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CONVERGENCE OF LINEAR MARKOV ITERATIONS

CARKOVA Viktorija, (LV), GOLDŠTEINE Jolanta, (LV), SWERDAN Myhailo, (UA)

Abstract. This paper deals with the linear difference equation in $\mathbb{R}^n$ with coefficients dependent on the Markov chain. It is proved that covariance matrices of solutions can be analyzed using powers of a positive operator in a Banach space with a reproducing cone. This property permits to formulate the necessary and sufficient mean square stability condition as a spectral problem or a problem of positive solvability of a specially constructed linear operator equation. The paper discusses three possible approaches for convergence analysis of defined by difference equation iterative procedure: mean square stability analysis using the second Lyapunov method; mean square stability analysis using Lyapunov index; reducibility method for moments of solutions, which permits approximately to reduce mean square stability problem to analysis of equation with constant coefficients.

Key words and phrases. Stochastic difference equations, Markov dynamical systems, mean square stability.

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1 Introduction

The paper deals with asymptotic stability problem for linear difference equations with Markov coefficients. A linear difference equation in $\mathbb{R}^n$ defined by equality:

$$x_t = A(y_t)x_{t-1}, t \in \mathbb{N}, \tag{1}$$

where $\{A(y), y \in \mathbb{Y}\}$ is continuous $n \times n$ matrix function on the metric compact $\mathbb{Y}$, $\sup_y ||A(y)|| = const < \infty$; $\{y_t, t \in \mathbb{N}\}$ is a homogeneous Feller Markov chain with phase space $\mathbb{Y}$, invariant measure $\mu(dy)$, and transition probability $p(y, dz)$. Under initial conditions $x_k = x$, $y_k = y$ the vector function $x_t(k, x, y) = X(t, k, y)x$, where $X(t, k, y) := \prod_{m=k+1}^{t} A(y_m)$, satisfies the difference equation (1) for any $t \geq k$.

This paper discusses three possible approaches for convergence analysis of iterative procedure (1) with Markov coefficients:

- mean square stability analysis of (1) using the second Lyapunov method;
- mean square stability analysis using Lyapunov index for (1);
- reducibility method for moments of equation (1) solutions, which permits to reduce (1) to equation with constant coefficients.

2 Mean square stability analysis using the second Lyapunov method

The most powerful tool for asymptotic stability analysis of dynamical system is the Second Lyapunov method. One should choose a nonnegative function $V(x, y)$, satisfying an equality $V(x, y) = 0$ if and only if $x = 0$ (called Lyapunov function) and analyze an expectation of difference by virtue of the above system and Markov chain (or the Lyapunov operator $\mathbf{L}$) defined by equality $(\mathbf{L}V)(x, y) := \mathbb{E}\{V(x_t, y_t)/x_{t-1} = x, y_{t-1} = y\} - V(x, y)$. If there exists such Lyapunov function that $|x|^p \leq V(x, y) \leq c_1|x|^p$, $(\mathbf{L}V)(x, y \leq -c_2|x|^p$ for any x, y and some positive p, c_1, c_2 then with increasing of t to infinity any solution of the above difference equations exponentially tends to zero with probability one. The main idea of this approach is to choose for (1) the Lyapunov function as a quadratic form $V(x, y) := (v(y)x, x)$ and then to analyze spectral properties of linear operator defined by an equality $((\mathbb{A}v)(y)x, x) := (\mathbf{L}V)(x, y)$.

Definition 2.1 *The equation (1) is called as exponentially mean square stable if there exist such constants $c > 0$ and $\lambda \in (0, 1)$ that*

$$\mathbb{E}|x_t(k, x, y)|^2 \leq c\lambda^{t-k}|x|^2 \tag{2}$$

for any $y \in \mathbb{Y}$, $x \in \mathbb{R}^n$, $k \in \mathbb{N}$ and $t \geq k$.

