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APPLICATION OF THE BOUNDED STOCHASTIC PROCESSES WITH TERMINAL CONDITION IN EXCHANGE RATE MODELING

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Abstract. In our article we use regulated stochastic proceses for modelling of fundamental determinants of exchange rate. It is assumed in the model that exchange rate is a function of the fundamental and time. Another essential assumptions of the model is that the fundamental process is bounded inside the band [-f,f] and that terminal condition for the exchange rate holds. Using the Ito's lemma, we obtain parabolic partial differential equation for exchange rate. We specified fundamental in two ways: as regulated Brownian motion and as regulated Ornstein-Uhlenbeck process. For the case of Brownian motion fundamental the analytical solution of the problem was obtained, whereas for the Ornstein-Uhlenbeck fundamental an analytical solution does not exist, therefore we had to use numerical method for solving of the problem.Our model corresponds to the situation, when a country is going to join a currency area at some moment t in the future. The issue is actual for the new EU member countries, intending to entry Euro zone.

Key words. Ornstein-Uhlenbeck process, Ito's lemma, Kummer function

Mathematics Subject Classification: 60H30

1 Introduction

One of the most important problems in macroeconomics is developing of currency exchange rate models. Models help to understand exchange rate dynamics as well as interaction of an exchange rate with other macroeconomic variables. There are two main types of exchange rate regimes: free-floating and fixed to some currency or currency basket. So-called target zones are hybrid of the two. When maintaining target zone policy, monetary authorities commit themselves to keep an exchange rate to definite currency or basket of currencies inside definite bands around some parity value.

The most successful exchange rate target zone model was proposed by P.Krugman (see[4]). Krugman has assumed, that the exchange rate, similarly to the prices of other assets, depends linearly on aggregated "fundamental", including various fundamental determinants of a currency rate (such as the domestic output, the money supply, the foreign interest rates, the price levels etc.) and on the expected change in future exchange rate. The fundamental contains two components: exogenous and stochastic component, called "velocity" and money supply, controlled by the central bank. Krugman assumed that velocity is Brownian motion without a drift. In his model Krugman substantially used assumption that target zone exists forever. This assumption is not realistic, because a target zone typically exists finite period of time.

In our work we generalize Krugman's model for the case when target zone exists finite time, taking into account a terminal condition on exchange rate. An example of such situation is ERM II, which suppose that countries participating in this mechanism have to keep ±15% wide target zone against euro. After successful fulfillment of the latter and other Maastriht criterions, these countries will abandon national currencies and will enter euro currency zone. We specified fundamental in two ways: as regulated Brownian motion and as regulated Ornstein-Uhlenbeck process. For the case of Brownian motion fundamental the analytical solution of the problem was obtained, whereas for the Ornstein-Uhlenbeck fundamental an analytical solution does not exist, therefore we had to use numerical method for solving of the problem.

The structure of the paper is as follows. In section 2 target zone model with terminal condition and Brownian motion fundamental is derived and solved. Section 3 contains results for the case of Ornstein-Uhlenbeck fundamental. Section 4 concludes.

2 Target zone model with a terminal condition.

The standard monetary exchange rate model was employed ([4]). In the model it is assumed that exchange rate $e(\tau)$ depends on fundamental $f(\tau)$, and on expected change in the exchange rate:

$$e(\tau) = f(\tau) + \alpha E_\tau[de(\tau)/d\tau], \qquad \alpha > 0, \tag{1}$$

where $E_\tau[de(\tau)/d\tau] = \lim_{s\to 0+} \dfrac{\mathrm{E}_\tau[e(\tau+s)] - e(\tau)}{s}$, and E_τ is the mathematical expectation operator conditioned on the information available up to time τ.

It is assumed usually that the fundamental may be written as the sum of two components: $f = m + v$, where m represents money supply and v is a composite money demand shock called velocity. It is supposed that velocity is an exogenous stochastic process whereas the money supply is a stochastic process under direct control of monetary authorities. So the stochastic process for the fundamental obeys

$$df(\tau) = dm(\tau) + dv(\tau),$$

where dm represents the effect of foreign exchange market interventions on domestic money supply. Velocity is assumed to follow a Brownian motion without drift and instantaneous standard deviation σ:

$$dv(\tau) = \sigma dw(\tau),$$

where $w(\tau)$ is a standard Wiener process with $E_\tau(dw) = 0$ and $E_\tau(dw^2) = d\tau$.

