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ON CONTINUOUS STOCHASTIC MODELING OF HETEROSKEDASTIC CONDITIONAL VARIANCE

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Abstract. The proposal continuous stochastic differential equation for conditional variance is constructed as a diffusion approximation of discrete ARCH process. In contrast to classical auto regressive models with independent random perturbations our paper deals with uncertainty given as a stationary ergodic Markov chain. The method is based on stochastic analysis approach to finite dimensional difference equations with proportional to small parameter ε increments. Writing a point-form solution of this difference equation as vertexes of a time-dependent continuous broken line given on the segment [0,1] with ε-dependent scaling of intervals between vertexes and tending ε to zero we apply probabilistic limit theorems for dynamical systems with rapid Markov switching. The distribution of stationary solution of resulting stochastic equation may be successfully used for analysis of initial discrete model. This method permits to discuss a correlation effect on log of cumulative excess return with stochastic volatility. model-based analysis shows that it is important to take into account possible serial residual correlation in conditional variance process. The proposed method is applied to investigate the GARCH(1,1) and GARCH(2,1) processes under assumption that random variables are serially correlated. As a result it is possible to find continuous stochastic differential equations these processes converge to in distribution.

Key words and phrases. Markov dynamical system, diffusion approximation, ARCH model

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1 Introduction

Many econometrical studies [1] and [3] have documented that financial time series tend to be highly heteroskedastic. This has many implications for many areas of macroeconomics and finance, including the term structure of interest rates, option pricing and dynamic capital-asset pricing theory. In the same time econometricians have also been very active in developing models of conditional heteroskedasticity. The most widely used models of dynamic conditional variance have been the ARCH models first introduced by [2]. In most general form, a univariate ARCH model makes conditional variance at time t a function of exogenous and lagged endogenous variables, time, parameters and past residuals.

In contrast to the stochastic differential equation models so frequently used in theoretical finance literature, ARCH models are discrete time stochastic difference equation systems. Empirics have favored the discrete time approach of ARCH as virtually all time series data are recorded only on discrete time intervals and a discrete time ARCH likelihood function is usually easy to compute and maximize. By contrast, the likelihood of a nonlinear stochastic differential equation system observed at discrete intervals can be very difficult to derive, especially when there are unobservable state variables (like conditional variance) in the system.

Substantial work has been done on relation between the continuous time nonlinear stochastic differential systems, used so much in theoretical literature, and the ARCH stochastic difference equation systems, favored by empirics. Although the two literatures have developed quite independently there have been attempts to reconcile the discrete and continuous models. Nelson [4] is one of the first to partially bridge the gap by developing conditions under which ARCH stochastic difference equation systems converge in distribution to Ito processes as the length of the discrete time intervals goes to zero.

In his work [4] investigates the GARCH(1,1)-M process of [3] for the log of cumulative excess return Y_t:

$$r_{t+1} = r_t + \varepsilon\gamma\sigma_t^2 + \varepsilon\sigma_t Z_t \tag{1}$$
$$\sigma_{t+1}^2 = \varepsilon^2(\omega - \theta\sigma_t^2) + \varepsilon\alpha\sigma_t^2 Z_t \tag{2}$$

He derives diffusion approximation equation in a form of linear stochastic equation

$$d\sigma_t^2 = (\omega - \theta\sigma_t^2)dt + \alpha\sigma_t^2 dw(t) \tag{3}$$

and shows that in continuous time the stationary distribution for the GARCH(1,1) conditional variance process is an inverted two parametric gamma.

In our paper we change the assumption about independence of random process in equations (1)-(2) and assume that random coefficients in are serially correlated

$$r_{t+1} = r_t + c\sigma_t^2 + \sigma_t y_t \tag{4}$$
$$\sigma_{t+1}^2 = \varepsilon^2(\omega - \theta\sigma_t^2) + \varepsilon\alpha\sigma_t^2 y_t \tag{5}$$

where discrete random process y_t satisfies AR(1) equation

$$y_t = \rho y_{t-1} + \sqrt{1-\rho^2} Z_t \tag{6}$$

For analysis of the above equations we apply method and results of the paper [5], which are briefly stated in the first section. In the second section we have derived the stochastic approximation equation for (5) under condition (6) in a form of stochastic equation

$$d\sigma_t^2 = (\omega + (\alpha^2\kappa(\rho) - \theta)\sigma_t^2)dt + \alpha\sqrt{1 - 2\kappa(\rho)}\sigma_t^2 dw(t) \tag{7}$$