Let $\mathbb{V}$ be the Banach space of symmetric uniformly bounded continuous $n \times n$ matrix functions $\{q(y), y \in \mathbb{Y}\}$ with norm

$$||q|| := \sup_{y\in\mathbb{Y},||x||=1} |(q(y)x, x)|.$$

To derive mean square stability conditions for (1) special constructed operator equation for quadratic functionals $(q(y)x, x)$ with $q \in \mathbb{V}$ is used, where $(.,.)$ denotes a scalar product. Using matrix $A(y)$ and transition probability one can define on $\mathbb{V}$ the linear continuous operator

$$(\mathbf{A}\, q)(y) := \int_{\mathbb{Y}} A^T(z)q(z)A(z)p(y, dz), \tag{3}$$

where top index T denotes transposition. It is easy to see that the above defined operator leaves as invariant the cone [5]

$$\mathbb{K} := \{q \in \mathbb{V} : \inf_{y\in\mathbb{Y},||x||=1} (q(y)x, x) \geq 0\}$$

with a set of inner points

$$\overset{\mathbf{o}}{\mathbb{K}} := \{q \in \mathbb{V} : \inf_{y\in\mathbb{Y},||x||=1} (q(y)x, x) > 0\}.$$

This cone permits to put space $\mathbb{V}$ in partial order using "inequality" $q_1 << q_2$ if $q_2 - q_1 \in \mathbb{K}$. Obviously that $q \in \overset{\mathbf{o}}{\mathbb{K}}$ if and only if there exists a such positive constant $c(q)$ that $q >> c(q)I$ where I is the matrix unit of the space $\mathbb{V}$.

Lemma 2.2 *For any* $q \in \mathbb{V},\ t > k \geq 0,\ y \in \mathbb{Y},$ *and* $x \in \mathbb{R}^n$

$$\left(\left(\mathbf{A}\right)^t q\right)(y)x, x\Big) = \mathbb{E}\left\{(q(y_{t+k})x_{t+k}(k,x,y), x_{t+k}(k,x,y))/y_k = y\right\}.$$

Using the definition of Cauchy matrix family one can rewrite the assertion of Lemma 1 in the matrix form

$$(\mathbf{A}^t q)(y) = \mathbb{E}\left\{X^T(t+k,k,y)q(y_{t+k})X(t+k,k,y)/y_k = y\right\}. \tag{4}$$

Theorem 2.3 *The next assertions are equivalent:*

(i) equation (1) is exponentially mean square stable;

(ii) there exists such $q \in \overset{\mathbf{o}}{\mathbb{K}}$ *that*

$$\mathbf{A}q - q = -I; \tag{5}$$

(iii) maximal positive real spectrum point $\mathbf{r}\{\mathbf{A}\}$ *of operator* $\mathbf{A}$ *is less than one.*

Proof. $(i) \longrightarrow (ii)$. On the basis of an equality

$$\left\|\mathbb{E}\left\{X^T(t,0,y)X(t,0,y)\right\}\right\| = \sup_{|x|=1}\left|(\mathbb{E}\left\{X^T(t,0,y)X(t,0,y)\right\}x, x)\right| =$$

$$\sup_{|x|=1}\left|\mathbb{E}\left\{(X(t,0,y)x, X(t,0,y)x)\right\}\right| = \sup_{|x|=1}\mathbb{E}\left\{|x_t(0,x,y)|^2\right\}$$

and mean square stability of (1) there exists the matrix function defined by formula

$$q(y) := \sum_{t=0}^{\infty}\mathbb{E}\left\{X^T(t,0,y)X(t,0,y)\right\}$$

Because of identity $X(k,k,y) \equiv I$ and equality

$$\sum_{t=0}^{\infty}\mathbb{E}\left\{X^T(t,0,y)X(t,0,y)\right\} = I + \sum_{t=1}^{\infty}\mathbb{E}\left\{X^T(t,0,y)X(t,0,y)\right\}$$

one can write inequality $q >> I$. Therefore $q \in \overset{\mathbf{o}}{\mathbb{K}}$. To complete a proof of the first assertion one can apply formula (4) with matrix function $q(y) \equiv I$ and to write the equalities

$$\mathbf{A}q(y) - q(y) = \mathbf{A}\left(\sum_{t=0}^{\infty}\mathbf{A}^t I\right) - \sum_{t=0}^{\infty}\mathbf{A}^t I =$$

$$\sum_{t=0}^{\infty}\mathbf{A}^{t+1}I - \sum_{t=0}^{\infty}\mathbf{A}^t I = -I.$$

$(ii) \longrightarrow (iii)$. Let $q \in \overset{\mathbf{o}}{\mathbb{K}}$ satisfies (5). There exists such positive number $c(q)$ that $c(q)I << q << ||q||I$ and one can get from the equation (5) inequality $\mathbf{A}q - q << -q/||q||$ or $\mathbf{A}^t q << r^t q$ for any $t \in \mathbb{N}$ where $r = 1 - ||q||^{-1} \in (0,1)$. Therefore

$$\mathbf{A}^t I << \frac{1}{c(q)}\mathbf{A}^t q << \frac{r^t}{c(q)}q << ||q||\frac{r^t}{c(q)}I$$

for any $t \in \mathbb{N}$, t.i.

