7[th] International Conference
APLIMAT 2008
Faculty of Mechanical Engineering - Slovak University of Technology in Bratislava
Section: Differential equations and their applications

THE CONSTRUCTION OF LYAPUNOV-KRASOVSKY QUADRATIC FUNCTIONAL FOR NEUTRAL STOCHASTIC DIFFERENTIAL EQUATIONS

Carkov Evgenijs, (LV), Bereza Vitaly, (UI)

Abstract. In current thesis the method of construction of Lyapunov-Krasovsky quadratic functional for analysis of exponential stability of trivial solution of neutral stochastic differential equation is reviewed.

Let on stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$ the stochastic process $x(\cdot)$ is given with values from $\mathbf{R}^1$ as strong solution of neutral stochastic differential equiation (NSDE)

$$d[x\,(t) - cx\,(t-1)] = [ax\,(t) + bx\,(t-1)]dt + \gamma x\,(t)\,dw\,(t)\,,\ c^2 \neq 1. \tag{1}$$

$$x(\theta) = \varphi(\theta),\ \theta \in [-1, 0]. \tag{2}$$

In current thesis the method of construction of Lyapunov-Krasovsky quadratic functional $v(t, \varphi)$ for analysis of exponential stability of trivial solution of NSDE (1), (2) is reviewed.

Let the $\mathbf{C}^*(Q)$ is a space of countably additive symmetric matrix measures on $Q \equiv \{-1 \leq \theta_1 \leq 0,\ -1 \leq \theta_2 \leq 0\}$, where $\mathbf{M}_n$ is the space of symmetric $n \times n$-measured matrixes and $\mathcal{C} \equiv \mathbf{C}(Q \to \mathbf{M}_n)$, i.e. space of continuous matrix functions $q(\theta_1, \theta_2)$, which the condition of symmetry $q(\theta_1, \theta_2)^\top = q(\theta_2, \theta_1)$ are satisfy for all $\{\theta_1, \theta_2\} \in Q$. Let's define the scalar product of elements $q \in \mathcal{C}$ and $\mu \in \mathcal{C}^*$ by equality

$$[\mu, q] \equiv \int_{-1}^{0}\int_{-1}^{0} \mathrm{tr}(q(\theta_2, \theta_1)\mu(d\theta_1, \theta_2)),$$

where tr is a matrix track. Each element $\mu \in \mathcal{C}^*$ may be regarded as linear continuous operator, what operates from space $\mathbf{C}_n([-1, 0])$ to space $\mathbf{C}_n^*([-1, 0])$ in accordance to rule

$(\mu\varphi)(A) \equiv \int\limits_{-1}^{0} \mu(A, d\theta)\varphi(\theta)$, where A is the element of σ-algebra of mincuspidal subsets of interval $[-1,0]$. Let us denote by $< l, \varphi >$ the scalar product of elements $l \in \mathbf{C}_n^*([-1,0])$ and $\varphi \in \mathbf{C}_n([-1,0]) : < l, \varphi > \equiv \int\limits_{-1}^{0} l(d\theta)^\top \varphi(\theta)$. By using of above formulas, it's possible to set into consideration the bilinear functional on $\mathbf{C}_n([-1,0])$, what is defined for any $\mu \in \mathcal{C}^*$ as

$$< \mu\varphi, \psi > \equiv \int\limits_{-1}^{0}\int\limits_{-1}^{0} \varphi(\theta_2)^\top \mu(d\theta_1, \theta_2)\psi(\theta_1).$$

Similarly, if $q \in \mathcal{C}$ and $x \in \mathbf{C}_n^*([-1,0])$, then it's could consider the operator

$$q : \mathbf{C}_n^*([-1,0]) \to \mathbf{C}_n([-1,0]),$$

what operates by the rule $(qx)(\theta) \equiv \int\limits_{-1}^{0} q(\theta, s)x(ds)$ and defines bilinear form

$< x, qy > (\theta) \equiv \int\limits_{-1}^{0}\int\limits_{-1}^{0} x(d\theta_1)^\top q(\theta_1, \theta_2)y(d\theta_2)$ in $\mathbf{C}_n^*([-1,0])$ for any $x, y \in \mathbf{C}_n^*([-1,0])$.