with coefficients dependent on correlation parameter ρ from (6). Analyzing ergodic property and stationary solution of this equation we have shown that it is important to take into account possible serial correlation in conditional variance process. Under the same assumption that random variables are serially correlated we analyze GARCH(2,1) conditional variance process

$$r_{t+1} = r_t + c\sigma_t^2 + \sigma_t y_t \tag{8}$$

$$\sigma_{t+1}^2 = \varepsilon^2(\omega - \theta\sigma_t^2) + \varepsilon\alpha\sigma_t^2 y_t \tag{9}$$

$$y_t = \omega_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \beta_1 y_{t-1} Z_t \tag{10}$$

Equations (8) to (10) will be rewritten in a vectorial form and stochastic differential equations will be developed using results found in [5].

2 Diffusion Approximation of Discrete Markov Dynamical Systems

2.1 Related Work

The problem of asymptotic analysis of dynamical systems under small random perturbations has been discussed in many mathematical and engineering papers. Apparently, A.V. Skorokhod was the first mathematician, which has proved that the probabilistic limit theorems may be successfully used to approximate distributions of solutions of random dynamical systems by the solutions of stochastic differential equations on any finite time interval (see bibliography in [15],[16], and [19]). The above result at once has met with wide application in engineering and economical papers (see [11], [8] [6], [12], [4] and references there). It should be mentioned that in spite of the fact that the above result has been developed for the analysis of equations on a finite time interval, the averaging and diffusion approximation procedures have been applied in many applications for asymptotic stability analysis of possible stationary solutions, that is, for analysis of differential equations as $t \to \infty$. To prove the validity of this approach for random dynamical systems with continuous trajectories the researchers had to use not only a special type of limit theorem (see for example [10] and [7]) but also a stochastic version of the Second Lyapunov method developed for stochastic Ito differential equations in [19]. But most of dynamical systems of the recent Economics (see, for example, [13], [8], [12], [4] and review there) require an extension of the above "smooth" models to allow the phase motion to have a jump type discontinuity. Some of results permissive to resolve this problem have been developed by author in [18] for dynamical systems with switching in Markov time moments. Proposal paper is devoted to similar approach to discrete Markov dynamical systems. This problem is very important in contemporary financial econometrics for analysis of ARCH type stationary iterative procedures (see, for example, [12] and [4]).

2.2 Probabilistic limit theorems and equilibrium stochastic stability

Let $p(y, dz)$ is transition probability of Markov chain y_t and $\mathcal{P}$ is Markov operator

$$(\mathcal{P}v)(y) := \int_{\mathbb{Y}} v(z)p(y, dz)$$

defined on the space $\mathbb{C}(\mathbb{Y})$ of bounded continuous functions. We will assume that the spectrum $\sigma(\mathcal{P})$ has the simple eigenvalue 1, $\sigma(\mathcal{P})\setminus\{1\} \subset \{z \in \mathbb{C} : |z| < \rho < 1\}$, and probability distribution $\{\mu(dy)\}$ is the solution of the equation $\mathcal{P}^*\mu = \mu$, where $\mathcal{P}^*$ is conjugate operator. Averaging procedure by the above invariant measure of any dependent on Markov process vector or matrix will be denoted with overline. Under these conditions one can extend [9] the potential of the above Markov process and to define the linear continuous operator by equality

$$(\Pi v)(y) := \sum_{k=0}^{\infty}(\mathcal{P}^k v)(y) \tag{11}$$

on the space $\bar{\mathbb{C}}(\mathbb{Y})$ of continuous functions $v \in \mathbb{C}(\mathbb{Y})$ with zero average $\bar{v} := \int\limits_{\mathbb{Y}} v(y)\mu(dy)$. This means that the equation $\mathcal{P}g - g = -v$ with $v \in \bar{\mathbb{C}}(\mathbb{Y})$ has unique solution (11) in $\bar{\mathbb{C}}(\mathbb{Y})$. Using the above Markov chain one can define on the segment $[0,1]$ step processes

$$s \in [t\varepsilon^2, (t+1)\varepsilon^2) : Y_\varepsilon(s) := y_t \tag{12}$$

If $\mathfrak{F}^t \subset \mathfrak{F},\ t \geq 0$ is minimal filtration for stationary process y_t then for any $t \geq 0$ and $s \in [t\varepsilon^2, (t+1)\varepsilon^2)$ random vectors $X_\varepsilon(s)$ and $Y_\varepsilon(s)$ are $\mathfrak{F}^t$-measurable. To avoid cumbersome formulae we will denote conditional expectation $\mathbf{E}\{\xi/\mathfrak{F}^t\}|_{x_t=x, y_t=y}$ in abridged form $\mathbf{E}^t_{x,y}\{\xi\}$.