Assume that foreign exchange market interventions, which directly affect the money supply, are undertaken to prevent the fundamental to move outside a specified band for the fundamental. Hence we assume that there exist lower and upper bounds for the fundamental, such that the fundamental fulfills

$$\underline{f} < f(\tau) < \overline{f}.$$

These interventions can be represented by:

$$dm = dL - dU,$$

where $dL \geq 0$, $dU \geq 0$. dL represents increase in money supply in case $f = \underline{f}$, and dU represents reductions in money supply in case $f = \overline{f}$. Once the fundamental moves inside the band, the interventions cease. This implies that the fundamental is a regulated Brownian motion process [2]. So the fundamental follows the equation

$$df(\tau)=\sigma dw - dU + dL.$$

Assume, that exchange rate is a function of fundamental f and time τ, e.g. $e = e(\tau, f)$.
We also assume that $e(\tau, f)$ obeys "*smooth pasting conditions*" (see [4],[2]):

$$\frac{\partial e}{\partial f}(\tau, \underline{f}) = \frac{\partial e}{\partial f}(\tau, \bar{f}) = 0 \tag{2}$$

"Regulators" U and L are monotonous continuous processes, hence, processes with bounded variation. Therefore, m is a process with bounded variation, too. This implies that f is an Itô process [5].

Now consider that function $e: R_+ \times R \to R$ is continuously differentiable with respect to the first argument and twice continuously differentiable with respect to the second argument. Applying Itô's lemma, we obtain:

$$de = (\frac{\partial e}{\partial \tau} + \frac{\sigma^2}{2}\frac{\partial^2 e}{\partial f^2})d\tau + \frac{\partial e}{\partial f}\sigma dw . \tag{3}$$

Taking mathematical expectation from both sides of equation (3) and noting that $E_\tau(dw)=0$, we have:

$$E_\tau(de) = (\frac{\partial e}{\partial \tau} + \frac{\sigma^2}{2}\frac{\partial^2 e}{\partial f^2})d\tau .$$

Which imply:

$$\frac{E_\tau(de)}{d\tau} = \frac{\partial e}{\partial \tau} + \frac{\sigma^2}{2}\frac{\partial^2 e}{\partial f^2} . \tag{4}$$

Substituting (4) into equation (1) gives

$$e = f + \alpha\left(\frac{\partial e}{\partial \tau} + \frac{\sigma^2}{2}\frac{\partial^2 e}{\partial f^2}\right)$$

By re-arranging the equation terms we get

$$\frac{\partial e}{\partial \tau} + \frac{\sigma^2}{2}\frac{\partial^2 e}{\partial f^2} - \frac{1}{\alpha}e = -\frac{1}{\alpha}f \ . \tag{5}$$

Suppose that at some definite moment T in the future a country, which maintained exchange rate target zone regime, will enter currency zone of pegging currency. In such a case at time T exchange rate doesn't depend on f. This means, that the *terminal condition*:

$$e(T,f) = 0 \tag{6}$$

holds. It is obvious, that this condition is necessary to avoid arbitrage opportunity at the moment of entering the currency zone.

We apply transformation $t = T - \tau$ and rewrite the equation (5) in the form:

$$\frac{\partial e}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 e}{\partial f^2} + \frac{1}{\alpha}e = \frac{1}{\alpha}f \tag{7}$$

Condition (6) will change to

$$e(0,f) = 0. \tag{8}$$

smooth pasting conditions are not affected by the transformation:

$$\frac{\partial e}{\partial f}(t, \underline{f}) = \frac{\partial e}{\partial f}(t, \bar{f}) = 0 \tag{9}$$

The system (7)-(9) is an initial-boundary problem for the parabolic equation. We will seek for solution in the form:

$$e(t,f) = e^*(t,f) + \hat{E}(f), \tag{10}$$

where $\hat{E}(f)$ is a solution to the following stationary problem:

$$\frac{\alpha\sigma^2}{2}\frac{\partial^2 \hat{E}(f)}{\partial f^2} + f = \hat{E}(f), \tag{11}$$

$$\frac{\partial \hat{E}}{\partial f}(\underline{f}) = \frac{\partial \hat{E}}{\partial f}(\overline{f}) = 0 \ . \tag{12}$$

In the case of a symmetric fundamental band, $(\underline{f} = -\overline{f})$, there is a symmetric solution (see [4]):

$$\hat{E}(f) = f - \frac{\operatorname{sh}(\beta f)}{\beta \operatorname{ch}(\beta \overline{f})} \tag{13}$$

where $$\beta = \sqrt{\frac{2}{\alpha\sigma^2}}.$$

e(t,f) from equation (10) is substituted in equation (7) and we obtain for e^*:

$$\frac{\partial e^*}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 e^*}{\partial f^2} - \frac{\sigma^2}{2}\frac{\partial^2 \hat{E}}{\partial f^2} + \frac{1}{\alpha}e^* + \frac{1}{\alpha}\hat{E} = \frac{1}{\alpha}f \tag{14}$$