For any two elements $\varphi, \psi \in \mathbf{C}_n([-1,0])$ and $x, y \in \mathbf{C}_n^*([-1,0])$ it's could define tensor product

$$\forall\theta_1 \in [-1,0],\ \forall\theta_2 \in [-1,0] : (\varphi \otimes \psi)(\theta_1, \theta_2) \equiv \psi(\theta_1)\varphi(\theta_2)^\top.$$

Let's denote $\mathbf{K} \equiv \{q \in \mathcal{C} : < x, qx > \geq 0\ \forall x \in \mathbf{C}_n^*([-1,0])\}$ [1], $\mathbf{K}_0 = \{\varphi \otimes \psi,\ \varphi \in \mathcal{C},\ \psi \in \mathcal{C}\}$, $\mathcal{E}^*$ is the domain of linear continuous functionals; $\mathbf{K}^* \equiv \{\mu \in \mathcal{E}^* :\ [\mu, q] \geq 0\ \forall q \in \mathbf{K}\}$ [2]. Let's note, that elements $\mu \in \mathbf{K}^*$ may be defined as positive square-law functionals $< \mu\varphi, \varphi >$. The cone $\mathbf{K}^*$ allows to enter the partial ordering in space $\mathcal{E}^*$: we shall write $\mu \triangleright \nu$ if $\mu - \nu \in \mathbf{K}^*$, that is $[\mu\varphi, q] \geq [\nu, q]$ for all $q \in \mathbf{K}$. Let's consider the positive quadratic functional χ_0, what is defined by formula $< \chi_0\varphi, \varphi > = |\varphi(0)|^2$ for any $\varphi \in \mathbf{C}_n([-1,0])$.

Let's denote $\mathbf{K}_0^* \equiv \{\mu \in \mathbf{K} :\ \exists c > 0,\ \mu \triangleright c\chi_0\}$.

Theorem 1 *The trivial solution of (1), (2) is exponential stable if and only if, when for any $\nu \in \mathbf{K}^*$ the solution of equation $A^*\mu = -\nu$ exists for any $\mu \in \mathbf{K}^*$, where*

$$(Aq)(\theta_1, \theta_2) \equiv \begin{cases} \left(\dfrac{\partial}{\partial\theta_1} + \dfrac{\partial}{\partial\theta_2}\right) q(\theta_1, \theta_2), \qquad if\ -1 \leq \theta_2 \leq \theta_1 < 0, \\ \dfrac{\partial}{\partial\theta_2} q(0, \theta_2) + c\dfrac{\partial}{\partial\theta_2} q(\theta_2, -1) + aq(0, \theta_2) + bq(\theta_2, -1) \\ if\ -1 \leq \theta_2 < \theta_1 = 0, \\ \left(2a + \gamma^2\right) q(0,0) + c\dfrac{\partial}{\partial\theta_2} q(0, -1) + 2bq(0, -1), \qquad if\ \theta_2 = \theta_1 = 0. \end{cases}$$

Theorem 2 *The trivial solution of (1), (2) is exponential stable if and only if, when the strictly positive defined functional μ exists, what satisfies the equation $A^*\mu = -\chi_0$.*

Leaning on above theorems [3],[4], the search of Lyapunov functional for (1), (2) problem is based on resolving of equation $(A)q = \varphi \otimes \varphi$, that may be reduced to looking for symmetric function $q(\theta_1, \theta_2)$, that the differential equation on partial derivatives $(A)q = -\varphi(\theta_1)\varphi(\theta_2)$ satisfies. The next step is the using of found q for resolving of Lyapunov equation in class of quadratic functional $v(\varphi) \equiv < \mu\varphi, \varphi > \equiv [\mu, \varphi \otimes \varphi]$.

On first step we obtain the equation

$$\left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right) q\left(\theta_1, \theta_2\right) = \varphi\left(\theta_1\right)\varphi\left(\theta_2\right), \tag{3}$$

where $q\left(\theta_1, \theta_2\right) = q\left(\theta_2, \theta_1\right)$. Let us make use of change of variables $\theta_1 = t - s,\ \theta_2 = t + s,\ f\left(t, s\right) \equiv q\left(t - s, t + s\right):$

$$\frac{\partial}{\partial} f\left(t, s\right) = \varphi\left(t - s\right)\varphi\left(t + s\right).$$