2.3 Derivation of diffusion approximation formula

In this subsection we will assume that $\bar{f}_1(x) \equiv 0$. Using the solution $x_t, t \in \mathbb{N}$ of difference equation

$$x_{t+1} = x_t + \varepsilon f_1(x_t, y_t) + \varepsilon^2 f_2(x_t, y_t), \tag{13}$$

with initial condition $x_0 = x$ and Markov process y_t one can define the broken lines by formulae

$$s \in [t\varepsilon^2, (t+1)\varepsilon^2] : \ X_\varepsilon(s) = (x_{t+1} - x_t)(s\varepsilon^{-2} - t) + x_t \tag{14}$$

for all $t \in [0, N(\varepsilon^{-2})]$, where $N(\alpha)$ is integer part of number α, and step process

$$s \in [t\varepsilon^2, (t+1)\varepsilon^2) : Y_\varepsilon(s) := y_t \tag{15}$$

for all $t \in [0, N(\varepsilon^{-2})]$. Not so difficult to be certain of Markov properties for the pare $\{X_\varepsilon(s), Y_\varepsilon(s), 0 \leq s \leq 1\}$. Therefore under assumption that $\varepsilon \to 0$ one can apply the Skorokhod limit theorems from [15] and [16] for sequences of Markov processes and look for diffusion approximation of $\{X_\varepsilon(s), 0 \leq s \leq 1\}$ if the latter exists. Much as it has been done in [18] for jump type Markov processes in our case for any arbitrary twice continuous differentiable on x function $v(x)$ one has to look for Lyapunov function in a form of decomposition

$$v^\varepsilon(x,y) := v(x) + \varepsilon[((\Pi f_1)(x,y), \nabla)v](x,y) + \varepsilon^2\hat{v}(x,y) \tag{16}$$

with some smooth function $\hat{v}(x,y)$. Here and further $\nabla v(x)$ is gradient and $(\cdot,\cdot)$ is scalar product in $\mathbb{R}^d$. Now one should compute derivative

$$\begin{aligned}&(\mathbf{L}(\varepsilon)v^\varepsilon)(x,y) := \\ &\lim_{\delta \downarrow 0} \frac{1}{\delta}\mathbf{E}^t_{x,y}\{v^\varepsilon(X^\varepsilon(s+\delta), Y^\varepsilon(s+\delta)) - v^\varepsilon(X^\varepsilon(s), Y^\varepsilon(s))\} = \\ &\frac{1}{\varepsilon^2}\mathbf{E}^t_{x,y}\{v^\varepsilon(x_{t+1}, y_{t+1}) - v^\varepsilon(x,y) + o(\varepsilon^2)\}\end{aligned} \tag{17}$$

for all $x \in \mathbb{R}^d, y \in \mathbb{Y}, t \geq 0$ and $s \in [t\varepsilon^2, (t+1)\varepsilon^2)$, and chose in (16) function $\hat{v}(x,y)$ in such a way as to exist limit

$$\lim_{\varepsilon \to 0} (\mathbf{L}(\varepsilon) v^{\varepsilon})(x,y) = (\mathbf{L}v)(x) \tag{18}$$

As it will be shown later right side of the above equation has a form of diffusion operator applied to function $v(x)$:

$$(\mathbf{L}v)(x) = \{(a(x), \nabla) + (\sigma(x)\nabla, \nabla)\} v(x) \tag{19}$$

with vector $a(x)$ and positive defined symmetric matrix $\sigma(x)$. To derive the above formula one has to present operator $\mathbf{L}(\varepsilon)$ accurate within $0(\varepsilon)$

$$\begin{aligned} \mathbf{L}(\varepsilon) = \tfrac{1}{\varepsilon^2}(\mathcal{P} - I) + \tfrac{1}{\varepsilon}(f_1(x,y), \nabla)\mathcal{P} + \\ (f_2(x,y), \nabla)\mathcal{P} + \tfrac{1}{2}(f_1(x,y), \nabla)^2 \mathcal{P} + 0(\varepsilon) \end{aligned} \tag{20}$$

to employ (20) to (16) and to decompose resulting function by powers of ε accurate within $0(\varepsilon)$:

