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MOMENT EQUATIONS FOR DISCRETE LINEAR MARKOV DYNAMICAL SYSTEMS

CARKOVA Viktorija, (LV), GOLDŠTEINE Jolanta, (LV)

Abstract. The paper analyzes the moments of linear difference equations with coefficients dependent on homogeneous ergodic Markov chain. A convenient for application method of asymptotic analysis for equations with near to constant coefficients is elaborated. Applying Kato perturbation theory and the second Lyapunov method an algorithm for decomposition of quadratic Lyapunov function by small parameter powers is proposed. If random perturbations are independent the proposal method enables to write necessary and sufficient stability conditions involving system coefficients.

Key words. Difference equations, Markov discrete systems, mean square stability.

Mathematics Subject Classification: 93E15

1 Introduction

This paper deals with difference equation with Markov coefficients defined in a space $\mathbb{R}^n$

$$x_t = A(\xi_t)x_{t-1}, \tag{1.1}$$

where $\{A(y),\ y \in \mathbb{Y}\}$ is bounded continuous $n \times n$ matrix function on the metric compact $\mathbb{Y}$, $\{\xi_t, t \in \mathbb{Z}\}$ is homogeneous Markov chain with phase space $\mathbb{Y}$, invariant measure $\mu(dy)$ and transition probability $P(y,dz)$. The transition probability has a Feller property, that is, if $\{u(y), y \in \mathbb{Y}\}$ is a function continuous then the function $\{(\mathcal{P}u)(y), y \in \mathbb{Y}\}$ which is defined by equality

$$(\mathcal{P}u)(y) =: \int_{\mathbb{Y}} u(z)P(y,dz) \tag{1.2}$$

is also continuous. There exists a unique probability measure on $\mathbb{Y}$ satisfying the equality

$$\left(\mathcal{P}^{*}\mu\right)(dz) =: \int_{\mathbb{Y}} \mu(dy) P(y,dz). \qquad (1.3)$$

Owing to exponential ergodicity assertion there exists such a positive number $\rho < 1$ that the spectrum of operator $\mathcal{P}$ defined on space $\mathbb{C}(\mathbb{Y})$ can be represented into a form

$$\sigma(\mathcal{P}) = \{1\} \cup \sigma_{\rho},\ \sigma_{\rho} \in \{\lambda \in C : |\lambda| < \rho\}. \qquad (1.4)$$

2 Analysis of the first moment equations

Let us consider that $\mathbb{R}^n$ is a linear space of *n*-dimensional column-vectors with scalar product

$$u \in \mathbb{R}^n,\ v \in \mathbb{R}^n :\ (u,v) = u^T v \qquad (2.1)$$

and that Markov sequence $\vec{\xi} := \{\xi_t,\ t \in \mathbb{Z}\}$ is defined on filtered probability space $\left(\Omega, \mathfrak{F}, \mathfrak{F}^t, \mathbf{P}\right)$, where $\{\mathfrak{F}^t\}$ minimal filtration with which it is harmonized. Let us denote

$$s \in \mathbb{R}:\ X_s^s = I;\quad t > s:\ X_s^t := \prod_{k=s+1}^{t} A(\xi_k). \qquad (2.2)$$

It is easy to see that a solution of (1.1) can be written in a form $x_t = X_t^s x_s$ for all $s \in \mathbb{R},\ t \geq s$. We can define an operator on the space of continuous *n*-dimensional representations $\mathbb{C}(\mathbb{Y} \to \mathbb{R}^n) := \mathbb{C}_n(\mathbb{Y})$ in a following way:

$$y \in \mathbb{Y}, u \in \mathbb{C}_n(\mathbb{Y}):\ (\mathbf{A}u)(y) = \int_{\mathbb{Y}} A^T(z) u(z) P(y,dz). \qquad (2.3)$$

Since the transition probability has a Feller property then $\mathbf{A}u \in \mathbb{C}_n(\mathbb{Y})$. It can be easy proved that the operator $\mathbf{A}$ is linear continuous operator on $\mathbb{C}_n(\mathbb{Y})$.

Lemma 1. For every $s \in \mathbb{R},\ t > 0,\ v \in \mathbb{C}_n(\mathbb{Y}),\ x \in \mathbb{R}^n$

$$E\left\{\left(X_s^{s+t}x, v(\xi_{s+t})\right) / \mathfrak{F}^s\right\} = \left(x, \left(\mathbf{A}^t v\right)(\xi_s)\right). \qquad (2.4)$$

Theorem 1. Let elements of sequence $\{\xi_t, t \in \mathbb{Z}\}$ be as independent identically distributed. Then

(i) operator $\mathbf{A}$ leaves as invariant subspace $\mathbb{R}^n \subset \mathbb{C}_n(\mathbb{Y})$ and operator $\mathbf{A}$ restriction $\overline{\mathbf{A}}$ on this subspace can be defined by equality

$$v \in \mathbb{R}^n :\ \overline{\mathbf{A}}v = \overline{A}^T v, \qquad (2.5)$$

where $\overline{A} = E\{A(\xi_0)\}$;