$$\sum_{t=0}^{m} \mathbf{A}^t I << \frac{||q||}{c(q)} \sum_{t=0}^{m} r^t I << \frac{||q||}{c(q)(1-r)} I$$

for any $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} \sup_{|x|=1, y\in\mathbb{Y}} \sum_{t=0}^{m} |((\mathbf{A}^t g)(y)x, x)| < \infty \tag{6}$$

for any $g \in \mathbb{V}$. Because linear operator $\mathbf{A}$ leaves the solid cone $\mathbb{K}$ as invariant there exists [5] such real spectrum point $\rho(\mathbf{A})$ that $\rho(\mathbf{A}) = sup\{|z|, z \in \sigma(\mathbf{A})\}$ and real eigenfunction $q_\rho \in \mathbb{K}$ corresponding to this spectrum point, t.i. $\mathbf{A}q_\rho = \rho(\mathbf{A})q_\rho$. Therefore if $\rho(\mathbf{A}) \geq 1$ one should write

$$\lim_{m\to\infty} \sup_{|x|=1, y\in\mathbb{Y}} \sum_{t=0}^{m} ((\mathbf{A}^t q_\rho)(y)x, x) = \infty.$$

This equality contradicts to (6).
$(iii) \longrightarrow (i)$. Because operator $\mathbf{A}$ leaves the above defined cone $\mathbb{K}$ as invariant, there exists [5] positive spectrum point $\mathbf{r}(\mathbf{A})$ satisfying equality $\mathbf{r}(\mathbf{A}) = \max \mathbf{R}e\{\sigma(\mathbf{A})\}$. Therefore, if $\mathbf{r}(\mathbf{A}) < 1$ then $\sigma(\mathbf{A}) \subset \{z \in \mathbb{C} : |z| < 1\}$ and there exist [5] such constants $c > 0,\ \lambda \in (0, 1)$ that $||\mathbf{A}^t|| \leq c\lambda^t$ for any $t \in \mathbb{N}$. Now one can write inequality

$$\mathbb{E}|x_{t+k}(k, x, y)|^2 = ((\mathbb{A}^t I)(y)x, x) \leq c\lambda^t |x|^2$$

and proof is complete.

More simple stability criterion one can reach assuming that the sequence $\{y_t, t \in \mathbb{N}\}$ consists of independent random variables with the same distribution $p(dy)$. In this case we will consider a contraction $\hat{\mathbf{A}}$ of the defined by (3) operator $\mathbf{A}$ on the space $\mathbb{M}_n$ of symmetric $n \times n$ real matrices

$$\hat{\mathbf{A}}q := \mathbb{E}\{A^T(y_t)qA(y_t)\} = \int_{\mathbb{Y}} A^T(y)qA(y)p(dy)$$

Using cone of the positive defined matrices $\overset{\mathbf{o}}{\mathbb{K}}_n := \mathbb{M}_n \bigcap \overset{\mathbf{o}}{\mathbb{K}}$

Corollary 2.4 *If the sequence $\{y_t, t \in \mathbb{N}\}$ consists of independent random variables with the same distribution $p(dy)$ the next assertions are equivalent:*

(i) equation (1) is exponentially mean square stable;

(ii) there exists such $q \in \overset{\mathbf{o}}{\mathbb{K}}_n$ that

$$\hat{\mathbf{A}}q - q = -I;$$

(iii) maximal positive real spectrum point $\mathbf{r}(\hat{\mathbf{A}})$ of operator $\hat{\mathbf{A}}$ is less than one.

3 Mean square Lyapunov index for Markov iterations

If (1) is equation with near to constant coefficients, i.e. matrix in the right part of equation (1) has a form

$$A(y,\varepsilon) = A_0 + \varepsilon A_1(y) + \varepsilon^2 A_2(y) + \ldots, \tag{7}$$