$$\begin{aligned} (\mathbf{L}(\varepsilon) v^{\varepsilon})(x,y) = \frac{1}{\varepsilon^2}(\mathcal{P} - I) v(x) + \\ \frac{1}{\varepsilon}[(f_1(x,y), \nabla) v(x) + (\mathcal{P} - I)((\Pi f_1)(x,y), \nabla) v(x)] + \\ (f_2(x,y), \nabla) v(x) + \frac{1}{2}(f_1(x,y), \nabla)^2 v(x)] + \\ (f_1(x,y), \nabla)\mathcal{P}[(f_1(x,y), \nabla) v(x)] + (\mathcal{P} - I)\hat{v}(x,y) + 0(\varepsilon) \end{aligned}$$

Therefore using obvious equalities $(\mathcal{P} - I)\Pi = -I$,
$(\mathcal{P} - I)v(x) = 0$ and formula (18) one can write equation

$$\begin{aligned} \mathbf{L}v(x) = (f_2(x,y), \nabla) v(x) + \frac{1}{2}(f_1(x,y), \nabla)^2 v(x) \quad + \\ (f_1(x,y), \nabla)[(\mathcal{P}\Pi f_1(x,y), \nabla) v(x)] + (\mathcal{P} - I)\hat{v}(x,y) \end{aligned}$$

with unknown function $\hat{v}(x,y)$. As it has been mentioned at the beginning of this subsection the above equation relative to $\hat{v}(x,y)$ has solution

$$\begin{aligned} \hat{v}(x,y) = \Pi\{(f_2(x,y), \nabla) v(x) + \frac{1}{2}(f_1(x,y), \nabla)^2 v(x) \quad + \\ (f_1(x,y), \nabla)[(\mathcal{P}\Pi f_1(x,y), \nabla) v(x)] - \mathbf{L}v(x)\} \end{aligned} \tag{21}$$

if and only if

$$\mathbf{L}v(x) = \{\overline{(f_2, \nabla)} + \frac{1}{2}\overline{(f_1, \nabla)^2} + \overline{(f_1, \nabla)[(\mathcal{P}\Pi f_1, \nabla)}\} v(x) \tag{22}$$

where overline denotes averaging by measure μ. This equation one can write in a form (19) using notations

$$\begin{aligned} a &= \overline{f}_2 + \overline{[\mathcal{P}\Pi D f_1]^T f_1} \\ \sigma &= \frac{1}{2}[\overline{f_1 f_1^T} + \overline{f_1 \mathcal{P}\Pi f_1^T} + \overline{(\mathcal{P}\Pi f_1) f_1^T}] \end{aligned} \tag{23}$$

where D is Frechet derivative by x and upper index T denotes transposition. To write this equation in a form

$$dX(t) = a(X(s))ds + \sum_{k=1}^{d} \sigma_k(X(s))dW_k(s) \tag{24}$$

with initial condition $X(0) = x_0$, where vector-functions $a(x)$ and $\sigma_k(x)$, $k = 1, 2, ..., d$ are defined based on averaging by measure μ of functions $f_j(x, y)$, $j = 1, 2$ and its derivatives, and $\{W_k, k = 1, 2, ..., d\}$ are independent standard Wiener processes, one has to find d dependent on x vectors σ_k defined by equation

$$\sum_{k=1}^{d} \sigma_k(x)\sigma_k^T(x) = \sigma(x)$$

As it has been mentioned in [19] this equation has solution for any positive defined matrix $\sigma(x)$.

2.4 Averaging and normalized deviations

Let us remind of assumption $\bar{f}_1(x) \equiv 0$ which permits in previous subsection to derive formulae (22) and (23). Otherwise one may not divide segment $[0, 1]$ by intervals of length ε^2 because $\Pi f_1(x, y)$ does not exist and therefore there are singularity in the definition of operator (17) as $\varepsilon \to 0$. To apply a diffusion approximation in this case one has to find solution of averaged equation

$$\bar{x}_{t+1} = \bar{x}_t + \varepsilon \bar{f}_1(\bar{x}_t) \tag{25}$$

and to derive an asymptotic formula for so called *normalized deviations*

$$z_t := \frac{x_t - \bar{x}_t}{\sqrt{\varepsilon}} \tag{26}$$

Substituting $x_t = \sqrt{\varepsilon} z_t + \bar{x}_t$ in (13)