Since $t := \dfrac{\theta_1 + \theta_2}{2} \leq s := \dfrac{\theta_2 - \theta_1}{2}$, then the solution of previous equation may be given in a form

$$f\left(t, s\right) = r\left(s\right) - \int_t^s \varphi\left(u - s\right)\varphi\left(u + s\right) du$$

with some function $r\left(s\right)$. Proceeding from these reasons, let's take the solution of (3) in a form

$$q\left(\theta_1, \theta_2\right) = r\left(\theta_2 - \theta_1\right) - \int_{\theta_2}^{\theta_2 - \theta_1} \varphi\left(u - \theta_2 + \theta_1\right)\varphi\left(u\right) du \tag{4}$$

Let's find so function r, that (4) satisfied the equation

$$\frac{\partial}{\partial\theta_2} q\left(0, \theta_2\right) + c\frac{\partial}{\partial\theta_2} q\left(\theta_2, -1\right) + aq\left(0, \theta_2\right) + bq\left(\theta_2, -1\right) = \varphi\left(0\right)\varphi\left(\theta_2\right)$$

for second step, that is under condition $-1 \leq \theta_2 < 0 = \theta_1$, and edge condition

$$\left(2a + \gamma^2\right) q\left(0, 0\right) + c\frac{\partial}{\partial\theta_2} q\left(0, -1\right) + 2bq\left(0, -1\right) = |\,\varphi\left(0\right)|^2, \tag{5}$$

$$\frac{\partial}{\partial\theta_2} r\left(\theta_2\right) + c\left[\frac{\partial}{\partial\theta_2}\left(r\left(-1 - \theta_2\right) - \int_{-1}^{-1-\Theta_2} \varphi\left(u + \theta_2 + 1\right)\varphi\left(u\right) du\right)\right] + ar\left(\theta_2\right) +$$
$$+ b\left(r\left(-1 - \theta_2\right) - \int_{-1}^{-1-\Theta_2} \varphi\left(u + \theta_2 + 1\right)\varphi\left(u\right) du\right) = \varphi\left(0\right)\varphi\left(\theta_2\right). \tag{6}$$

Using a designation,

$$g\left(t\right) \equiv c\frac{\partial}{\partial t}\int_{-1}^{-1-t} \varphi\left(u + t + 1\right)\varphi\left(u\right) du + b\int_{-1}^{-1-t} \varphi\left(u + t + 1\right)\varphi\left(u\right) du + \varphi\left(0\right)\varphi\left(t\right),$$

we shall copy (6) as

$$\dot{r}\left(t\right) + c\dot{r}\left(-1 - t\right) + ar\left(t\right) + br\left(-1 - t\right) = g\left(t\right) \tag{6.1}$$

for function $\{r(t), \ -h \le t < 0\}$,

$$\left(2a+\gamma^2\right) r(0) + c\dot{r}(-1) + 2br(-1) = |\varphi(0)|^2. \tag{7}$$

By means of a designation $p(t) \equiv r(-1-t)$ we shall write the equation (6.1) in a form

$$\begin{aligned} \dot{r}(t) + c\dot{p}(t) &= -ar(t) - bp(t) + g(t), \\ \dot{p}(t) + c\dot{r}(t) &= ap(t) + br(t) - g(-1-t). \end{aligned} \tag{8}$$

Put $z(t) \equiv \int_{-1}^{-1-t} \varphi(u-t-1)\varphi(u)\,du$, then

$$g(t) \equiv \dot{z}(t) + z(t) + \varphi(0)\varphi(t).$$

Let's write system (8) in equivalent form

$$\begin{aligned} \dot{r}(t) &= \left(\frac{-a-bc}{1-c^2}\right) r(t) - \left(\frac{b+ac}{1-c^2}\right) p(t) + \frac{g(t)+cg(-1-t)}{1-c^2}, \\ \dot{p}(t) &= \left(\frac{ac+b}{1-c^2}\right) r(t) + \left(\frac{bc+a}{1-c^2}\right) p(t) - \frac{cg(t)+g(-1-t)}{1-c^2}. \end{aligned} \tag{9}$$

Let's denote

$$A \equiv \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}, \qquad B \equiv \begin{pmatrix} -a & -b \\ b & a \end{pmatrix}, \qquad d \equiv \begin{pmatrix} g(t) \\ -g(-1-t) \end{pmatrix}.$$

Let's find characteristic constants of system (9):

$$\det\left(A^{-1}B - \lambda E\right) = 0,$$

$$\lambda_1 = \sqrt{\frac{a^2-b^2}{1-c^2}}, \lambda_2 = -\sqrt{\frac{a^2-b^2}{1-c^2}}.$$

