



## TIME-OPTIMAL ADAPTIVE CONTROL OF DYNAMIC SYSTEMS

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**Abstract.** The process of control over a complex object is hampered by the fact that such an object is usually evolving in some uncertain way; its parameters change randomly. To solve this task, it is common to involve the adaptation methods for identification of the objects with changing parameters. Controls of this kind are usually called the control with an adaptable model. This paper solves the task of the associative control of an object, whose parameters are changing in an uncertain way. To enable efficient control, it is necessary to constantly correct model's parameters, that is, to have the associative control with the parametric adaptation. Thus the task of control of an object comprises the algorithm of estimation of object's parameters and algorithm of control of this object. The solution of this task relates to the class of associative algorithms. This paper provides the results of investigations of the dynamic system with variable parameters. To perform the control efficiently, this system would be supported by time-optimal high speed adaptive control using the algorithms of object's parameter identification.

**Keywords:** time-optimal adaptive control, parametric identification, identifier

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### 1 Problem definition

It is necessary to develop an adaptive control algorithm, which converses a system from any initial state into the origin in the least time, for the predetermined linear dynamic system with time changing parameters

$$\begin{aligned}\dot{\mathbf{X}}(t) &= \mathbf{A}(t)\mathbf{X}(t) + \mathbf{B}(t)\mathbf{U}(t) + \mathbf{T}_d(t) \\ \mathbf{Y}(t) &= \mathbf{C}(t)\mathbf{X}(t) + \mathbf{v}(t),\end{aligned}\tag{1.1}$$

where  $\mathbf{A}$  - ( $n \times n$ ) state matrix,  $\mathbf{B}$  - ( $m \times r$ ) control matrix,  $\mathbf{C}$  - ( $m \times n$ ) matrix of measurable variables,  $\mathbf{X}$  - ( $n \times 1$ ) state vector of the system,  $\mathbf{Y}$  - ( $m \times 1$ ) vector of measurable variables,  $\mathbf{U}$  - ( $n \times r$ ) control vector,  $\mathbf{T}_d(t)$ ,  $\mathbf{v}(t)$  - vectors of external actions operating on the object and on the measurable system.

Let us describe the setting of the system (1.1) in brief in respect to physics.

Let  $\mathbf{z}(t)$  be a vector that consists of  $m$  components  $z_1(t), z_2(t), \dots, z_m(t)$ . The vector  $\mathbf{z}(t)$  is tentatively called desired output. The system (1.1) is to be controlled in such a fashion that output vector of measurable variables  $\mathbf{y}(t)$  would be “close” to the vector  $\mathbf{z}(t)$ . In this case vector of control error  $\mathbf{e}(t)$  can be recorded as  $\mathbf{e}(t)=\mathbf{z}(t)-\mathbf{y}(t)$ . In other words, such a control  $\mathbf{u}(t)$  is to be generated that could provide a minimal error value  $\mathbf{e}(t)$ . Assuming that control  $\mathbf{u}(t)$  is not limited by magnitude, the cases in which value  $\mathbf{u}(t)$  is unboundedly increased, are possible. In order to avoid such extreme situations in object control contour, it is necessary to include limitations that would exclude overstated requirements for the system controller.

All the above mentioned physical requirements are to be considered in the mathematical functional. Performance quality of the adaptive control system will be evaluated based on quantitative index of the mathematical functional.

To solve the defined problem, as a mathematical functional the following quadratic functional is used:

$$H(u) = 1/2 \langle \mathbf{e}_1(T), \mathbf{F} \mathbf{e}_2(T) \rangle + 1/2 \int_0^T [\langle \mathbf{e}(t), \mathbf{Q}(t) \mathbf{e}(t) \rangle + \langle \mathbf{u}(t), \mathbf{R}(t) \mathbf{u}(t) \rangle] dt, \quad (1.2)$$

where  $\mathbf{F}$  - ( $m \times m$ ) constant positively semi-determined matrix;  $\mathbf{Q}(t)$  - ( $m \times m$ ) positively semi-determined matrix;  $\mathbf{R}(t)$  - ( $r \times r$ ) positively determined matrix;  $T$  - eventual time.

Two vectors dot product is indicated with  $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$ .

Such a definition of the problem (a functioning object control problem) is a classical definition.