$$z_{t+1} = z_t + \delta g_1(\bar{x}_t, y_t) + \delta^2 [Df_1(\bar{x}_t, y_t)] z_t + o(\delta^2), \tag{27}$$

where $\delta = \sqrt{\varepsilon}, g_1(x, y) = f_1(x, y) - \bar{f}_1(x)$, one can apply to system (25)-(27) approach of previous subsection. The sequence (26) gives rise to random processes

$$Z^\delta(s) = \frac{X^\delta(s) - \bar{X}^\delta(s)}{\delta}$$

where $X^\delta(s)$ and $\bar{X}^\delta(s)$ are defined in the same way like (14) for all $s \in [t\delta^2, (t+1)\delta^2)$ and any $t \in [0, N(\delta^{-2})]$. After substitution $Z^\delta(s)$ instead of $X_\varepsilon(s)$, $[Df_1(\bar{X}(s), Y^\delta(s))]Z^\delta(s)$ instead of $f_2(X_\varepsilon(s), Y_\varepsilon(s))$ and $g_1(\bar{X}(s), Y^\delta(s))$ instead of $f_1(X_\varepsilon(s), Y_\varepsilon(s))$ in corresponding formulae and vanishing δ one can approximate probability distribution $\mathbf{P}_\delta^Z$ of process $Z^\delta(s)$ by probability distribution $\mathbf{P}^Z$ of process Z satisfying stochastic differential equation equation

$$dZ(s) = D\bar{f}_1(\bar{X}(s))Z(s)ds + \sum_{k=1}^{d} \sigma_k(\bar{X}(s))dW_k(s)$$

with initial condition $\bar{X}(0) = x_0$, where $\{W_k(s), k = 1, 2, ..., d\}$ are independent standard Wiener processes, and vectors $\{\sigma_k, k = 1, 2, ..., d\}$ satisfy an equality

$$\sum_{k=1}^{d} \sigma_k(x)\sigma_k^T(x) = [\overline{g_1 g_1^T} + \overline{g_1 \mathcal{P}\Pi g_1^T} + \overline{(\mathcal{P}\Pi g_1) g_1^T}](x)$$

Deterministic function $\bar{X}(s)$ one can find as the solution of ordinary differential equation

$$d\bar{X}(s) = \bar{f}_1(\bar{X}(s))ds$$

Roughly speaking for sufficiently small ε one can approximate distribution of the sequence $\{x_t, 0 \geq t \geq N(\varepsilon^{-1})\}$ by distribution of sequence $\{X(t\varepsilon) + \sqrt{\varepsilon}Z(t\varepsilon), 0 \geq t \geq N(\varepsilon^{-1})\}$.

2.5 Equilibrium asymptotic stability

As it has been mentioned in the Section 2 some of application iterative procedures analysis require asymptotic analysis of equation (13) as $t \to \infty$. For example discussing diffusion approximation approach to GARCH time series authors of papers [4] and [12] indicate this problem in view of the approximation and asymptotic stability analysis of stationary conditional variance. In previous section we have derived an approximate distribution of sequence $\{x_t, 0 \leq t \leq N\}$ for any finite integer number N by distribution of solution of stochastic differential equation $\{X(s), 0 \leq s \leq 1\}$ but for the above mentioned asymptotic analysis as $t \to \infty$ one has to deal with equation (24) with unrestrictedly large s. Besides there is a problem of legality results which are based on the diffusion approximation as $s \to \infty$. This subsection is devoted to the above problem.
Let point $x = 0$ be an equilibrium of iteration procedure (13), i.s. $f_1(0, y) \equiv 0$ and $f_2(0, y) \equiv 0$. If for any $\eta > 0$ there exists such a neighborhood $U_r := \{x \in \mathbb{R}^d : |x| < r\}$ that any starting in U_η solution x_t of (13) does not leave U_r and tends to zero as $t \to \infty$ with probability greater than $1 - \eta$ the above equilibrium is called *asymptotic stochastically stable*. As it has been shown in [14] for equilibrium stability analysis one can employ the second Lyapunov method with Lyapunov operator defined by formula

$$(\mathcal{L}v)(x, y) := \mathbf{E}^0_{x,y}\{v(x_1, y_1)\} - v(x, y)$$

and Lyapunov functions satisfying inequality

$$|x|^p < v(x, y) < c|x|^p$$

with some positive p a $c \geq 1$. If there exists such a Lyapunov function $v(x, y)$ that