Taking into account (8)-(10), (11), (12), we arrive at the following system:

$$\frac{\partial e^*}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 e^*}{\partial f^2} + \frac{1}{\alpha}e^* = 0, \tag{15}$$

$$\frac{\partial e^*}{\partial f}(\bar{f}, t) = \frac{\partial e^*}{\partial f}(-\bar{f}, t) = 0, \tag{16}$$

$$e^*(0, f) = -\hat{E}(f). \tag{17}$$

The system (15)-(17) describes an initial-boundary problem for the parabolic equation. We construct the solution of the problem (15)-(17) as Fourier series:

$$e^*(t, f) = \sum_{k=1}^{\infty} C_k \exp\left(-\left(\lambda_k + \frac{1}{\alpha}\right)t\right)\sin\left(\frac{\sqrt{2\lambda_k}}{\sigma}f\right), \tag{18}$$

where $\lambda_k = \dfrac{\left(\dfrac{\pi}{2} + \pi k\right)^2 \sigma^2}{2\bar{f}^2}$ and C_k are Fourier coefficients for the function $(-\hat{E}(f))$:

$$C_k = \frac{(-1)^k 2\bar{f}}{(\pi/2 + \pi k)^2 + \dfrac{2\bar{f}^2}{\alpha\sigma^2}} - \frac{(-1)^k 2\bar{f}}{(\pi/2 + \pi k)^2}. \tag{19}$$

3 Mean-reverting fundamentals

Assume that fundamentals follow regulated Ornstein-Uhlenbeck process. Then stochastic differential equation for fundamental has the following form:

$$df(\tau) = -\rho(f(\tau) - \mu)d\tau + \sigma dw(\tau) - dU + dL, \tag{20}$$

where ρ and σ are positive constants; μ is the long-run level of f and ρ is the speed of adjustment of the process towards μ. dU and dL are as described in the previous section and reflect the effect of marginal interventions on money supply, while the term $-\rho(f(\tau) - \mu)d\tau$ describes intramarginal interventions.

Applying Itô's lemma to (20) and performing manipulations, similar to that of section 1, we arrive to the system:

$$\frac{\partial e}{\partial t} + \rho\big(f(t) - \mu\big)\frac{\partial e}{\partial f} - \frac{\sigma^2}{2}\frac{\partial^2 e}{\partial f^2} + \frac{1}{\alpha}e = \frac{1}{\alpha}f(t) \tag{21}$$

$$e(0,f) = 0 \tag{22}$$

$$\frac{\partial e}{\partial f}(t,\underline{f}) = \frac{\partial e}{\partial f}(t,\overline{f}) = 0 \,. \tag{23}$$

We will seek for solution in the form:

$$e(t,f) = e^*(t,f) + \hat{E}(f)\,, \tag{24}$$

where $\hat{E}(f)$ is a solution to the following stationary problem:

$$\frac{\alpha\sigma^2}{2}\frac{\partial^2 \hat{E}(f)}{\partial f^2} - \alpha\rho(f-\mu)\frac{\partial \hat{E}(f)}{\partial f} - \hat{E}(f) = -f \tag{25}$$

$$\frac{\partial \hat{E}}{\partial f}(\underline{f}) = \frac{\partial \hat{E}}{\partial f}(\overline{f}) = 0 \,. \tag{26}$$

It is not difficult to obtain the general solution of the stationary equation (25):

$$\hat{E}(f) = C_1 M\left(\left[\frac{1}{2\alpha\rho}\right],\left[\frac{1}{2}\right],\frac{\rho(\mu-f)^2}{\sigma^2}\right) + C_2\frac{\sqrt{\rho}(\mu-f)}{\sigma}M\left(\left[\frac{1+\alpha\rho}{2\alpha\rho}\right],\left[\frac{3}{2}\right],\frac{\rho(\mu-f)^2}{\sigma^2}\right) + \frac{\alpha\rho\mu+f}{1+\alpha\rho}$$

where $M([a],[b],[z])$ is the confluent hypergeometric Kummer function ([1]); C_1 and C_2 are integration constants.