In numerous publications on investigation of optimal control systems all problems related to practical realization of optimal control algorithms are examined [1,3]. Indeed, optimal control does exist and is explicitly defined [1] by

$$\mathbf{u}(t) = \mathbf{R}^{-1}(t) \mathbf{B}^T(t) [\mathbf{g}(t) - \mathbf{K}(t) \mathbf{x}(t)]. \quad (1.3)$$

True symmetric positively determined ( $n \times n$ ) matrix  $\mathbf{K}(t)$  is the solution of matrix differential Riccati equation:

$$\dot{\mathbf{K}}(t) = -\mathbf{K}(t) \mathbf{A}(t) - \mathbf{A}^T(t) \mathbf{K}(t) + \mathbf{K}(t) \mathbf{B}(t) \mathbf{R}^{-1}(t) \mathbf{B}^T(t) \mathbf{K}(t) - \mathbf{C}^T(t) \mathbf{Q}(t) \mathbf{C}(t) \quad (1.4)$$

with boundary condition  $\mathbf{K}(T) = \mathbf{C}^T(T) \mathbf{F} \mathbf{C}(T)$ .

The second part of the problem definition for dynamic system's optimal control is providing its time-optimal high speed. Thus the requirements for the chosen quality control functional (1.2) expand. Additional requirements, on the one hand, are to consider time variations of the dynamic system's parameters (coefficients of the matrices  $\mathbf{A}(t)$  and  $\mathbf{B}(t)$ ), and adaptation of control algorithm that forms control signals (1.3) vector both to the changes of the system's internal parameters and to the variable in time system's (1.1) external influences, on the other hand.

It is planned to achieve realization of these additional requirements by including an identifier into control contour of the dynamic system [4,5].

The following tasks are set for the identifier algorithm:

- solution of parametric identification problem for the controlled dynamic system, i.e. calculation of the current values of state matrix  $\mathbf{A}(t)$  and control matrix  $\mathbf{B}(t)$  in terms of the measured consequences of state vector  $\mathbf{X}(t)$  and control vector  $\mathbf{U}(t)$  in the defined time interval;
- calculation of gain matrix  $\mathbf{K}(t)$  in each renovated interval of matrices  $\mathbf{A}(t), \mathbf{B}(t)$ ;

- modification of the control algorithm in the required time interval both at system's parameters modification and at the system being modified by external disturbing factors.

## 2 Dynamic system analysis based on its functioning

After a detailed consideration of the defined task let us make a short dynamic system analysis. The presence of time-optimal high speed adaptive system in the dynamic system is predetermined by its operation mode. Let us consider special motion of a solid body (a satellite or any spacecraft) [1]. Its schematic position is shown in Figure 2.1.

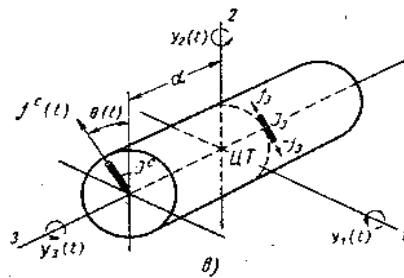


Figure 2.1. Spacecraft model

Three axes are put through the center of mass (indicated as 1, 2, and 3 accordingly). The axes will be fixed to the object. Let us introduce these definitions:

$I_1, I_2$  и  $I_3$  - the inertial moments with respect to the axes;  
 $y_1(t), y_2(t)$ , and  $y_3(t)$  - object's angular velocities relative to the axes.

The introduced coordinate origin is rigidly bound to the object and all measurements will be taken with respect to the coordinate origin.

In the absence of external influences the equations of motion for this body are described with a differential equation system (Euler equations). We will choose an object with one symmetry axis (axis 3) in order to illustrate application of optimal control theory. That means that moments of inertia regarding axes 1 and 2 are equal, i.e.  $I = I_1 = I_2 \neq I_3$ . Considering this relationship, Euler equations take the following form:

$$\begin{aligned} I \dot{y}_1(t) &= (I - I_3)y_3(t)y_2(t) \\ I \dot{y}_2(t) &= (I_3 - I)y_3(t)y_1(t) \\ I_3 \dot{y}_3(t) &= 0. \end{aligned} \tag{2.1}$$

Let us assume that we formed moment vector  $\mathbf{u}(t)$  and the equations of motion (2.1) took the following form:

$$\begin{aligned} I \dot{y}_1(t) &= (I - I_3)y_3(t)y_2(t) + u_1(t) \\ I \dot{y}_2(t) &= (I_3 - I)y_3(t)y_1(t) + u_2(t) \\ I_3 \dot{y}_3(t) &= u_3(t). \end{aligned} \tag{2.2}$$

Control moments can be obtained in different ways. In this case jet engines will be used for this purpose. A rotary (vernier) engine  $J^c$  fixed on the object in such a manner that it can rotate within plane that is perpendicular to the axis 3 will be used for object control.

