) (impulse system with Markov switching) was proved in works of Y.Tsarkov. For impulse systems with look-ahead Markov switching (as the system (6)-(7)) the diffusion approximation was proved in my thesis [2] So we have the

THEOREM (about diffusion approximation)

If all above mentioned conditions are fulfilled, then the solution of the system (6)-(7) Z(t) weakly converges, as $\varepsilon \to 0$, *to the solution* $\tilde{Z}(t)$ *of the following diffusion equation:*

$$d\tilde{Z}(t) = b_1(\tilde{Z})dt + \sigma_1(\tilde{Z})dw_1(t)$$

$$b_1(Z) = \int_Y G_1\{D_Z \Pi[F_1(Z, Y, v)]\} \mu(dv),$$

$$\frac{1}{2}\sigma_1^2(Z) = \int_Y \frac{a(v)}{2} \int_Y g_1^2(Z, Y, v, \tilde{v}) P(v, d\tilde{v}) \mu(dv) + \int_Y G_1\{\Pi[F_1(Z, Y, v)]\} \mu(dv),$$

where $F_1(Z, Y, v) = a(v) \int_Y g(Z, Y, \tilde{v}) P(v, d\tilde{v})$,

Π *is a potential and*

$$G_1 u(Z,Y,v) = a(v)\int_{Y} g(Z,Y,v,\tilde{v}) u(Z,Y,v,\tilde{v}) P(v,d\tilde{v}).$$

3 Results

Let take, that $F_2(y) \equiv 0$, i.e.

$$\frac{q_1 q_2}{q_1+q_2}\left(2\alpha_0 + 2\beta y + \alpha_1 \frac{q_1^2+q_2^2}{q_1 q_2}\right) \equiv 0, \qquad (10)$$

We apply the diffusion approximation for the system (8)-(9) and obtain the diffusion equation in the form:

$$d\tilde{Y}(t) = b_2(\tilde{Y})dt + \sigma_2(\tilde{Y})dw_2(t)$$

After some mathematical manipulations, which are not difficult, but rather cumbersome, one can obtain the coefficients b_2 and σ_2, linear dependent of y.

$$b_2(y) = B_1 + B_2 y,$$

where

$$B_1 = \frac{q_1 q_2 \alpha_0}{(q_1+q_2)^2}\left(\alpha_1\left(\frac{q_2^2}{q_1} - \frac{q_1^2}{q_2}\right) + \beta(q_2 - q_1)\right),$$

$$B_2 = \frac{q_1 q_2}{(q_1+q_2)^2}\left(\alpha_1 \frac{q_1}{q_2} + \beta\right)\left(\alpha_1\left(\frac{q_2^2}{q_1} - \frac{q_1^2}{q_2}\right) + \beta(q_2 - q_1)\right),$$

$$\sigma_2(y) = S_1 + S_2 y,$$

where

$$S_1 = \left(2\alpha_0 + \alpha_1 \frac{q_1^2+q_2^2}{q_1 q_2}\right)\sqrt{\frac{q_1 q_2}{q_1+q_2}},$$

$$S_2 = 2\beta\sqrt{\frac{q_1 q_2}{q_1+q_2}}.$$

Similar assume $F_1(z) \equiv 0$, i.e.

$$\frac{q_1 q_2}{q_1+q_2}\left(2A(y)z - \sqrt{\frac{q_2}{q_1}}\sqrt{\alpha_0 + \left(\alpha_1 \frac{q_1}{q_2} + \beta\right)y} + \sqrt{\frac{q_1}{q_2}}\sqrt{\alpha_0 + \left(\alpha_1 \frac{q_2}{q_1} + \beta\right)y}\right) \equiv 0$$

and we obtain the diffusion equation for the system (6)-(7):

$$d\tilde{Z}(t) = b_1(\tilde{Z})dt + \sigma_1(\tilde{Z})dw_1(t)$$

After some transformations taking into account condition (10) we obtain:

$$b_1(z) = \frac{q_1 q_2 (q_2 - q_1)}{q_1+q_2} A(y) z;$$

$$\sigma_1(z) = A(y)z\sqrt{\frac{2q_1q_2(1-q_1-q_2)}{q_1+q_2}}$$

Let notate

$$D_1 = \frac{q_1q_2(q_2-q_1)}{q_1+q_2},$$

$$D_2 = \sqrt{\frac{2q_1q_2(1-q_1-q_2)}{q_1+q_2}}$$

Then our diffusion equations will be following:

$$d\widetilde{Z}(t) = D_1A(Y)\widetilde{Z}\,dt + D_2A(Y)\widetilde{Z}\,dw_1(t), \quad (11)$$

$$d\widetilde{Y}(t) = \left(B_1 + B_2\widetilde{Y}\right)dt + \left(S_1 + S_2\widetilde{Y}\right)dw_2(t). \quad (12)$$

