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SECOND LYAPUNOV METHOD FOR ASYMPTOTIC ANALYSIS OF LINEAR MARKOV IMPULSE DYNAMICAL SYSTEM

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Abstract. The object of investigations is n-dimensional linear impulse dynamical system with Markov switching. Its characteristics are dependent on a small positive parameter. It is proven, that the covariation operator family of this system can be represented as a positive semigroup in the specially chosen Banah space with a reproducing cone. This property permits to formulate necessary and sufficient condition for the mean square stability as a problem of positive solvability of the specially constructed Lyapunov equation. An algorithm for decomposition of the solution of above Lyapunov equation in a Laurent's series in terms of powers of this parameter is proposed.

Key words. Impulse differential equation, Markov dynamical system, Lyapunov equation, mean square stability

Mathematics Subject Classification: 34D05, 34E10, 34F05.

1 Introduction

Let $\{y_\varepsilon(t),\ t \ge 0\}$ be series of right continuous homogeneous Markov processes [1] on a countable space $\mathrm{Y} \subset R$ depending on parameter $\varepsilon \in (0,1)$ with weak infinitesimal operator Q defined on any element of the space **V** of bounded mapping $v\colon \mathrm{Y} \to R$

$$Qv(y) := a(y)\sum_{z\in \mathrm{Y}}\big(v(z) - v(y)\big)p(y,z) \tag{1}$$

with bounded uniformly positive intensity

$$\forall y \in \mathrm{Y}: \quad 0 < a_1 \le a(y) \le a_2$$

Let $\{\tau_j,\ j \in \mathrm{N}\}$ be switching moments of the above Markov process. Then we will describe the series of Markov Impulse Dynamical Systems in R^n with:
the phase coordinates $x_\varepsilon(t)$ of this systems satisfy:

- the differential equation

$$\frac{dx_\varepsilon}{dt} A(y_\varepsilon(t), \varepsilon) x_\varepsilon \tag{2}$$

for all $t \in (\tau_{j-1}, \tau_j),\quad j \in N$;

- and the jump equation

$$x_\varepsilon(t) = x_\varepsilon(t-0) + B(y_\varepsilon(t), y_\varepsilon(t-0), \varepsilon) x_\varepsilon(t-0) \tag{3}$$

for all $\{\tau_j,\ j \in \mathrm{N}\}$, where the matrices $A(y,\varepsilon)$, $B(z,y,\varepsilon)$ are defined as the series

$$A(y,\varepsilon) = A_0 + \sum_{k=1}^{\infty} A_k(y)\varepsilon^k, \qquad B(z,y,\varepsilon) = \sum_{k=1}^{\infty} B_k(z,y)\varepsilon^k$$

with matrix coefficients satisfying the inequalities

$$\sup_{y \in Y} \| A_k(y) \| := \alpha_k < \infty, \qquad \sup_{z,y \in Y} \| B_k(z,y) \| := \beta_k < \infty, \quad k \in N \tag{4}$$

and also the series composed of α_k, β_k are convergent.
It is easy to make sure of existence and uniqueness of the above defined process $x_\varepsilon(t)$ for all $t \geq 0$.

LEMMA 1. [2] The pair $\{x_\varepsilon(t), y_\varepsilon(t)\}$ jointly is the Markov process on the phase space with the weak infinitesimal operator

$$(L_\varepsilon v)(x,y) = (A(y,\varepsilon)x, \nabla)v(x,y) + (Qv)(x,y) + (G_\varepsilon v)(x,y)$$

where

$$(G_\varepsilon v)(x,y) = a(y) \sum_{z \in \mathrm{Y}} (v(x + B(z,y,\varepsilon)x, z) - v(x,z)) p(y,z)$$

(.,.) is scalar product and ∇ is operator-gradient in R^n.

For the analysis of mean square stability conditions of the system (2)-(3) for all sufficiently small $\varepsilon > 0$ we use the second Lyapunov method with special constructed quadratic functional and Kato perturbation theory. With a view to develop this approach the semigroup of the linear continuous shift operators of conditional covariance matrices of the solutions of (2)-(3) is analysing. It is proven that the exponential mean square stability problem of (2)-(3) can be formulated as the problem of the existence of a positive solution of Lyapunov equation in the space of quadratic functionals. This result allows to propose the simple asymptotic algorithm of exponential mean square stability analysis of the MIDS with a small parameter ε.

2 LYAPUNOV EQUATION FOR QUADRATIC FUNCTIONALS

Let us denote by **Q** the space of the symmetric $n \times n$ matrix-valued continuous functions $\{q(y), \ y \in \mathrm{Y}\}$ with the subset $K := \{q \in \mathbf{Q}: \ (q(y)x, x) \geq 0, \ \forall x \in R^n, \ \forall y \in \mathrm{Y}\}$ of nonnegative-definite matrices. It is clearly to see that according to the norm, defined by

$$\| q \| := \sup\{|(q(y)x, x)| : \ y \in \mathrm{Y}, |x| = 1\}$$

the space **Q** is Banah space. For given elements q_1 and q_2 of this space we shall write $q_1 >> q_2$ if $q_1 - q_2 \in K$. It easy to prove that the set K is a reproducing cone in the space **Q** with the set of inner points of K defined as

$$\dot{K} := \{q \in K : \exists c > 0, q >> cI\}.$$

The solution of (2)-(3) with initial conditions $x_\varepsilon(0) = x, \quad y_\varepsilon(0) = y$ may be written in the form $x_\varepsilon(t,s,x,y) = X_\varepsilon(t,s,y)x$, where the family of matrix-valued functions $\{X_\varepsilon(t,s,y), \quad t \geq s \geq 0\}$ satisfies equations (2)-(3) for all $t \geq s \geq 0$ and initial conditions $X_\varepsilon(s,s,y) \equiv I$, $y_\varepsilon(0) = y$.
For any $\varepsilon \in [0,1]$ one can introduce the one-parameter family of operators

$$(T_\varepsilon(t)q)(y) = \mathrm{E}_y^{(s)}\{X_\varepsilon^T(t+s,s,y)q(y_\varepsilon(t+s))X_\varepsilon(t+s,s,y)\}.$$