To make the task more specific, the chosen criterion is establishment time of the three velocities' indicated values.

Thus, the physical definition of the task is the following: to reduce the initial velocities  $y_1(0)$  and  $y_2(0)$  down to zero in minimal time while preserving velocity  $y_3(t)$  equal to the defined value  $\hat{y}_3$ , i.e.

$$y_3(t) = y_3(0) = \hat{y}_3 = \text{const} \quad \text{at } t > 0. \quad (2.3)$$

Differential equation for velocities  $y_1(t)$  and  $y_2(t)$  can be obtained by inserting equation (2.3) into equation (2.2). For more convenient consideration, angular moments  $x_1(t)$  and  $x_2(t)$  can be inserted into the equations with the help of the following relationships:

$$x_1(t) = Iy_1(t); \quad x_2(t) = Iy_2(t). \quad (2.4)$$

Then from the equations (2.2), (2.3), and (2.4) it follows that angular moments obey the differential equations

$$\begin{aligned} \dot{x}_1(t) &= \omega x_2(t) + u_1(t) \\ \dot{x}_2(t) &= -\omega x_1(t) + u_2(t), \\ \sqrt{u_1^2(t) + u_2^2(t)} &\leq \alpha F^c, \end{aligned} \quad (2.5)$$

where  $\alpha$  - distance from engine mounting point to the object's center of mass,  $\omega$  - constant, equal to

$$\omega = \frac{(I - I_3)}{I} \hat{y}_3 \quad \text{radian/sec.} \quad (2.6)$$

It is necessary to generate control vector  $\mathbf{u}(t)$  that transfers  $\mathbf{x}(t)$  into zero in a minimal time.

To define such time-optimal high speed, is necessary to solve a set of equations (2.5). The solution shows that the vector is equal to

$$\mathbf{u}(t) = -2 \frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|} = -2 \frac{\omega \mathbf{x}(t)}{\|\omega \mathbf{x}(t)\|} \quad (2.7)$$

or, equivalently,

$$\begin{aligned} u_1(t) &= -\frac{2x_1(t)}{\sqrt{x_1^2(t) + x_2^2(t)}}; \\ u_2(t) &= -\frac{2x_2(t)}{\sqrt{x_1^2(t) + x_2^2(t)}}. \end{aligned} \quad (2.8)$$

$\theta(t)$  identifies an angle between engine axis  $J^c$  and axis 2:

$$\theta(t) = \text{arctg} \frac{x_2(t)}{x_1(t)} = \text{arctg} \frac{y_2(t)}{y_1(t)}. \quad (2.9)$$

Minimal time  $t_d$  required for transforming  $(\omega x_1, \omega x_2)$  into  $(0,0)$  by the control vector  $\mathbf{u}(t)$  (2.7), is equal to (if  $\alpha F^c = 2$ ) [1].

$$t_d = \frac{1}{\alpha F^c} \sqrt{(\omega x_1)^2 + (\omega x_2)^2} = \frac{1}{2} \sqrt{(\omega x_1)^2 + (\omega x_2)^2}. \quad (2.10)$$

Let us make some preliminary observations on practical implementation of time-optimal high speed control scheme with application of vernier engine.

First of all, the set limitation on angular velocity  $y_3(t)$ , that previously was equal to defined value  $\hat{y}_3$ , can be removed. Let's remove this limitation and measure three angular velocities  $y_1(0)$ ,  $y_2(0)$  and  $y_3(0)$  at the initial time moment  $t = 0$ ; moreover the initial value is not equal to the desired value:

$$y_3(0) \neq \hat{y}_3 \text{ and } y_1(0) \neq 0; y_2(0) \neq 0. \quad (2.11)$$

Angular velocity  $y_3(t)$  can be independently controlled with engine  $J_3$ . Since  $y_3(t)$  is defined by the equation

$$y_3(t) = y_3(0) + 1/I_3 \int_0^t u_3(\tau) d\tau, \quad (2.12)$$

$\omega(t)$  possesses the value:  $\omega(t) = [(I-I_3)/I]y_3(t)$  according to (2.6)

$$\omega(t) = \frac{(I-I_3)}{I} y_3(t). \quad (2.13)$$

Hence differential equations (2.5) in respect to the angular moments  $x_1(t)$  and  $x_2(t)$  take the form of

$$\begin{aligned} \dot{x}_1(t) &= \omega(t)x_2(t) + u_1(t) \\ \dot{x}_2(t) &= -\omega(t)x_1(t) + u_2(t). \end{aligned} \quad (2.14)$$