$$(\mathcal{L}v)(x, y) < -\gamma|x|^p$$

with $\gamma \in (0, 1)$ then [14] equilibrium is asymptotic stochastically stable and $\mathbf{E}_{x,y}\{|x_t|\} \leq M|x|^p \exp\{-\rho t\}$ with some positive constants M and ρ. Besides under smoothness assumptions of the Section 1 on vectors $f_1(x, y)$ and $f_2(x, y)$, this equilibrium is asymptotic stochastically stable if and only if [14] the same property has the trivial solution of its linear approximation

$$\tilde{x}_{t+1} = \tilde{x}_t + \varepsilon\tilde{f}_1(\tilde{x}_t, y_t) + \varepsilon^2\tilde{f}_2(\tilde{x}_t, y_t) \tag{28}$$

where $\tilde{f}_j(x,y) = (Df_j)(0,y)x$, $j = 1,2$. Therefore for asymptotic analysis of (13) as $t \to \infty$ one can apply formulae (19) with (16), (21), (22), and (23) substituting linear on $x \in \mathbb{R}^d$ functions $\tilde{f}_j(x,y)$ instead of $f_j(x,y), j = 1,2$ and rewriting equation (24) in a form of linear stochastic Ito equation

$$d\tilde{X}(s) = A\tilde{X}(s))ds + \sum_{k=1}^{d} B_k \tilde{X}(s) dW_k(s) \tag{29}$$

The same result like mentioned above for Markov iterations (28) one can find in [19] for stochastic differential equation (29): trivial solution of (29) is asymptotic stochastically stable if and only if there exists such twice continuous differentiable Lyapunov function $V(x)$ that

$$|x|^p \le V(x) \le h_1|x|^p, \quad \mathbf{L}\, V(x) \le -h_2|x|^p \tag{30}$$

and $||D^l \bigtriangledown v(x)|| \le h_3|x|^{p-l-1}$, $l = 1,\, 2,\, 3$ for some $p > 0$, positive constants $h_j,\ j = 1,2,3.$ and any $x \in \mathbb{R}^d$. Now for analysis of asymptotic behaviour of linear iteration (28) one can apply the second Lyapunov method with function

$$\begin{aligned} & V^\varepsilon(x,y) := \\ V(x) \quad + \quad & \varepsilon[((\Pi\tilde{f}_1)(x,y), \bigtriangledown)V](x,y) + \varepsilon^2\hat{V}(x,y) \end{aligned} \tag{31}$$

where $V(x)$ satisfies inequalities (30) and

$$\begin{aligned} \hat{V}(x,y) = \Pi\{(\tilde{f}_2(x,y), \bigtriangledown)V(x) + \frac{1}{2}(\tilde{f}_1(x,y), \bigtriangledown)^2 V(x) \quad + \\ (\tilde{f}_1(x,y), \bigtriangledown)[(\mathcal{P}\Pi\tilde{f}_1(x,y), \bigtriangledown)V(x)]\} \end{aligned} \tag{32}$$

Owing to linearity of functions $\tilde{f}_j(x,y), j = 1,2$ and definition of $\mathbf{L}V(x)$ for all sufficiently small $\varepsilon > 0$ there exist such positive constants $h_j, j = \overline{4,9}$ that the above defined functions satisfy inequalities

$$\begin{aligned} h_4|x|^p \le |\hat{V}(x,y)| \le h_5|x|^p, \\ h_6|x|^p \le |[((\Pi\tilde{f}_1)(x,y), \bigtriangledown)V](x,y)| \le h_7|x|^p \\ |V^\varepsilon(x,y) - V(x)| \le \varepsilon h_8|x|^p \end{aligned}$$

and

$$|(\mathbf{L}(\varepsilon)V^\varepsilon)(x,y) - \mathbf{L}\, V(x)| < \varepsilon h_9|x|^p$$

Therefore if the trivial solution of diffusion approximation is asymptotically stable then there exists Lyapunov function satisfying (30) and for stability analysis of (28) one can use function (31):

$$\begin{aligned} (\mathcal{L}V^\varepsilon)(x,y) \quad &= \quad \varepsilon^2(\mathbf{L}(\varepsilon)V^\varepsilon)(x,y) \le \\ \le \varepsilon^2\mathbf{L}\, V(x) \quad &\le \quad \varepsilon^2(-h_2 + \varepsilon h_9)|x|^p \end{aligned}$$

This inequality convinces of asymptotical stochastic stability for trivial solution of difference equation (28) if ε is sufficiently small.

