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ON DELAYED STOCHASTIC EXPONENT

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Abstract. The paper deals with Itô first order linear stochastic functional differential equation. It is shown that for time asymptotic analysis of solutions one can successfully employ derived by authors covariation method. This method contemplates to construct the linear operator semigroup, which is acting in the partially ordered space of countable additive symmetric measures. The generator of this semigroup helps to find a positive quadratic functional, that defines necessary and sufficient condition of mean square exponential decreasing for solutions of the above equation.

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1 Introduction

There are many mathematical models of financial econometrics (see [5] and references there) dealing with the scalar linear Itô stochastic differential equation

$$dx(t) = ax(t)dt + \sigma x(t)dw(t) \tag{1}$$

where $\{w(t),\ t \geq 0\}$ is the standard Wiener process, adjusted with filtered probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The researchers mostly refer to the equation (1) the same as to its nontrivial solution as *the stochastic exponent*. Applying the Girsanov's theorem [5] on any finite interval $[0, T]$ one can pass on to martingale probabilistic measure $\mathbb{P}^*$ which is equivalent to $\mathbb{P}$ and to reduce the above equation to the martingale form

$$dx(t) = \sigma x(t)dw^*(t) \tag{2}$$

where $\{w^*(t)\}$ is the standard Wiener process according to $\mathbb{P}^*$. This transformation permits to derive many very important formulae disregarding to the value of coefficient a in the initial equation (1). But to analyze the stochastic dynamic in time of any real object (for example term structure of securities on stock exchange [5]) one has to revert to the initial equation (1) and to deal with the solution of this equation

$$x(t+s) = x(s)\exp\{t\lambda(t)\} \tag{3}$$

where

$$\lambda(t) = a - \frac{\sigma^2}{2} + \sigma\frac{w(t+s) - w(s)}{t} \tag{4}$$

Applying the law of large numbers to the Wiener process [5] we may insist that owing to equality $\lim_{T\to\infty} \mathbb{P}^*\{\sup_{t>T} \lambda(t) > c\} = 0$ for any $c > 0$ the solutions of the equation (1) may exponentially decrease even for $a > 0$ if $a < \sigma^2/2$. Not so difficult also to find the first and the second moments of the process (3):

$$\mathbb{E}x(t+s) = \mathbb{E}x(s)\exp\{at\}, \quad \mathbb{E}|x(t+s)|^2 = \mathbb{E}|x(s)|^2 \exp\{t(2a+\sigma^2)\} \tag{5}$$

These formulae and the formula (3) allow to draw a conclusion that even in the case of positive a owing to presence of volatility $\sigma > 0$ the stochastic exponent may be infinitesimal with probability one as time goes to infinity but its average and variance come to infinitely large. It is not so realistic for the real dynamics of stocks an other securities. That is why last time some of the papers propose different linear stochastic models of price dynamics taking into account possible delay in some characteristics of the real market. In our paper we refer only one [10] most typical of the above mentioned papers, where the authors have discussed the time asymptotic of stock price $\{S(t)\}$ writing its mathematical model in a following form ([10], (6.1)):

$$dS(t) = [aS(t) + \mu S(t-\tau)]dt + \sigma S(t-\rho)dw(t) \tag{6}$$

Applying the second Lyapunov method with specially constructing quadratic Lyapunov-Krasovsky type [8] functional they derive the formula ([10], (6.4))