The dynamic system described by equations with variable parameters (2.14) is an optimal system to which control law formulated in [1] is applicable: "The unique time-optimal high speed control  $\mathbf{u}(t)$ , i.e. such a control that minimizes time functional of the transient phenomenon and fuel consumption functional, is equal to  $\mathbf{u}(t) = -m \frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|}$ ,  $\|\mathbf{u}(t)\| \leq m$  at any  $t$  value". Therefore, the system with vernier engine can be applied to transform  $y_1(t)$  and  $y_2(t)$  into zero in a minimal time and with minimal fuel consumption simultaneously performing control of velocity  $y_3(t)$ . Moreover, time-optimal high speed control system with vernier engine does not depend on speed of rotation  $y_3(t)$ .

### 3 Dynamic system analysis based on adaptive control with parametric identification

The results of dynamic system (a space vehicle) analysis in respect to its movement optimization subject to operation physical conditions and providing time-optimal high speed, as well as the recommendations on optimal control algorithm composition allow us to proceed to the problem of closed-loop control system that implements adaptive control principle based on parametric identification.

Adaptive control principle for a real object that operates in transient conditions proposes formation (correction) of regulator algorithm in time interval of disturbance influence on the object, at which the value of quality control criterion functional exceeds permissible limits.

As it was mentioned earlier [5], the task of adaptive control of the closed-loop control based on parametric identification algorithm application includes:

- measurement of the controlled object state and control vectors at each step of controlled coordinates discretization;

- choice of a proper quality of adaptive control efficiency criterion functional, which is calculated on a specified stage of information update about state and object control vectors;
- controlled system parameter evaluation (coefficients of state matrix  $\mathbf{A}(t)$  and control matrix  $\mathbf{B}(t)$ ) within the applicable time period;
- formation (correction) of system regulator algorithm subject to computational value of the selected optimal control quality functional and the controlled system parameter evaluation.

A list of solvable tasks at optimal adaptive control of closed-loop automated control system based on parametric identification algorithms practically includes requirements that were defined while analyzing the space vehicle optimal control task with application of vernier engine. Figure 3.1 shows the flow chart of closed-loop control based on parametric identification.

Solution of the above mentioned tasks for time-optimal high speed of the adaptive control for the selected flow chart for a dynamic loop system, the controlled object of which is spacecraft, will be performed using algorithms of IT system MATLAB [4].

Let the dynamics of the controlled system be described by (1.1). It is necessary to synthesize an adaptive control algorithm that transfers system from any initial state into origin of coordinates in a minimal time. Control object is described by Euler equations in form (2.2).

Let us introduce more convenient coordinates  $x_1(t)$  and  $x_2(t)$  and define  $x_1(t) = \omega / (K_1 y_1(t))$ ;  $x_2(t) = 1 / (K_1 y_2(t))$ . Then the state equation of the open loop will look as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K_1 u(t) \end{bmatrix}$$