(ii) for every $s \in \mathbb{Z}$, $t > s$ and every equation (1.1) $\mathfrak{F}^t$-harmonized solution $\{x_t, t \geq 0\}$ the following equation is into force

$$E\{x_t\} = \bar{A}^{t-s} E\{x_s\} \tag{2.6}$$

The equation (1.1) is called *reducible in the mean* if there exists such a continuous matrix function $\{B(y), y \in \mathbb{Y}\}$ and a matrix Λ that for any $s \in \mathbb{R}$ and $t > s$ the following equality is into force

$$E\{B(\xi_t) x_t / \mathcal{F}^s\} = \Lambda^{t-s} B(\xi_s) x_s . \tag{2.7}$$

We will analyze reducibility in the mean of equation (1.1) in case when matrix function $\{A(y), y \in \mathbb{Y}\}$ is near to constant and can be represented in a form of uniformly converged row

$$A(y) = A_0 + \varepsilon A(y, \varepsilon) := A_0 + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k A_{k+1}(y) \tag{2.8}$$

where $\varepsilon \in (0,1)$ is a small parameter. Corresponding to the matrix (2.7) the operator (2.3) can be rewritten like sum $\mathbf{A}(\varepsilon) = \mathbf{A}_0 + \varepsilon \mathbf{A}(\varepsilon)$ hereto the operator $\mathbf{A}_0$ leaves as invariant the subspace $\mathbb{R}^n$ and it can be represented as tensor product of operators $\mathbf{A}_0 = \mathcal{P} \otimes A_0^T$:

$$h \in \mathbb{C}(\mathbb{Y}), g \in \mathbb{R}^n : \ \mathbf{A}_0 (h \otimes g) = \mathcal{P}h \otimes A_0^T g \tag{2.9}$$

where $\mathcal{P}$ is Markov operator defined by equality (1.2).
Owing to exponentially ergodicity the operator $\mathbf{A}_0$ spectrum is in a form

$$\sigma(\mathbf{A}_0) = \{\lambda_1 \lambda_2 : \lambda_1 \in \sigma(\mathcal{P}), \lambda_2 \in \sigma(A_0)\} = \sigma(A_0) \cup \sigma_\rho , \tag{2.10}$$

where $\sigma_\rho(A_0) := \{\lambda_1 \lambda_2 : \lambda_1 \in \sigma(\mathcal{P}), \lambda_2 \in \sigma_\rho\}$ and $\sigma(A_0) \cap \sigma_\rho = \varnothing$. This allows for further analysis to use the asymptotic method based on decomposition of operator $\mathbf{A}(\varepsilon)$ spectral projector by powers of small parameter ε. It is proved that in such situation for a sufficiently small $\bar{\varepsilon} > 0$ and for every $|\varepsilon| < \bar{\varepsilon}$ difference equation is reducible in the mean, moreover the matrix function $\{B(y, \varepsilon), y \in \mathbb{Y}\}$ is a basis in operator $\mathbf{A}(\varepsilon)$ root subspace corresponding to the part of spectrum $\sigma_0(\varepsilon)$ defined by equality $\lim_{\varepsilon \to 0} \sigma_0(\varepsilon) = \sigma_0$, but matrix $\Lambda(\varepsilon)$ is a operator $\mathbf{A}(\varepsilon)$ restriction matrix on this root subspace. For every $|\varepsilon| < \bar{\varepsilon}$ a basis $\{B(y, \varepsilon), y \in \mathbb{Y}\}$ which is a $n \times n$-dimensional matrix function and a constant $n \times n$ matrix $\Lambda(\varepsilon)$ are defined in a unique way by the equality

$$y \in \mathbb{Y}, |\varepsilon| < \bar{\varepsilon} : \ (\mathbf{A}(\varepsilon) B)(y, \varepsilon) = B(y, \varepsilon) \Lambda^T(\varepsilon) . \tag{2.11}$$

In the algorithm of finding a basis $B(y,\varepsilon)=\{b_1(y,\varepsilon),b_2(y,\varepsilon),\dots,b_n(y,\varepsilon)\}$ and matrices $\Lambda(\varepsilon)$ the decompositions of these matrices by powers of small parameter ε

$$\Lambda(\varepsilon):=\Lambda_0+\varepsilon\sum_{k=0}^{\infty}\Lambda_{k+1},\qquad B(y,\varepsilon):=B_0+\varepsilon\sum_{k=0}^{\infty}B_{k+1}(y) \tag{2.12}$$

and operator $\mathbf{A}(\varepsilon)$

$$\mathbf{A}(\varepsilon):=\mathbf{A}_0+\varepsilon\sum_{k=0}^{\infty}\varepsilon^k\mathbf{A}_{k+1} \tag{2.13}$$

are used, where $(\mathbf{A}_j v)(y)=\int_{\mathbb{Y}}A_j^T(z)v(z)P(y,dz)$. Step by step finding Λ_1, $B_1(y)$, Λ_2, $B_2(y)$ and Λ_3, $B_3(y)$ and so on we can obtain needed accuracy for matrix $\Lambda(\varepsilon)$ decomposition. As $\mathbb{Y}$ is compact and matrices $\{B_j(y), j=1,2,\dots\}$ are continuous, the elements of obtained basis $B:=I+\varepsilon B_1+\varepsilon^2 B_2+\dots$ are independent for sufficiently small ε.