3 Markov type GARCH models

3.1 Continuous Stochastic Model of Conditional Variance Dynamics

In papers [12] and [4] the authors discuss a problem of diffusion approximation for very popular in contemporary econometrics GARCH (General AutoRegressive Conditional Heteroscedastic) process for conditional time series variance. The paper [4] deals with model given in a form of first order linear difference equation

$$\sigma_{t+1}^2 = \omega_h + \sigma_t^2[\beta_h + h^{-1}\alpha_h {}_hZ_t^2] \tag{33}$$

where h is small positive parameter, $\{{}_hZ_t, t \in \mathbb{Z}\}$ is sequence of i.i.d. random variables with zero mean, variance $\mathbf{E}\{{}_hZ_t^2\} = h$, and fourth moment $\mathbf{E}\{{}_hZ_t^4\} = 3h^2$. Under assumptions

$$1 - \alpha_h - \beta_h = h\theta + o(h), \omega_h = h\omega + o(h), \alpha_h = \frac{\sqrt{h}}{\sqrt{2}}\alpha + o(h)$$

author of paper [4] derives diffusion approximation equation in a form

$$d\sigma_t^2 = (\omega - \theta\sigma_t^2)dt + \alpha\sigma_t^2 dW(t) \tag{34}$$

To compare this result with our derived formulae one can denote

$$h = \varepsilon^2, x_t = \sigma_t^2, y_t = \frac{{}_hZ_t^2 - 1}{\sqrt{2h}}$$

and rewrite equation (33) in a form of difference equation (13) accurate within ε-items of second order

$$x_{t+1} = x_t + \varepsilon^2[\omega - \theta x_t] + \varepsilon\alpha y_t x_t \tag{35}$$

Let y_t be stationary Markov process with the same unconditional moments as $\frac{{}_hZ_t^2-1}{\sqrt{2h}}$, that is, $\mathbf{E}y_t = 0, \mathbf{E}y_t^2 = 1$ and correlation function $C(k) = \mathbf{E}\{y_t y_{t+k}\}$ for $k \in \mathbb{N}$. Following our proposal method of diffusion approximation one should for this equation calculate parameters (23) with $f_1(x, y) = \alpha y x$, $f_2(x, y) = \omega - \theta x$. By definition

$$a(x) = \omega - \theta x + \alpha^2 x \sum_{l=1}^{\infty} \left\{ \int_{\mathbb{Y}} \mathbf{E}_y^0\{y y_l\}\mu(dy) \right\} =$$

$$= \omega + \left[\alpha^2 \sum_{k=1}^{\infty} C(k) - \theta \right] x$$

$$\sigma^2(x) = \alpha^2 x^2 \int_{\mathbb{Y}} y^2 \mu(dy) + 2\alpha^2 x^2 \sum_{k=1}^{\infty} C(k) =$$

$$\alpha^2 x^2 \left[\mathrm{Var}\{y_t\} + 2\sum_{k=1}^{\infty} C(k) \right]$$

If $\{y_t, t \in \mathbb{Z}\}$ are independent random variables with zero mean and unit variance like it has been assumed in [4] we have derived equation (34) because $C(k) \equiv 0$. If $\kappa := \sum_{k=1}^{\infty} C(k) \neq 0$ one should apply diffusion approximation for GARCH(1,1)-process in a following form

$$d\sigma_t^2 = (\omega + (\alpha^2\kappa - \theta)\sigma_t^2)dt + \alpha\sqrt{1+2\kappa}\sigma_t^2 dW(t) \tag{36}$$

As it has been proved this equation one can use also for analysis of (33) as $t \to \infty$. According to [19] if

$$\alpha^2\kappa - \theta - \frac{\alpha^2(1+2\kappa)}{2} = -\theta - \frac{\alpha^2}{2} < 0$$

there exists stationary solution $\hat{s}_t^2$ of the above equation and deviations $z_t := s_t^2 - \hat{s}_t^2$ of any other solution from this stationary process exponentially tend to zero as $t \to \infty$. In spite of the fact that process y_t has nonzero correlation this result no differs from similar result of the paper [4]. But to approximate stationary process for GARCH(1,1) with Markov process y_t instead of i.i.d. sequence one has to deal with stationary solution of equation (36) where $\kappa \neq 0$. As it has been derived by E.Wong [17] for linear stochastic Ito equation (36) the stationary process defined σ_t^{-2} has density function $f(x) = \frac{s^r x^{(r-1)}}{\Gamma(r)} e^{-sx}$ where $r = 1 + \frac{2(\theta - \alpha^2\kappa)}{\alpha^2(1+2\kappa)}, s = \frac{2\omega}{\alpha^2(1+2\kappa)}$ or stationary variance $\hat{s}^2$ has distribution defined by formula

$$\mathbb{P}\{\hat{s}^2 < z\} = 1 - \int_0^{1/z} f(x)dx, \;\; f(x) = \frac{s^r x^{(r-1)}}{\Gamma(r)} e^{-sx} \tag{37}$$

This convince of possible considerable correlation affect on the asymptotic approximation of conditional variance stationary distribution.