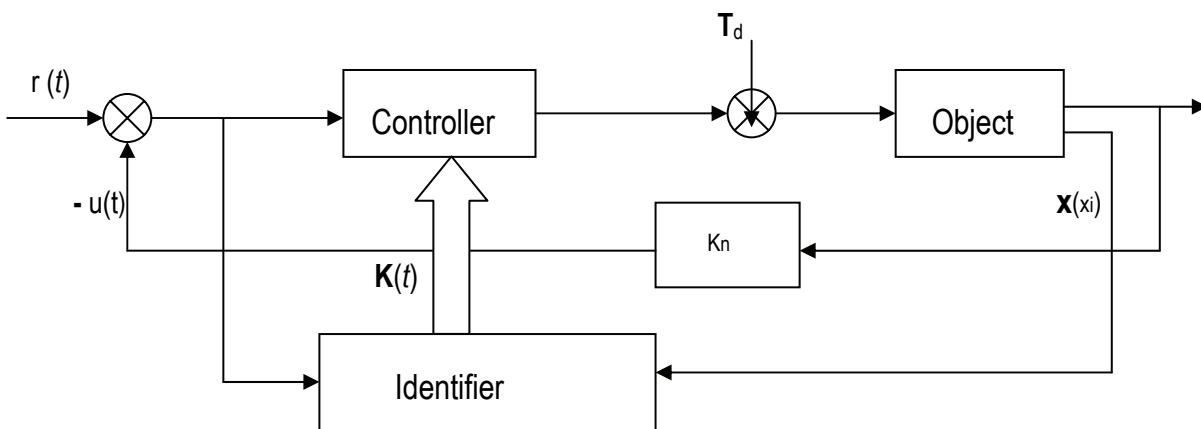


Figure 3.1. Spacecraft control system flow chart

A first-order integrating element with transfer function  $W(s)=5/s$  is introduced as the regulator for open loop. Let us transform the open loop into the closed one (3.1) by assigning a feedback factor (loop gain)  $K_b$  equal to 1 and parameter of the object  $\omega = 0.5$ . Presenting the closed loop in the state space we obtain:

$$\begin{aligned} \dot{X} &= \mathbf{A}X + \mathbf{B}u; \quad Y = \mathbf{C}X + \mathbf{D}u; \\ \dot{X} &= \begin{bmatrix} 0 & -0.25 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \quad Y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \end{aligned} \quad (3.2)$$

Figure 3.2 shows impulse transient function of the closed control loop (3.2). The function proves system instability.

Let us create a new control algorithm for the given closed loop using linear-quadratic algorithm in the system's feedback loop. Coordinates of the system state vector  $\mathbf{X}(t)$  are the entering signals of the algorithm. In order to generate such algorithm for system regulator, MATLAB system functions are used that employ our selected optimization criterion  $H(u)$  as relationship (1.2). As a result of such synthesis, we obtain optimal control algorithm in feedback loop that consists of four coefficients  $\mathbf{K}(t)$  of Riccati equation:

$$\mathbf{K}^T(t) = [44.72/s \quad 23.17 \quad 44.16 \quad 6.808].$$

In this case controlled system with a new-created regulator takes the following form in the state space:

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{u}, \quad \mathbf{Y} = \mathbf{C}\mathbf{X} + \mathbf{D}\mathbf{u}, \quad \mathbf{C} = [0 \ 1 \ 0 \ 0], \quad \mathbf{D} = [0 \ 0],$$

$$\dot{\mathbf{X}} = \begin{bmatrix} 0 & -0.25 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -23.17 & -45.16 & 0 & 5.59 \\ 0 & -8 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 6.808 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \quad (3.3)$$

Figure 3.3 shows impulse transient function of the closed control loop (3.3). The function proves that the system with linear-quadratic regulator in the feedback loop is also unstable.

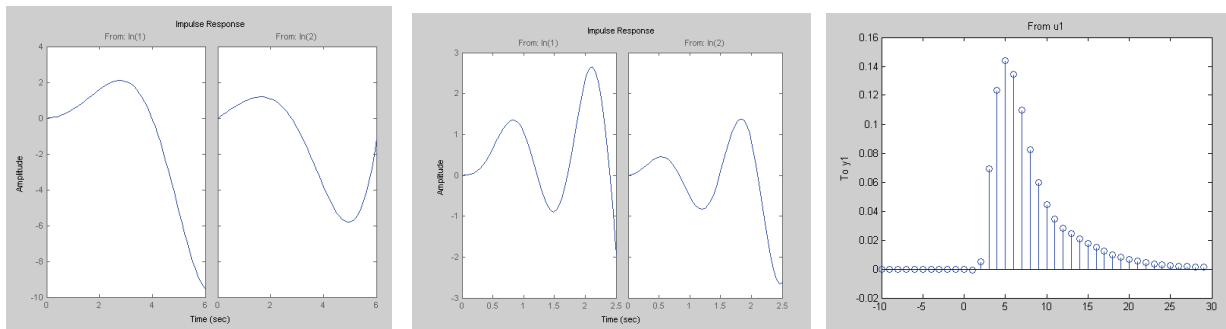


Figure 3.2. Impulse response (3.2) Figure 3.3. Impulse response (3.3) Figure 3.4. Impulse response (3.4)

Further complication of the controlled system regulator algorithm does not provide object's operation stability. In order to solve the set problem, we introduce identifier algorithms into the object control contour. The algorithms provide calculation of the system's state and control matrices parameters. Obtained evaluations of the system parameters are included into optimal values algorithm for Riccati vector (1.4). Vector  $\mathbf{K}(t)$  provides calculation of the optimal control vector  $\mathbf{u}(t)$  for the system (1.3), which provides the object's time-optimal high speed return from a random trajectory point into the origin. Implementing the previous algorithms minimization of the quality functional  $H(u)$  of optimal system's motion is provided, the numerical value  $H(u)$  of which is a result of modeling.

