7^{th} International Conference
APLIMAT 2008
Faculty of Mechanical Engineering - Slovak University of Technology in Bratislava
Section: Differential equations and their applications

THE ALGORITHM FOR ANALYSIS OF STABILITY OF LINEAR STOCHASTIC IMPULSE DYNAMICAL SYSTEM WITH MARKOV COEFFICIENTS

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Abstract. This article is the development of the research, reported on the previous APLIMAT conference. The difference is that now the phase coordinate of an impulse system satisfies a stochastic differential equation, not a linear differential equation. All the proofs became more complicated, but the decomposition of the solution in terms of powers of a small positive parameter is obtained and the algorithm of stability analysis based on this decomposition of the solution is proposed.

Key words. Impulse differential equation, stochastic differential equation, Markov dynamical system, Lyapunov equation, mean square stability

Mathematics Subject Classification: 34D05, 34E10, 34F05.

1 Introduction

Let $\{y_\varepsilon(t),\ t \ge 0\}$ be series of right continuous homogeneous Markov processes [1] on a countable space $\mathrm{Y} \subset R$ depending on parameter $\varepsilon \in (0,1)$ with weak infinitesimal operator Q defined on any element of the space **V** of bounded mapping $v\colon \mathrm{Y} \to R$

$$Qv(y) := a(y)\sum_{z\in\mathrm{Y}}(v(z)-v(y))p(y,z) \tag{1}$$

with bounded uniformly positive intensity

$$\forall y \in \mathrm{Y}: \quad 0 < a_1 \le a(y) \le a_2$$

Let $\{\tau_j,\ j \in \mathrm{N}\}$ be switching moments of the above Markov process. Then we will describe the series of Stochastic Markov Impulse Dynamical Systems in R^n with:
the phase coordinates $x_\varepsilon(t)$ of this systems satisfy:

- the stochastic differential equation

$$dx_\varepsilon = A(y_\varepsilon(t),\varepsilon)x_\varepsilon dt + C(y_\varepsilon(t),\varepsilon)x_\varepsilon dw(t) \tag{2}$$

for all $t \in (\tau_{j-1},\tau_j)$, $j \in N$; $w(t) = (w_1(t),...,w_n(t))$, $w_i(t)$, $i=1,...,n$ are Wiener processes.and the jump equation

$$x_\varepsilon(t) = x_\varepsilon(t-0) + B(y_\varepsilon(t), y_\varepsilon(t-0),\varepsilon)x_\varepsilon(t-0) \tag{3}$$

for all $\{\tau_j,\ j \in \mathrm{N}\}$, where the matrices $A(y,\varepsilon)$, $B(z,y,\varepsilon)$, $C(y,\varepsilon)$ are defined as the series

$$A(y,\varepsilon) = A_0 + \sum_{k=1}^{\infty} A_k(y)\varepsilon^k, \qquad B(z,y,\varepsilon) = \sum_{k=1}^{\infty} B_k(z,y)\varepsilon^k, \quad C(y,\varepsilon) = \sum_{k=1}^{\infty} C_k(y)\varepsilon^k$$

with matrix coefficients satisfying the inequalities

$$\sup_{y\in Y} \| A_k(y) \| := \alpha_k < \infty, \qquad \sup_{y\in Y} \| C_k(y) \| := \alpha_k < \infty, \qquad \sup_{z,y\in Y} \| B_k(z,y) \| := \beta_k < \infty, \quad k \in N \tag{4}$$

and also the series composed of α_k, β_k are convergent.
It is easy to make sure of existence and uniqueness of the above defined process $x_\varepsilon(t)$ for all $t \ge 0$.

Lemma 1. [2] The pair $\{x_\varepsilon(t), y_\varepsilon(t)\}$ jointly is the Markov process on the phase space with the weak infinitesimal operator

$$(L_\varepsilon v)(x,y) = (A(y,\varepsilon)x,\nabla)v(x,y) + (Qv)(x,y) + (G_\varepsilon v)(x,y) + \frac{1}{2}(C(y,\varepsilon)x,\nabla^2)v(x,y)$$

where $(G_\varepsilon v)(x,y) = a(y)\sum_{z\in \mathrm{Y}} (v(x + B(z,y,\varepsilon)x, z) - v(x,z))p(y,z)$

(.,.) is scalar product and ∇ is operator-gradient in R^n.