For symmetric case (when μ=0 and $\underline{f} = -\overline{f}$) the constant C_1=0 and the values of C_2 and $\bar{f}$ can be obtained from the system:

$$\begin{cases} \dfrac{\bar{f}}{1+\alpha\rho} - C_2\dfrac{\sqrt{\rho}\bar{f}}{\sigma}\cdot M\left(\left[\dfrac{1+\alpha\rho}{2\alpha\rho}\right],\left[\dfrac{3}{2}\right],\dfrac{\rho\cdot\bar{f}^2}{\sigma^2}\right) = \bar{e} \\ \dfrac{1}{1+\alpha\rho} - C_2\dfrac{2}{3}\dfrac{\sqrt{\rho}(1+\alpha\rho)\bar{f}^2}{\alpha\sigma^3}\cdot M\left(\left[\dfrac{1+3\alpha\rho}{2\alpha\rho}\right],\left[\dfrac{5}{2}\right],\dfrac{\rho\cdot\bar{f}^2}{\sigma^2}\right) - C_2\dfrac{\sqrt{\rho}}{\sigma}\cdot M\left(\left[\dfrac{1+\alpha\rho}{2\alpha\rho}\right],\left[\dfrac{3}{2}\right],\dfrac{\rho\cdot\bar{f}^2}{\sigma^2}\right) = 0 \end{cases}$$

For the non-stationary problem (21)-(23) analytical solution does not exist. The solution can be obtained only by numerical methods. For this purpose we used package Mathlab7.0. The numerical solution of the system (21)-(23) is shown on the figure 1. The system was solved for parameter values μ=0, ρ =1, σ = 0.1, α = 3, $\bar{e} = 0.01$ and $\underline{e} = -0.01$ (which corresponds to ±1% target zone about parity value).

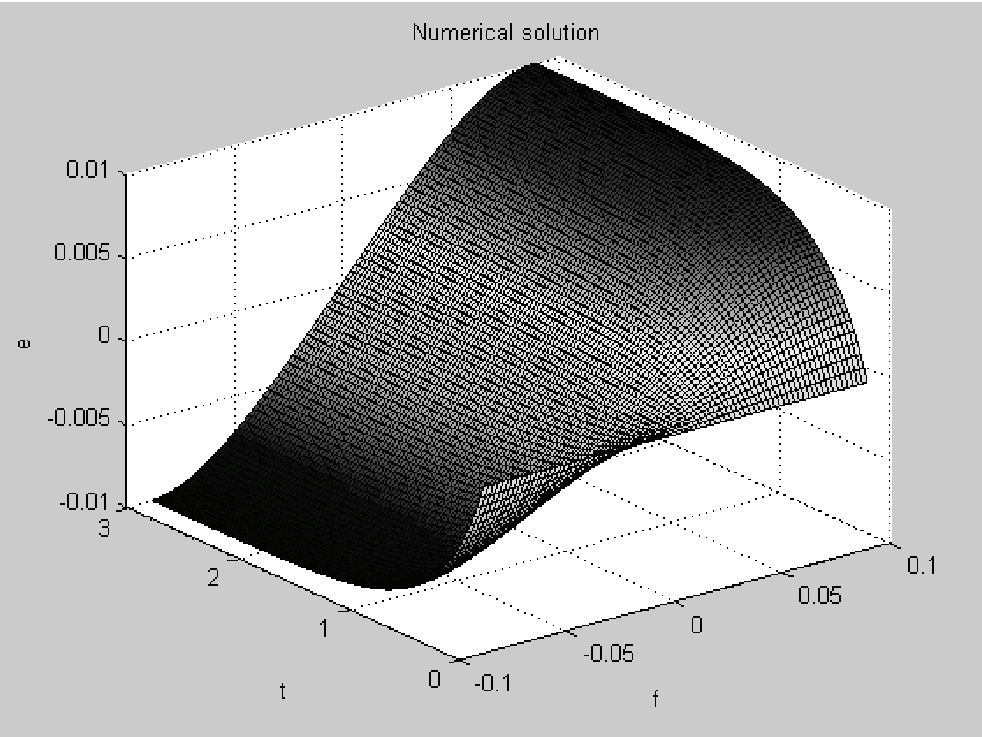

Figure 1. Exchange rate as function of fundamental and time.

4 Conclusions

In the presented work a generalization of Krugman's Target Zones model for the case of time-dependent solution with terminal condition has been proposed. Fixing of the terminal value of the exchange rate corresponds to joining a currency area.

The main assumptions of the model are that Central bank, using marginal interventions, does not allow the fundamentals to exceed the interval $[-\bar{f}, \bar{f}]$ bounds, so-called *"smooth pasting conditions",* and zero terminal condition.

We specified fundamental in two ways: as regulated Brownian motion and as regulated Ornstein-Uhlenbeck process. For the case of Brownian motion fundamental the analytical solution of the problem was obtained. For the Ornstein-Uhlenbeck fundamental only stationary problem has analytical solution; for the non-stationary problem an analytical solution does not exist, therefore we had to use numerical method for solving of the problem.

The model with fundamental, described by regulated Brownian motion, corresponds to the monetary policy with only marginal intervention. The other model with fundamental, described by regulated Ornstein-Uhlenbeck process corresponds to the monetary policy that allows intramarginal interventions. The proposed models can be used by monetary authorities, maintaining target zone currency policy and intending to enter a currency zone.

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