3.2 Diffusion Model of Stock Return with Stochastic Volatility

The simplest stock return S_t mathematical model involving assumption on conditional heteroskedasticity of interest rate h_t variance σ_t^2 under commonly used condition on risk neutrality of probabilistic measure $\mathbf{P}$ may be written ([2],[4]) as the system of two difference equation

$$S_{t+1} = S_t(1 + \varepsilon\sigma_t^2 y_{t+1}), \tag{38}$$

$$\sigma_{t+1}^2 = \sigma_t^2 + \varepsilon^2[\omega - \theta\sigma_t^2] + \varepsilon\alpha(y_{t+1}^2 - 1)\sigma_t^2 \tag{39}$$

where y_t is Gaussian random sequence with zero mean and unit variance. When it is considered that these random numbers do not independent we will use for y_t equation of type AR(1):

$$y_{t+1} = \rho y_t + \sqrt{1 - y_t^2}\xi_{t+1} \tag{40}$$

where $\{\xi_t\}$ is i.i.d. Gaussian sequence, $\mathbb{E}\xi_t = 0, \mathbb{E}\xi_t^2 = 1$. To employ formulae (refLL) let us denote $x_{1t} = S_t, x_{2t} = \sigma_t^2$ and

$$\vec{x}_t = \begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix}$$

and rewrite equations (38) in a vector form

$$\vec{x}_{t+1} = \vec{x}_t + \varepsilon y_{t+1} \begin{pmatrix} \sqrt{x_{2t}} & 0 \\ 0 & \alpha \end{pmatrix} \vec{x}_t + \varepsilon^2 \begin{pmatrix} 0 \\ \omega \end{pmatrix} - \varepsilon^2 \begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix} \vec{x}_t \tag{41}$$

Now one can use formula (refLL) with

$$f_1 = \begin{pmatrix} x_{1t} y_{1t} \sqrt{x_{2t}} \\ \alpha x_{2t}(y_t^2 - 1) \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ \omega - \theta x_{2t} \end{pmatrix}$$

applying in formulae (refLL) an averaging by invariant distribution $N(0,1)$ of Markov chain (40) and to write a final limit stochastic equation for vector $\vec{x}_t$:

$$d\vec{x}_t = a(\vec{x}_t)dt + b_1(\vec{x}_t)dw_1(t) + b_2(\vec{x}_t)dw_2(t)$$

where

$$a(\vec{x}) = \begin{pmatrix} x_1 x_2 \frac{\rho}{1-\rho} \\ \omega + (\alpha^2 \frac{\rho^2}{1-\rho^2} - \theta)x_2 + \frac{x_1^2}{2} \end{pmatrix} \tag{42}$$

vectors b_1, b_2 are defined by equality

$$b_1(\vec{x})b_1^T(\vec{x}) + b_2(\vec{x})b_2^T(\vec{x}) = \begin{pmatrix} x_1^2 x_2 \frac{3+\rho}{1-\rho} & 0 \\ 0 & 2\alpha^2 x_2^2 \frac{3+\rho^2}{1-\rho^2} \end{pmatrix} \tag{43}$$

This means that stock return S_t and conditional variance σ_t^2 have dynamics approximately describes by system of Ito stochastic differential equations

$$dS_t = S_t \sigma_t \frac{\rho}{1-\rho} dt + S_t \sigma_t \sqrt{\frac{3+\rho}{1-\rho}} dw_1(t), \tag{44}$$

$$d\sigma_t^2 = \{\omega + (\alpha^2 \frac{\rho^2}{1-\rho^2} - \theta)\sigma_t^2 + S_t^2 \frac{\rho}{2(1-\rho)}\}dt + \alpha\sigma_t^2 \sqrt{\frac{2(3+\rho^2)}{1-\rho^2}} dw_2(t) \tag{45}$$

where $w_1(t)$ and $w_2(t)$ are independent standard Wiener processes.

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