Let's find eigenvectors, which fit with found characteristic constants:

$$\begin{pmatrix} \frac{-a-bc}{1-c^2} & \frac{-b-ac}{1-c^2} \\ \frac{ac+b}{1-c^2} & \frac{bc+a}{1-c^2} \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix} = \lambda_1 \begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix}.$$

It's easy to see, that the roots of this system are

$$h_{11} = b + ac, \ h_{12} = -(bc + a + w),$$

where $w \equiv \sqrt{(a^2-b^2)(1-c^2)}$.
Let's find eigenvector to second characteristic constant:

$$\begin{pmatrix} \frac{-a-bc}{1-c^2} & \frac{-b-ac}{1-c^2} \\ \frac{ac+b}{1-c^2} & \frac{bc+a}{1-c^2} \end{pmatrix} \begin{pmatrix} h_{21} \\ h_{22} \end{pmatrix} = \lambda_2 \begin{pmatrix} h_{21} \\ h_{22} \end{pmatrix},$$

$$h_{21} = b + ac,\ h_{22} = w - (a + bc)\,.$$

Let's write the solutions of respective homogeneous system (9):

$$r_0\,(t) = c_1\,(b + ac) \exp\{\lambda_1 t\} + c_2\,(b + ac) \exp\{\lambda_2 t\}\,,$$

$$p_0\,(t) = c_1\,(-w - a - bc) \exp\{\lambda_1 t\} + c_2\,(w - a - bc) \exp\{\lambda_2 t\}\,.$$

Let's find the partial solution of (9), having taken advantage of a method of variation of constant:

$$c_1^{'} \exp\{\lambda_1 t\} + c_2^{'} \exp\{\lambda_2 t\} = \frac{g\,(t) + cg\,(-1 - t)}{(1 - c^2)\,(b + ac)},$$

$$c_1^{'}\,(-w - a - bc) \exp\{\lambda_1 t\} + c_2^{'}\,(w - a - bc) \exp\{\lambda_2 t\} = -\frac{cg\,(t) + g\,(-1 - t)}{1 - c^2}.$$

Then

$$c_1 = \frac{\int_{-1}^{t} \exp\{\lambda_2 s\}\,\alpha(s) ds}{2\,(1 - c^2)\,(b + ac)\,w},\ c_2 = \frac{\int_{-1}^{t} \exp\{\lambda_1 s\}\,\beta(s) ds}{2\,(1 - c^2)\,(b + ac)\,w},$$

where

$$\alpha\,(t) \equiv \left(1 - c^2\right)(bg\,(-1 - s) - ag\,(s)) + (g\,(s) + cg\,(-1 - s))\,w,$$

$$\beta\,(t) \equiv (g\,(s) + cg\,(-1 - s))\,w - \left(1 - c^2\right)(bg\,(-1 - s) - ag\,(s))\,.$$

So, we obtained the global solution of system (8):

$$r\,(t) = (b + ac)\,(c_1 \exp\{\lambda_1 t\} + c_2 \exp\{\lambda_2 t\}) +$$

$$+\frac{\exp\{\lambda_1 t\} \int_{-1}^{t} \exp\{\lambda_2 s\}\,\alpha\,(s)\,ds + \exp\{\lambda_2 t\} \int_{-1}^{t} \exp\{\lambda_1 s\}\,\beta\,(s)\,ds}{2\,(1 - c^2)\,w},$$

$$p\,(t) = (-w - a - bc) \exp\{\lambda_1 t\} c_1 + (w - a - bc) \exp\{\lambda_2 t\}\,c_2 +$$

$$+\frac{\Big\{(-w - a - bc) \exp\{\lambda_1 t\} \int_{-1}^{t} \exp\{\lambda_2 s\}\,\alpha\,(s)\,ds}{2\,(1 - c^2)\,(b + ac)\,w} +$$

$$+\frac{(w - a - bc) \exp\{\lambda_2 t\} \int_{-1}^{t} \exp\{\lambda_1 s\}\,\beta\,(s)\,ds\Big\}}{2\,(1 - c^2)\,(b + ac)\,w},$$