By definition of the semigroup $T_\varepsilon(t)$ leaves the cone K invariant because if $(q(y)x, x) \geq 0$ for any $y \in Y, \quad x \in R^n$ then

$$(T_\varepsilon(t)q)(y) = \mathrm{E}_y\left(X_\varepsilon^T(t+s,s,y)q(y_\varepsilon(t+s))X_\varepsilon(t+s,s,y)\right) \geq 0 .$$

LEMMA 2. The family of operators $\{T_\varepsilon(t), \ t \geq 0\}$ forms a strongly continuous semigroup with an infinitesimal operator defined by

$$\begin{aligned}(\mathrm{A}_\varepsilon)q(y) = {} & A^T(y,\varepsilon)q(y) + a(y)\sum_{z \in \mathrm{Y}} B^T(z,y,\varepsilon)q(z)p(y,z) + \\ & + A(y,\varepsilon)q(y) + a(y)\sum_{z \in \mathrm{Y}} B(z,y,\varepsilon)q(z)p(y,z) + \\ & + a(y)\sum_{z \in \mathrm{Y}} B^T(z,y,\varepsilon)q(z)B(z,y,\varepsilon)p(y,z) + Qq(y)\end{aligned}$$

LEMMA 3. If the spectrum $\sigma_\varepsilon := \sigma_\varepsilon(\mathrm{A})$ of the operator A_ε is situated in the half-plane $C_\alpha := \{C : \Re\lambda \leq \alpha\}$ and there exists an eigenvalue with the real part equal to α, then α also is an eigenvalue.

THEOREM 1. The solution of (2)-(3) is exponentially square stable if and only if there exists a positive number ρ such that $\sigma_\varepsilon \subset C_{-\rho}$.

THEOREM 2. The solution of (2)-(3) is exponentially mean square stable if and only if there exists $q \in \dot{K}$ and $r \in \dot{K}$ such that

$$\mathrm{A}_\varepsilon q = -r \tag{5}$$

This equation can be called the Lyapunov equation for MIDS (2)-(3).

COROLLARY. MIDS (2)-(3) is exponentially mean square stable if and only if there exists a solution $q \in \dot{K}$ of equation (5) with $r=I$.

3 RESOLUTION OF LYAPUNOV EQUATION

Let us represent the decomposition of the operator A_ε in the following form

$$\mathrm{A}_\varepsilon = G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + \ldots$$

where the linear continuous operators G_0, G_1 and G_2 are defined on an arbitrary element for $q \in \mathrm{Q}$ by equalities:

$$(G_0 q)(y) = A_0^T(y)q(y) + q(y)A_0(y) + (Qq)(y),$$

$$(G_1 q)(y) = A_1^T(y)q(y) + q(y)A_1(y) + a(y)\sum_{z\in \mathrm{Y}}[B_1^T(z,y)q(z) + q(z)B_1(z,y)]p(y,z),$$

$$(G_2 q)(y) = A_2^T(y)q(y) + q(y)A_2(y) + a(y)\sum_{z\in \mathrm{Y}}[B_2^T(z,y)q(z) + q(z)B_2(z,y)]p(y,z) + \\ + a(y)\sum_{z\in \mathrm{Y}} B_1^T(z,y)q(z)B_1(z,y)p(y,z)$$

LEMMA 4. Let the real eigenvalue of the operator G_0 $\lambda_0 \notin \sigma(\mathrm{A}_\varepsilon)$ for all sufficiently small $\varepsilon > 0$. Under the above assumptions there exists positive ε_0 such that the solution q^ε of the equation

$$\mathrm{A}_\varepsilon q^\varepsilon - \lambda_0 q^\varepsilon = f$$

for any $f \in Q$ and $\varepsilon \in (0, \varepsilon_0)$ has the form

$$q^\varepsilon = \sum_{k=-d}^{\infty} \varepsilon^k q_k$$

with some $d \in \mathrm{N}$.

Let us assume that $\lambda_0 = 0$ and let

$$\hat{q}(\varepsilon) = \sum_{k=-d}^{0} \varepsilon^k q_k \ ,$$

$$\tilde{q} = \sum_{k=0}^{\infty} \varepsilon^k q_{k+1} \ ,$$

where $q^\varepsilon = \hat{q}(\varepsilon) + \varepsilon \tilde{q}(\varepsilon)$ is the solution of the equation

$$\mathrm{A}_\varepsilon q^\varepsilon = -I \qquad (6)$$

The proposed algorithm uses the method of equating the coefficients corresponding to the same powers of the parameter ε in the equation (6) and the well known Fredholm alternative.

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