Using MATLAB programs leads to the following results based on the algorithms of time-optimal adaptive system control problem.

Controlled closed loop in the state space has the form:  $\mathbf{X}(t+T_s) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{u}(t), \mathbf{Y} = \mathbf{C}\mathbf{X} + \mathbf{D}\mathbf{u},$

$$\mathbf{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.35119 & -1.2958 & 1.8787 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -0.00092749 \\ 0.0047605 \\ 0.069107 \end{bmatrix} [u_1], \quad \mathbf{X}(0) = \begin{bmatrix} 4.7215 \\ 4.7954 \\ 4.824 \end{bmatrix}$$

$$y(t) = [1 \ 0 \ 0] \mathbf{X}(t) + [0] \mathbf{U}(t). \quad (3.4)$$

Figure 3.4 shows impulse transient function of the closed control loop (3.4). The function proves that the system with linear-quadratic regulator in the feedback loop and identification algorithm that calculates parameters of the controlled system in the process of its operation provides system's stability.

Let us define stability margin of the modeled systems (at different regulator algorithms) at amplitude (Gm) and phase (Pm) on appropriate frequencies Wcg and Wcp. For the controlled system in which evaluations of the system parameters in control algorithm are not used we obtain: Gm = 10.8 dB; Pm = Inf; Wcg = 3.1416 rad/sec.; Wcp = NaN.

The controlled system is unstable as shown in Figure 3.5.

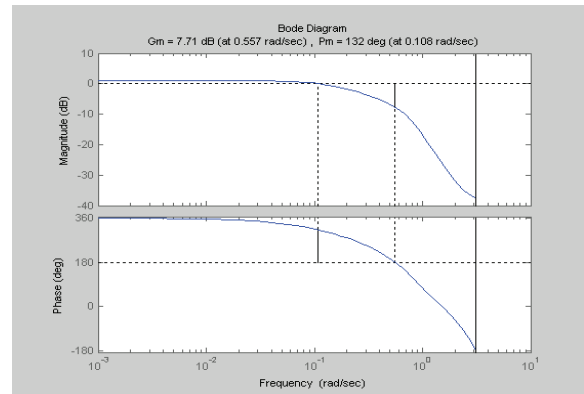
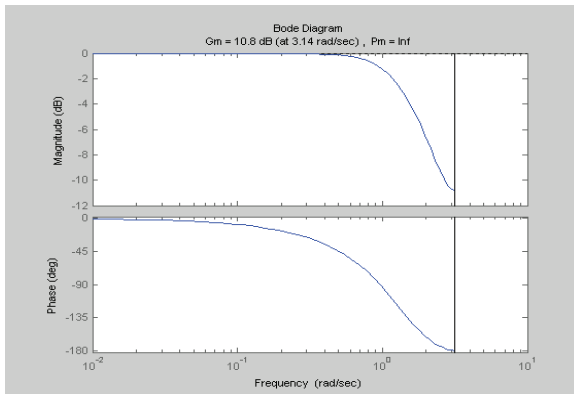


Figure 3.5. Magnitude and Phase response (unstable)      Figure 3.6. Magnitude and Phase response (stable)

For the controlled system in which evaluations of the system parameters in control algorithm are used, we obtain: Gm = 7.71 dB; Pm = 132 deg; Wcg = 0.5571 rad/sec.; Wcp = 0.1080 rad/sec. The controlled system is stable as can be seen from Figure 3.6.

## 4 Conclusions

For the evaluation of the control contour for the reviewed dynamic system is evaluation of high speed time of such an adaptive control system. This parameter  $t_d$  value is calculated according to relation (2.10). As a result of the performed modeling of time-optimal high speed adaptive system with linear-quadratic regulator in the feedback loop and parametric identification algorithm the high speed time  $t_d$  was 2.0501 seconds. Axis rotation angle of the vernier engine  $J^c$  in regard to the axis 2 of the controlled object (for expression (2.9)) is:  $\theta(t) = 0.7890$  rad/s = 45 degrees. Comparing efficiency of the offered optimal adaptive system for automatic control of a space vehicle with similar systems that implement control algorithms based on maximum Pontryagin principle [1], the benefit is more than 2.6 times.



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