Let's calculate $\dot{r}\,(t)$:

$$\dot{r}\,(t) = (b + ac)\,\lambda_1\,(c_1 \exp\{\lambda_1 t\} - c_2 \exp\{\lambda_2 t\}) + \frac{1}{2\,(1 - c^2)\,w} \times$$

$$\times \Big[\lambda_1 \exp\{\lambda_1 t\} \int_{-1}^{t} \exp\{\lambda_2 s\}\,\alpha\,(s)\,ds + \alpha\,(t) - \lambda_1 \exp\{\lambda_2 t\} \times$$

$$\times \int_{-1}^{t} \exp\{\lambda_1 s\}\,\beta\,(s)\,ds + \beta\,(t)\Big]\,.$$

In view of inequality $p(0) = r(-1)$, we shall obtain:

$$c_1\left(-w-a-bc-(b+ac)\exp\{\lambda_2\}\right)+c_2\left(w-a-bc-(b+ac)\exp\{\lambda_1\}\right)=$$

$$=\frac{(w+a+bc)\int_{-1}^{0}\exp\{\lambda_2 s\}\,\alpha(s)\,ds-(w-a-bc)\int_{-1}^{0}\exp\{\lambda_1 s\}\,\beta(s)\,ds}{2\left(1-c^2\right)(b+ac)\,w}.$$

According to edge condition (7) we shall deduce:

$$c_1\left((b+ac)\left(2a+\gamma^2\right)+\left[c\,(b+ac)\,\lambda_1+2b\,(b+ac)\right]\exp\{\lambda_2\}\right)+$$

$$+c_2\left((b+ac)\left(2a+\gamma^2\right)-\left[c\,(b+ac)\,\lambda_1+2b\,(b+ac)\right]\exp\{\lambda_1\}\right)=$$

$$=\frac{-\left(2a+\gamma^2\right)\left(\int_{-1}^{0}\exp\{\lambda_2 s\}\,\alpha(s)\,ds+\int_{-1}^{0}\exp\{\lambda_1 s\}\,\beta(s)\,ds\right)}{2\left(1-c^2\right)w}+$$

$$+\frac{2\left(1-c^2\right)w\,|\,\varphi(0)\,|^2+c\,(\alpha(-1)+\beta(-1))}{2\left(1-c^2\right)w}.$$

Having taken advantage by Kramer formulas, we shall receive, that

$$c_1=\left[\left(\sqrt{\left(a^2-b^2\right)\left(1-c^2\right)}+a+bc\right)\int_{-1}^{0}\exp\left\{-\sqrt{\frac{a^2-b^2}{1-c^2}}s\right\}\alpha(s)\,ds\times\right.$$

$$\times\left(2a+\gamma^2-\left(c\sqrt{\frac{a^2-b^2}{1-c^2}}+2b\right)\exp\left\{\sqrt{\frac{a^2-b^2}{1-c^2}}\right\}\right)-$$

$$-\left(\sqrt{\left(a^2-b^2\right)\left(1-c^2\right)}-a-bc-(b+ac)\exp\left\{\sqrt{\frac{a^2-b^2}{1-c^2}}\right\}\right)\times$$

$$\times\left(2\left(1-c^2\right)\sqrt{\left(a^2-b^2\right)\left(1-c^2\right)}\,|\,\varphi(0)\,|^2+c\,(\alpha(-1)+\beta(-1))\right)-\left(2a+\gamma^2\right)\times$$

$$\left.\times\left(\int_{-1}^{0}\exp\left\{-\sqrt{\frac{a^2-b^2}{1-c^2}}s\right\}\alpha(s)\,ds+\int_{-1}^{0}\exp\left\{\sqrt{\frac{a^2-b^2}{1-c^2}}s\right\}\beta(s)\,ds\right)\right]\Bigg/$$

$$\Bigg/\left[\left(\sqrt{\left(a^2-b^2\right)\left(1-c^2\right)}-a-bc-\right.\right.\left[\left(\sqrt{\left(a^2-b^2\right)\left(1-c^2\right)}-a-bc-\right.\right.$$

$$\left.-(b+ac)\exp\left\{-\sqrt{\frac{a^2-b^2}{1-c^2}}\right\}\right)\left(2a+\gamma^2-\left(c\sqrt{\frac{a^2-b^2}{1-c^2}}+2b\right)\exp\left\{\sqrt{\frac{a^2-b^2}{1-c^2}}\right\}\right)-$$

$$-\left(\sqrt{\left(a^2-b^2\right)\left(1-c^2\right)}-a-bc-(b+ac)\exp\left\{\sqrt{\frac{a^2-b^2}{1-c^2}}\right\}\right)\times$$