4 Discussion

One interesting example can be obtained, if we take $A_1(y)=A_1$ and the parameters of infinitesimal matrix are equal, i.e. $q_1 = q_2 = q$: $Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix}$. Then the drift coefficients in both diffusion equations are equal to zero. The diffusion equations have the form:

$$d\widetilde{Z}(t) = A\widetilde{Z}\sqrt{q(1-2q)}\,dw_1(t)$$

$$d\widetilde{Y}(t) = \sqrt{2q}\,\beta\widetilde{Y}\,dw_2(t)$$

These equations are independent. One can analyse their solutions by using simulation in MATHEMATICA. On the following Figures 1 and 2 you can see five simulations for each diffusion process ($\widetilde{Y}$ and $\widetilde{Z}$) if ***q=0.4, A=1, β=-0.90***

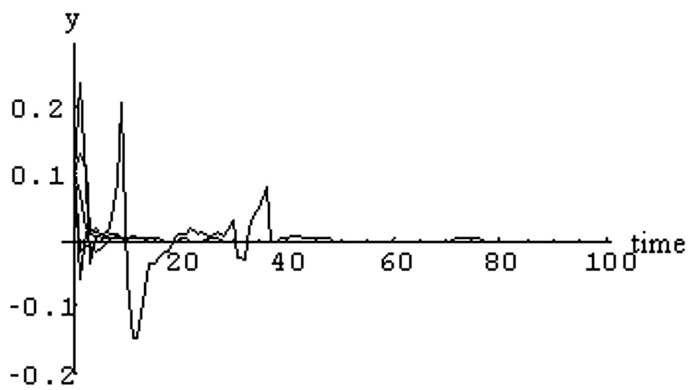

Figure 1.

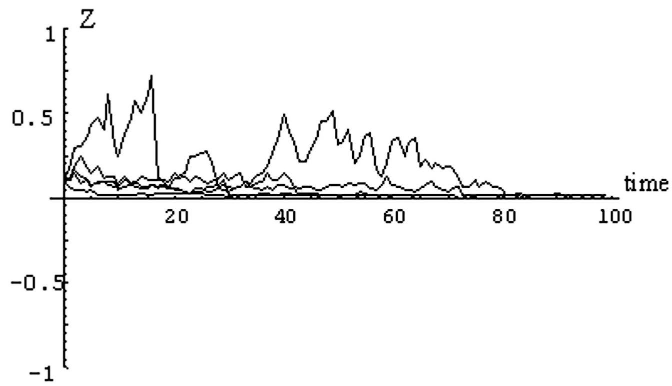

Figure 2.

The most of the trajectories converge to zero, but not all of them. We obtain them after more simulations.
We can analyse the solutions of equations (11) and (12) too. Let take $\beta = -0.9;\ A = 1;\ q_1 = 0.6;\ q_2 = 0.3;\ \alpha_0 = 0.1;\ \alpha_1 = 0.25$
Trajectories of equation (12) oscillate near some average value, as for example in the Figure 3.

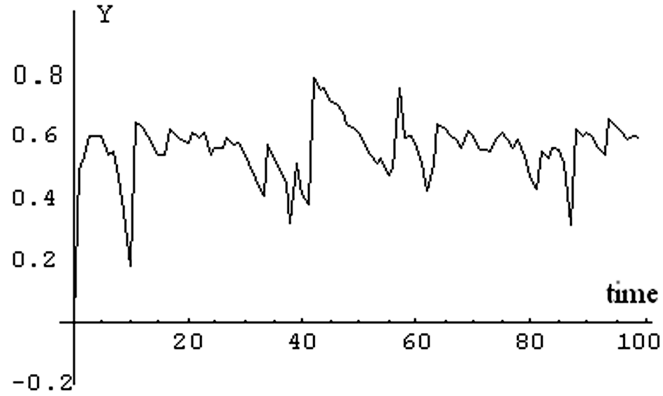

Figure 3.

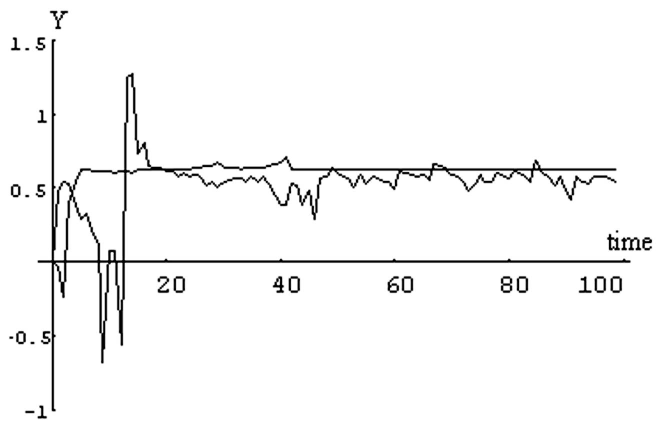

Figure 4.

What the value is it, becomes clear, if we draw the solution of the equation (12) together with the same equation, but without any drift as in Figure 4.

Conclusion

So, we reduced the solutions of the initial system of equations (1)-(5) to the solutions of diffusion equations, the behaviour of which one can analyse by modelling, for example, in the package of MATHEMATICA.
By the analysis of the solutions of these diffusion equations under different values of the initial parameters it is possible to find out different conformities to the law.

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