$$a + \frac{\sigma^2}{2} + |\mu| < 0 \tag{7}$$

which guarantees an exponential decreasing of any solution of (6) with probability one. It should be mentioned that the equation of type (6) as a model of price dynamics has been discussed in our papers [9]. According to the adaptive Samuels-Marshall price equilibrium model the term $aS(t)$ reflects a dependence of price on demand and then for realistic market the coefficient a should be positive. This assertion conflicts with (23) and therefore the results of the paper [10] are inapplicable for the above market models. However even for positive a under some assumptions on μ and σ any solution of equation (6) may be exponentially decreasing with probability one. Besides there are many papers published by A.V. Swishchyk (see references in [11]) where the author is trying to construct the quadratic type Lyapunov functional which might be useful for stability analysis of stochastic functional differential equations. Of course it will be preferable to have a functional enabling at least to derive necessary and sufficient conditions for stability analysis of the linear deterministic part of stochastic equations. In our opinion any citing in [11] paper could not have done it. Unfortunately the authors do not know about our proposal approach to this problem which have been expound in [6]. As it has been shown in [7] and [8] the proposed in [6] method permits to construct sufficiently smooth quadratic Lyapunov-Krasovsky functionals which allow not only to derive necessary and sufficient conditions for deterministic linear functional differential equations but also may be successfully applied to the discussed by paper [10] problem. In this paper we will specify this method to equation (6). As a result we derive not only a necessary functional but also write out necessary and sufficient conditions of exponential mean square decreasing of any solution of equation (6).

2 Quadratic Lyapunov-Krasovsky functionals

Let $\mathcal{C}(Q)$ be the space of continuous functions $q(\theta_1,\,\theta_2)$ on $Q := \{-\tau \le \theta_1 \le 0, -\tau \le \theta_2 \le 0\}$, satisfying symmetry condition $q(\theta_1,\,\theta_2) \equiv q(\theta_2,\,\theta_1)$ with norm $||q|| := \sup\limits_{\{\theta_1,\,\theta_2\}\in Q} |q(\theta_1,,\,\theta_2)|$. By Riesz theorem the set of linear continuous functionals $\mathcal{C}^*(Q)$ is isometrically isomorphic to the space of countably additive measures on the space of Borel subsets Σ_Q of square Q. The scalar product of elements $q \in \mathcal{C}(Q)$ and $\mu \in \mathcal{C}^*(Q)$ is defined by the equality

$$[\mu,\, q] := \iint\limits_Q q(\theta_1,\theta_2)\mu(d\theta_1,d\theta_2) \tag{8}$$

Since the integral in the right-band side of the last equality has sense for any measurable symmetric function $q \in \mathbf{B}(Q)$ also as well we keep the above notation for this case. Using the above formula (8) one can define quadratic functional on the space $\mathbf{C}([-\tau,0])$

$$[\mu,\, \varphi\otimes\varphi] := \iint\limits_Q \varphi(\theta_1)\varphi(\theta_2)\mu(d\theta_1,d\theta_2) \tag{9}$$

where function $\varphi\otimes\varphi \in \mathcal{C}(Q)$ is a tensor product $(\varphi\otimes\varphi)(\theta_1,\theta_2) := \varphi(\theta_1)\varphi(\theta_2)$. As it has been proved in [8] to apply the second Lyapunov method for asymptotic stability analysis of (6) one can use the subset $\overset{\circ}{\mathbb{K}}$ of positive defined quadratic functionals

$$\overset{\circ}{\mathbb{K}} := \{\exists c > 0,\, \forall\varphi \in \mathcal{C}(Q) : [\mu,\, \varphi\otimes\varphi] \ge c|\varphi(0)|^2\} \tag{10}$$

According to the proposed by our paper [8] method one have to define corresponding to equation (6) linear closed operator on the space $\mathcal{C}(Q)$

$$(\mathbb{A}q)(\theta_1,\theta_2) = \begin{cases} \left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right) q(\theta_1,\theta_2), & \text{if } -\tau \le \theta_2 \le \theta_1 < 0,\\ \frac{\partial}{\partial\theta_2} q(0,\theta_2) + aq(0,\theta_2) + bq(\theta_2,-\tau,), & \text{if } -\tau \le \theta_2 < \theta_1 = 0,\\ (2a+\sigma^2)q(0,0) + 2\mu q(0,-\tau), & \text{if } \qquad \theta_2 = \theta_1 = 0, \end{cases} \tag{11}$$

and to look for a solution of the equation

$$\mathbb{A}^*\mu = -\nu \tag{12}$$

for a chosen $\nu \in \mathcal{C}^*(Q)$. The solutions of (6) exponentially mean square decrease if and only if for any $\nu \in \overset{\circ}{\mathbb{K}}$ this equation has unique positive defined solution $\mu = -(\mathbb{A}^*)^{-1}\nu \in \overset{\circ}{\mathbb{K}}$. It should be mentioned that we need only to examine inequality (10) an therefore for a given ν one should find only in $[(\mathbb{A}^*)^{-1}\nu, \varphi\otimes\varphi]$. This permits to formulate an algorithm of mean square stability analysis of the equation (6) has the following form:

- solve the equation
$$\mathbb{A}q = \varphi\otimes\varphi \tag{13}$$
- make sure of nonpositivity of $[\nu, \mathbb{A}^{-1}(\varphi\otimes\varphi)]$ for any $\varphi \in \mathbf{C}([-\tau,0])$;

- if $\exists\varphi \in \mathbf{C}([-\tau, 0]) : [\nu, \mathbb{A}^{-1}(\varphi \otimes \varphi)] > 0$ then there exists a solution of (6) with exponentially increasing second moment;
- otherwise find

$$\delta((\mathbb{A}^*)^{-1}\nu) := [\nu, \mathbb{A}^{-1}(\mathbf{1}_0 \otimes \mathbf{1}_0)] \tag{14}$$

where

$$\mathbf{1}_0(\theta) := \begin{cases} 0, & \text{for } -\tau \le \theta < 0; \\ 1, & \text{for } \quad \theta = 0 \end{cases}$$

The inequality $\delta((\mathbb{A}^*)^{-1}\nu) < 0$ guarantees an exponential decreasing of any solution of (6).

3 Lyapunov equation and exponential decreasing of delayed stochastic exponent

According to reasoning of previous section the equation we need to solve has a form $\mathbb{A}q = \varphi \otimes \varphi$. Owing to symmetry condition one can be looking for solution only in the triangle $-\tau \le \theta_2 \le \theta_1 \le 0$ and next to extend by symmetry $q(\theta_1, \theta_2) \equiv q(\theta_2, \theta_1)$ for other points of the set Q. Then at first we should find the solution of equation

$$\left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right) q(\theta_1, \theta_2) = \varphi(\theta_1)\varphi(\theta_2) \tag{15}$$

for $-\tau \le \theta_2 \le \theta_1 < 0$. It is not difficult to find this solution in a form

$$q(\theta_1, \theta_2) = r(\theta_2 - \theta_1) - \int_{\theta_2}^{\theta_2-\theta_1} \varphi(u - \theta_2 + \theta_1)\varphi(u)du. \tag{16}$$

with an arbitrary smooth function $\{r(t), -\tau \le t < 0\}$. Now we pass on to equation

$$\frac{\partial}{\partial\theta_2} q(0, \theta_2) + aq(0, 0) + \mu q(\theta_2, -\tau) = \varphi(0)\varphi(\theta_2) \tag{17}$$

for $-\tau \le \theta_2 < 0$ and to solve this equation satisfying boundary codition

$$(2a + \sigma^2)q(0, 0) + 2\mu q(0, -\tau) = |\varphi(0)|^2. \tag{18}$$

Substituting (16) in (17) and (17) we have equation for unknown function $r(t)$

$$\dot{r}(t) + ar(t) + \mu r(-\tau - t)) = g(t) \tag{19}$$

where $g(t) = \mu \int_{-1}^{-\tau-t} \varphi(u + \tau + t)\varphi(u)du + \varphi(t)\varphi(0)$, with boundary condition

$$(2a + \sigma^2)r(0) + 2\mu r(-\tau) = |\varphi(0)|^2 \tag{20}$$

Using notation $r(-\tau - t)) := h(t)$ one can rewrite the equation (19) as the system of two linear ordinary differential equations

$$\begin{aligned} \dot{r}(t) &= -ar(t) - \mu h(t) + g(t) \\ \dot{h}(t) &= ah(t) + \mu h(t) - g(-\tau - t) \end{aligned} \tag{21}$$