$$\left.\times\left(2a+\gamma^2+\left(c\sqrt{\frac{a^2-b^2}{1-c^2}}+2b\right)\exp\left\{-\sqrt{\frac{a^2-b^2}{1-c^2}}\right\}\right)\right],$$

$$c_2 = \left[\left(\sqrt{(a^2 - b^2)(1 - c^2)} + a + bc + (b + ac) \exp\left\{ -\sqrt{\frac{a^2 - b^2}{1 - c^2}} \right\} \right) \left(\left(2a + \gamma^2\right) \times \right.\right.$$

$$\times \left(\int_{-1}^{0} \exp\left\{ -\sqrt{\frac{a^2 - b^2}{1 - c^2}} s \right\} \alpha(s)\, ds + \int_{-1}^{0} \exp\left\{ \sqrt{\frac{a^2 - b^2}{1 - c^2}} s \right\} \beta(s)\, ds \right) -$$

$$\left. -2\left(1 - c^2\right) \sqrt{(a^2 - b^2)(1 - c^2)}\, |\varphi(0)|^2 - c\left(\alpha(-1) + \beta(-1)\right) \right) -$$

$$- \left(2a + \gamma^2 + \left(c\sqrt{\frac{a^2 - b^2}{1 - c^2}} + 2b \right) \exp\left\{ -\sqrt{\frac{a^2 - b^2}{1 - c^2}} \right\} \right) \left(\left(\sqrt{(a^2 - b^2)(1 - c^2)} + a + bc \right) \times \right.$$

$$\times \int_{-1}^{0} \exp\left\{ -\sqrt{\frac{a^2 - b^2}{1 - c^2}} s \right\} \alpha(s)\, ds - \left(\sqrt{(a^2 - b^2)(1 - c^2)} - a - bc \right) \times$$

$$\left.\left. \times \int_{-1}^{0} \exp\left\{ \sqrt{\frac{a^2 - b^2}{1 - c^2}} s \right\} \beta(s)\, ds \right) \right] \Big/$$

$$\Big/ \left[\left(-\sqrt{(a^2 - b^2)(1 - c^2)} - a - bc - (b + ac) \exp\left\{ \sqrt{\frac{a^2 - b^2}{1 - c^2}} \right\} \right) \times \right.$$

$$\times \left(2a + \gamma^2 - \left(c\sqrt{\frac{a^2 - b^2}{1 - c^2}} + 2b \right) \times \right.$$

$$\left. \times \exp\left\{ \sqrt{\frac{a^2 - b^2}{1 - c^2}} \right\} \right) - \left(\sqrt{(a^2 - b^2)(1 - c^2)} - a - bc - (b + ac) \exp\left\{ \sqrt{\frac{a^2 - b^2}{1 - c^2}} \right\} \right) \times$$

$$\left. \times \left(2a + \gamma^2 + \left(c\sqrt{\frac{a^2 - b^2}{1 - c^2}} + 2b \right) \exp\left\{ -\sqrt{\frac{a^2 - b^2}{1 - c^2}} \right\} \right) \right].$$

References

[1] KREJN M. A., RUTMAN M. A.: Linejnyj operator, ostavljajuij kak invariant konus v banachovom prostranstve, Uspechi matematiceskij nauk, 1947 1947. 3, N 1. S. 3-95

[2] REPIN JU. M.: Kvadraticeskije funkcionaly Ljapunova dlja system s zapazdyvanijem, Prikluenia. mat. mech.,1965. 29 N 3. S. 364-366

[3] CARKOV E. F., SVERDAN M. L.: Kvadratini funkcionali da funkcionalno-diferencialnych rivnja, Naukovyj visnik ernivekovo universitetu, Zbirnik nauk. prac. Matematika, ernivci: Ruta, 2002, Vyp. 150, S. 107 - 120.

[4] CARKOV E. F.: Sluajnyje vozmuenija funkcionano-diferencianych uravnenij. Riga, Zinatne, 1989.

Current address

Vitaly Bereza, PhD.
Department of Mathematical and Applied Statistics, Faculty of Applied Mathematics,

Yuriy Fedkovych Chernivtsi National University,
Kotsjubynskyi Str., Chernivtsi, 58032, Ukraine
e-mail: vbereza@mail.ru

Jevgenijs CARKOVS, professor
Probability & Statistics Dpt., Riga Technical University,
Meza iela 1/4, LV-1048, phone: +371 67089517,
e-mail: carkovs@livas.lv, carkovs@latnet.lv.