3 Covariance analysis

In this section the dynamics of the second moments' matrix of linear difference equation (1.1) solution are analyzed, that is, behavior of matrix

$$Q_t:=E\{x_t x_t^T\} \tag{3.1}$$

as matrix function of argument *t*. The space $\mathbb{M}_n$ of real $n\times n$ matrices can be viewed as n^2-dimensional Euclide space $\mathbb{R}^{n^2}$ with scalar product $[q,g]:=Sp\{qg^T\}$. Set $\hat{\mathbb{M}}_n$ of symmetric $n\times n$ matrices in a form $q:=(q_{ij})_{i,j=1,\dots,n}$ creates a linear closed subspace in $\mathbb{M}_n$. $\hat{\mathbb{M}}_n$ can be identified using Euclide space $\mathbb{R}^{\frac{n(n+1)}{2}}$ with vectors in a form

$$\vec{q}:=\left(q_{11},q_{12},\dots,q_{1n};\ q_{22},q_{23},\dots,q_{2n};\ q_{(n-1)(n-1)},q_{(n-1)n};\ q_{nn}\right)^T \tag{3.2}$$

and scalar product $(q,g):=q^T g$. Using these denotations and equation (1.1) one can write a linear difference equation in $\mathbb{M}_n$ for matrix sequences $(xx)_t:=x_t x_t^T$:

$$(xx)_t=A(\xi_t)(xx)_{t-1}A^T(\xi_t):=\vec{A}(\xi_t)(xx)_{t-1} \tag{3.3}$$

and use the results from previous section. The defined family $\vec{A}(\xi_t)$ of linear operators in (3.3) for every fixed argument ξ_t leaves as invariant space $\hat{\mathbb{M}}_n$, and therefore instead of (3.3) we can analyze a corresponding linear difference equation in $\mathbb{R}^{\frac{n(n+1)}{2}}$.

Let denote as $\mathbb{V}$ the Banach space of symmetric $n\times n$ matrix functions $\{q(y), y\in\mathbb{Y}\}$ with a norm $\|q\| := \sup_{y\in\mathbb{Y},\|x\|=1} \left|(q(y)x,x)\right|$. Using a matrix function $\{A(y), y\in\mathbb{Y}\}$ and Markov chain transition probabilities one can define a linear continuous operator on $\mathbb{V}$:

$$(\mathbf{A}q)(y) := \int_{\mathbb{Y}} A^T(z)q(z)A(z)P(y,dz). \tag{3.4}$$

Analyzing this operator we can make a judgment about the dynamics of the second moments' matrix of linear difference equation (1.1) solution. This operator leaves as invariant [7] a cone of positive defined matrix functions defined by equality $\mathbb{K} := \left\{ q\in\mathbb{V} : \inf_{y\in\mathbb{Y},\|x\|=1} (q(y)x,x) \geq 0 \right\}$ with a set of inner points $\overset{\circ}{\mathbb{K}}$. This cone $\mathbb{K}$ allows partly to order the space $\mathbb{V}$ using an inequality $q_1 << q_2 \Leftrightarrow q_2 - q_1 \in \mathbb{K}$. It is clear that $q\in\overset{\circ}{\mathbb{K}}$ then and only then when exists such a $c(q)$ that $q >> c(q)I$, where I is matrix unit. This ordering allows rather easy to investigate behavior of the second moments of the (1.1) solution when $t\to\infty$. Let denote $x_{t+k}(k,x,y)$ the solution of (1.1) satisfying the initial conditions $x_k = x$, $\xi_k = y$ and $X(t+k,k,y)$ for matrices (2.2) if $\xi_k = y$. If unconditional second moment of (3.1) solution exponentially decreases at $t\to\infty$, that is,

$$\exists C>0, \exists\lambda\in\{\mathcal{C} : |z|<1\}, \forall x\in\mathbb{R}^n, \forall y\in\mathbb{Y}, \forall k\in\mathbb{Z}, \forall t\geq 0 : E\left\{\left\|x_{t+k}(k,x,y)\right\|^2\right\} \leq C\lambda^t |x|^2$$

then [1] (1.1) is mean square exponentially stable. This property can be easy established analyzing a real positive spectrum of operator (3.4) [1].

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Current address

Viktorija Carkova, Assoc. prof.
University of Latvia, 8 Zellu street, Riga, LV-1002, Latvia,
tel.number: +371 7674363, e-mail: tsarkova@latnet.lv

Jolanta Goldšteine, M.Sc. math.
University of Latvia, 8 Zellu street, Riga, LV-1002, Latvia,
tel. number: +371 2 949 8160 , e-mail: jolanta_g@osi.lv