It is not so difficult to find solution of (21) satisfying boundary condition (20). Substituting this solution into (16) we will have solution of equation (16) in an explicit form. Now for a given $\nu \in \overset{\circ}{\mathbb{K}}$ one can find the Lyapunov-Krasovsky quadratic functional for equation (6): $v(\varphi) := [\nu, q] = [\nu, \mathbb{A}^{-1}(\varphi \otimes \varphi)]$. For example if $\tau = 1,\ \omega^2 := \mu^2 - a^2 > 0$ choosing measure $\nu := \delta_0$ as delta-function with one-point support $\{0, 0\}$ (that is defined by equality $\forall q \in \mathcal{C}(Q)$: $[\delta_0, q] = q(0, 0)]$) one can define the desired functional explicitly [8]:

$$
\begin{aligned}
v(\varphi) \quad &= \quad \frac{f_1(\varphi) 2\mu \sin\omega + f_2(\varphi)(\mu \sin\omega - \omega)}{(a + \mu\cos\omega)(2\mu\sin\omega) - (2a + \sigma^2 + 2\mu\cos\omega)(\mu\sin\omega - \omega)} - \\
&- \quad \frac{\sin\omega}{\omega}\left[\int_{-1}^{0} |\varphi(s)|^2 ds + \varphi(-1)\varphi(0)\right]
\end{aligned}
\tag{22}
$$

where

$$
\begin{aligned}
f_1(\varphi) \quad &= \quad a\left(\frac{1}{\omega}\int_{-1}^{0} \sin(\omega s)(\mu g(-1-s) - a g(s))ds - \right. \\
&- \quad \frac{1}{\omega}\sin(\omega)\left[\left(\int_{-1}^{0} |\varphi(s)|^2 ds\right) + \varphi(-1)\varphi(0)\right] + \\
&+ \quad \left.\int_{-1}^{0} \cos(\omega s) g(s) ds\right) + \omega \int_{-1}^{0} \sin(\omega s) g(s) ds + \\
&+ \quad \int_{-1}^{0} \cos(\omega s)(\mu g(-1-s) - a g(s))ds - \\
&- \quad \cos(\omega)\left[\left(\int_{-1}^{0} |\varphi(s)|^2 ds\right) + \varphi(-1)\varphi(0)\right], \\
f_2(\varphi) \quad &= \quad |\varphi(0)|^2 - \int_{-1}^{0} \cos(\omega s) g(s) ds + \\
&+ \quad \frac{1}{\omega}\left(\int_{-1}^{0} \sin(\omega s)(\mu g(-1-s) - a g(s))ds - \right. \\
&- \quad \left.\sin(\omega)\left[\left(\int_{-1}^{0} |\varphi(s)|^2 ds\right) + \varphi(-1)\varphi(0)\right]\right).
\end{aligned}
$$

Now substituting $\varphi(\theta) = \mathbf{1}_0(\theta)$ in (22) one has to examine positivity of the obtained expression. It easy to calculate $f_1(\mathbf{1}_0) = 0,\ f_2(\mathbf{1}_0) = 1$ and therefore the necessary and sufficient condition for exponential mean square decreasing of delayed exponent in our case has a following form:

$$\frac{\mu \sin\omega - \omega}{(a + \mu\cos\omega)(2\mu\sin\omega) - (2a + \sigma^2 + 2\mu\cos\omega)(\mu\sin\omega - \omega)} < 0 \qquad (23)$$

The above constructed functional $v(\varphi)$ is also well-behaved for quasylinear stochastic functional differential equation analysis. As it has been proved in [7] if process $y(t)$ has stochastic differential then the process $v(y_t)$ where $y_t := \{y(t+\theta), -1 \le \theta \le 0\}$ also has stochastic differential. Besides $v(\varphi) = 0$ if and only if $\varphi(\theta) \equiv 0$. This means that one can find the stochastic differential for example for process $z(t) = [v(x_t)]^p$ with any sufficiently small $p > 0$ and analyze not only the second moment asymptotic but also the time asymptotic of solutions in terms of convergence by probability [6].

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