For the analysis of mean square stability conditions of the system (2)-(3) for all sufficiently small $\varepsilon > 0$ we use the second Lyapunov method with special constructed quadratic functional and Kato perturbation theory. With a view to develop this approach the semigroup of the linear continuous shift operators of conditional covariance matrices of the solutions of (2)-(3) is analysing. It is proven that the exponential mean square stability problem of (2)-(3) can be formulated as the problem of the existence of a positive solution of Lyapunov equation in the space of quadratic functionals. This result allows to propose the simple asymptotic algorithm of exponential mean square stability analysis of the MIDS with a small parameter ε.

2 Lyapunov equation for quadratic functionals

Let us denote by **Q** the space of the symmetric $n \times n$ matrix-valued continuous functions $\{q(y),\ y \in \mathrm{Y}\}$ with the subset $K := \{q \in \mathbf{Q}:\ (q(y)x,x) \ge 0,\ \forall x \in R^n,\ \forall y \in \mathrm{Y}\}$ of nonnegative-definite matrices. It is clearly to see that according to the norm, defined by

$$\| q \| := \sup\{|(q(y)x,x)|:\ y \in \mathrm{Y}, |x| = 1\}$$

the space **Q** is Banach space. For given elements q_1 and q_2 of this space we shall write $q_1 >> q_2$ if $q_1 - q_2 \in K$. It easy to prove that the set *K* is a reproducing cone in the space **Q** with the set of inner points of *K* defined as

$$\dot{K} := \{q \in K : \exists c > 0, q >> cI\}.$$

The solution of (2)-(3) with initial conditions $x_\varepsilon(0) = x, \quad y_\varepsilon(0) = y$ may be written in the form $x_\varepsilon(t,s,x,y) = X_\varepsilon(t,s,y)x$, where the family of matrix-valued functions $\{X_\varepsilon(t,s,y), \quad t \geq s \geq 0\}$ satisfies equations (2)-(3) for all $t \geq s \geq 0$ and initial conditions $X_\varepsilon(s,s,y) \equiv I$, $y_\varepsilon(0) = y$.
For any $\varepsilon \in [0,1]$ one can introduce the one-parameter family of operators

$$(T_\varepsilon(t)q)(y) = \mathrm{E}_y^{(s)}\left\{X_\varepsilon^T(t+s,s,y)q(y_\varepsilon(t+s))X_\varepsilon(t+s,s,y)\right\}.$$

By definition of the semigroup $T_\varepsilon(t)$ leaves the cone *K* invariant because if $(q(y)x,x) \geq 0$ for any $y \in Y, \quad x \in R^n$ then

$$(T_\varepsilon(t)q)(y) = \mathrm{E}_y\left(X_\varepsilon^T(t+s,s,y)q(y_\varepsilon(t+s))X_\varepsilon(t+s,s,y)\right) \geq 0 .$$

Lemma 2. The family of operators $\{T_\varepsilon(t), \ t \geq 0\}$ forms a strongly continuous semigroup with an infinitesimal operator defined by

$$\begin{aligned}(\mathrm{A}_\varepsilon)q(y) &= A^T(y,\varepsilon)q(y) + a(y)\sum_{z\in \mathrm{Y}} B^T(z,y,\varepsilon)q(z)p(y,z) + C^T(y,\varepsilon)q(y)C(y,\varepsilon) + \\ &+ q(y)A(y,\varepsilon) + a(y)\sum_{z\in \mathrm{Y}} q(z)B(z,y,\varepsilon)p(y,z) + \\ &+ a(y)\sum_{z\in \mathrm{Y}} B^T(z,y,\varepsilon)q(z)B(z,y,\varepsilon)p(y,z) + Qq(y)\end{aligned}$$

Lemma 3. If the spectrum $\sigma_\varepsilon := \sigma_\varepsilon(\mathrm{A})$ of the operator A_ε is situated in the half-plane $C_\alpha := \{C : \Re\lambda \leq \alpha\}$ and there exists an eigenvalue with the real part equal to α, then α also is an eigenvalue.

Theorem 1. The solution of (2)-(3) is exponentially square stable if and only if there exists a positive number ρ such that $\sigma_\varepsilon \subset C_{-\rho}$.

Theorem 2. The solution of (2)-(3) is exponentially mean square stable if and only if there exists $q \in \dot{K}$ and $r \in \dot{K}$ such that

$$\mathrm{A}_\varepsilon q = -r \tag{5}$$

This equation can be called the Lyapunov equation for MIDS (2)-(3).

Corollary. MIDS (2)-(3) is exponentially mean square stable if and only if there exists a solution $q \in \dot{K}$ of equation (5) with $r=I$.

3 Resolution of Lyapunov equation

Let us represent the decomposition of the operator A_ε in the following form

$$\mathrm{A}_\varepsilon = G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + ...$$

where the linear continuous operators G_0, G_1 and G_2 are defined on an arbitrary element for $q \in Q$ by equalities:

$$(G_0 q)(y) = A_0^T(y)q(y) + q(y)A_0(y) + (Qq)(y),$$

$$(G_1 q)(y) = A_1^T(y)q(y) + q(y)A_1(y) + a(y)\sum_{z\in Y}[B_1^T(z,y)q(z) + q(z)B_1(z,y)]p(y,z) + C_1^T(y)q(y)C_1(y),$$

$$(G_2 q)(y) = A_2^T(y)q(y) + q(y)A_2(y) + a(y)\sum_{z\in Y}[B_2^T(z,y)q(z) + q(z)B_2(z,y)]p(y,z) +$$
$$+ a(y)\sum_{z\in Y} B_1^T(z,y)q(z)B_1(z,y)p(y,z) + C_2^T(y)q(y)C_2(y)$$

Lemma 4. Let the real eigenvalue of the operator G_0 $\lambda_0 \notin \sigma(A_\varepsilon)$ for all sufficiently small $\varepsilon > 0$. Under the above assumptions there exists positive ε_0 such that the solution q^ε of the equation

$$A_\varepsilon q^\varepsilon - \lambda_0 q^\varepsilon = f$$

for any $f \in Q$ and $\varepsilon \in (0, \varepsilon_0)$ has the form

$$q^\varepsilon = \sum_{k=-d}^{\infty} \varepsilon^k q_k$$

with some $d \in N$.

Let us assume that $\lambda_0 = 0$ and let

$$\hat{q}(\varepsilon) = \sum_{k=-d}^{0} \varepsilon^k q_k,$$

$$\tilde{q} = \sum_{k=0}^{\infty} \varepsilon^k q_{k+1},$$

where $q^\varepsilon = \hat{q}(\varepsilon) + \varepsilon\tilde{q}(\varepsilon)$ is the solution of the equation

$$A_\varepsilon q^\varepsilon = -I \qquad (6)$$

The proposed algorithm uses the method of equating the coefficients corresponding to the same powers of the parameter ε in the equation (6) and the well known Fredholm alternative.

4 Example

Let consider the Markov process $y(t)$ with two states space $Y = \{0, 1\}$ defined by the infinitesimal matrix

$$Q = \begin{pmatrix} -\alpha & \alpha \\ 1-\alpha & \alpha - 1 \end{pmatrix},$$

where $\alpha \in (0;1)$. One can find elements of the transition probability matrix from the definition of infinitesimal operator

$$p(0,0)=0 \quad p(0,1)=1$$
$$p(1,0)=0 \quad p(1,1)=1$$

and intensities $a(0)=\alpha, \quad a(1)=1-\alpha$. Therefore the invariant measure of this Markov process is given by the equalities $\mu(0)=1-\alpha, \quad \mu(1)=\alpha$. Let the above Markov process be switching process for two dimensional stochastic Markov impulse dynamical system given by the equations:

$$\begin{cases} dx = \left(A_0(y(t)) + \varepsilon A_1(y(t))\right)x\,dt + C_1(y(t))\,dw(t) & t \in \left(\tau_{j-1}, \tau_j\right), \quad j \in N \qquad (7) \\ x(t) = \left(I + \varepsilon B_1(y(t), y(t-))\right)x(t-) & t \in \left\{\tau_j, \ j \in \mathrm{N}\right\} \qquad (8) \end{cases}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ -\nu^2 & 0 \end{pmatrix} \qquad A_1(y) = Ay = \begin{pmatrix} 0 & 0 \\ 0 & -2\delta \end{pmatrix} y$$

$$B_1(z,y) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}(1-z)y \qquad C_1(y) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} y$$

In the first step we must find constant matrices p and q satisfying the equations: $\begin{cases} G^* p(y) = 0 \\ G q(y) = 0 \end{cases}$.

So we solve the system:

$$\begin{cases} \begin{pmatrix} 0 & 1 \\ -\nu^2 & 0 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 & -\nu^2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -\nu^2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} + \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\nu^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{cases}$$

Under condition $Tr\{pq\}=1$ we obtain the matrices

$$p := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{\nu^2}{2} \end{pmatrix} \qquad q := \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\nu^2} \end{pmatrix}.$$

Then we should calculate $< p, G_1 q >$. Due to $\mu(1)=\alpha$, $a(1)=1-\alpha$, $A_1(0)=0$, $B_1(z,0)\equiv 0$, $B_1(1,1)\equiv 0$ one can write:

$$< p, G_1 q >= \alpha\, Tr\{p(G_1 q)(1)\} = \alpha\, Tr\left\{p\left(A^T q + qA + (1-\alpha)\left(B^T q + qB\right) + C^T q C\right)(1)\right\}$$

1) If $< p, G_1 q > \neq 0$, then $d=1$ and $q_{-1}(y) \equiv \gamma q$ with $\gamma = -\dfrac{Tr\{p, I\}}{< p, G_1 q >}$. We find the matrix $q_0(y)$ satisfying the equality $(G_0 q_0)(y) = -I - \gamma(G_1 q)(y)$ and take the matrix $\hat{q}(\varepsilon)$ in a form

$q_0(y)+\frac{\gamma q}{\varepsilon}$. Because matrix q is positively definite the above matrix for sufficiently small $\varepsilon > 0$ is positively definite if and only if $\gamma > 0$. So the mean square stability condition is

$$Tr\{p(A^T q + qA + (1-\alpha)(B^T q + qB) + C^T qC)(1)\} < 0$$

2) If $< p, G_1 q >= 0$, we put $d > 1$ and find a solution of the equation

$$(G_0 q_{-1})(y) := A_0^T q_{-1}(y) + q_{-1}(y)A_0 + (Qq_{-1})(y) = -(G_1 q)(y)$$

and check a normal solvability condition of the equation

$$(G_0 q_0)(y) = -I - \gamma((G_1 q_{-1})(y) - (G_2 q)(y))$$

with some constant γ. If $< p, G_1 q_{-1} > + < p, G_2 q > \neq 0$ from the above condition one could find the number

$$\gamma = -\frac{Tr\{p, I\}}{< p, G_1 q_{-1} > + < p, G_2 q >}.$$

Now due to inequality $Tr\{p, I\} > 0$ one can conclude that the mean square stability condition of the solution of stochastic Markov impulse dynamical system (7)-(8) is

$$< p, G_1 q_{-1} > + < p, G_2 q > < 0.$$

3) But if $< p, G_1 q_{-1} > + < p, G_2 q >= 0$, we continue the procedure.

Acknowledgement

The paper was supported by grant from The Latvian Academy of Sciences no. 05-1880 with title "*Asymptotic methods for the stochastic analysis of Markov impulse systems with application to the financial econometrics* ".

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