the paper proposes an algorithm, which reduces the performances of the equation (1) second moments dynamics to analysis of the operator $\mathbf{A}(\varepsilon)$ in finite dimensional subspace $\mathbb{V}(\varepsilon) \subset \mathbb{V}$. This subspace as well as the restriction matrix $\Lambda(\varepsilon)$ of the operator $\mathbf{A}$ on it may be defined by the specially constructed basis $\mathbf{B}(\varepsilon)$, analytically dependent on ε. The maximal by modulus real eigenvalue $\rho(\varepsilon)$ of matrix $\Lambda(\varepsilon)$ for sufficiently small $\varepsilon > 0$ coincides with similar eigenvalue of operator $\mathbf{A}(\varepsilon)$. By terminology of [2] this number defines mean square Lyapunov index by formula $\lambda_2(\varepsilon) = \lim\limits_{t\to\infty} \sup\limits_{k,y,|x|=1} \frac{1}{2t} ln\mathbb{E}\{|x_t(k,x,y)|^2\}$ and this number defines behavior of the second moment $\mathbb{E}\{|x_t(k,x,y)|^2\}$ as $t \to \infty$: if $\lambda_2(\varepsilon) < 0$ sequence $\mathbb{E}\{|x_t(k,x,y)|^2\}$ exponentially decreases, if $\lambda_2(\varepsilon) > 0$ - exponentially increases.

Let $\sigma(\mathbf{A})$ be the spectrum and $r(\mathbf{A})$ be the spectral radius of operator $\mathbf{A}$. Substituting matrix (7) in formula (3) one can decompose the operator family $\mathbf{A}(\varepsilon)$ by power of ε: $\mathbf{A}(\varepsilon) = \mathbf{A}_0 + \varepsilon\mathbf{A}_1 + \varepsilon^2\mathbf{A}_2 + \ldots$ with some bounded operators $\mathbf{A}_k$, $k = 1, 2, \ldots$ and $\mathbf{A}_0 q := \int\limits_{\mathbb{Y}} A_0^T q(z) A_0 p(y, dz)$. It means that operator family $\mathbf{A}(\varepsilon)$ analytically depends on parameter ε and for finding mean square Lyapunov index $\lambda_2(\varepsilon)$ we can apply methods and results of perturbation theory of linear continuous operators [4] for decomposition of finite dimension spectral point $r(\mathbf{A}(\varepsilon))$. Using the definition of the operator $\mathbf{A}_0$ we can write that $\sigma(\mathbf{A}_0) = \{\lambda_1 \cdot \lambda_2 \cdot \lambda_3 : \lambda_{1,2} \in \sigma(A_0), \lambda_3 \in \sigma(P)\}$. According to this formula spectral radius of operator $\mathbf{A}_0$ is spectral point which corresponds $r(\mathbf{A}_0) = \{\max|\lambda|^2 : \lambda \in \sigma(A_0)\}$, and besides $r(\mathbf{A}_0) \in \sigma(\mathbf{A}_0)$. Owing to analyticity of operator-family $\mathbf{A}(\varepsilon)$ for sufficiently small values of ε there exists [4] part of spectrum σ_ε of the operator $\mathbf{A}(\varepsilon)$ satisfying the equality $\lim\limits_{\varepsilon\to 0} \sigma_\varepsilon = \{r^2(A_0)\}$ where $r(A_0) = \max\{|\lambda| : \lambda \in \sigma(A_0)\}$. The root subspace $\mathbb{V}(\varepsilon) \subset \mathbb{V}$ corresponding to the part of spectrum given by the above formula has the same dimension $m = \dim \mathbb{V}(0)$ for all sufficiently small $\varepsilon \geq 0$ [4]. A basis $\mathbf{B}(\varepsilon)$ can be constructed in $\mathbb{V}(\varepsilon)$ [4] of the form $\mathbf{B}(\varepsilon) = \mathbf{P}(\varepsilon)\mathbf{B}^0$, where $\mathbf{P}(\varepsilon)$ is the total projector in $\mathbb{V}(\varepsilon)$ and $\mathbf{B}^0 \subset \hat{M}(\mathbb{R}^n)$, where $\hat{M}(\mathbb{R}^n)$ is a set of symmetric $n \times n$ matrices, because all corresponding to $r(\mathbf{A}_0)$ eigen-elements of the operator $\mathbf{A}_0$ are symmetric $n \times n$ matrices. Because the projector $\mathbf{P}(\varepsilon)$ is an analytic function of ε [4] one can look for the basis as decomposition $\mathbf{B}(\varepsilon) = \mathbf{B}^0 + \varepsilon\mathbf{B}^1 + \varepsilon^2\mathbf{B}^2 + \ldots$. Let $\Lambda(\varepsilon)$ be the matrix of restriction of the operator $\mathbf{A}(\varepsilon)$ on the subspace $\mathbb{V}(\varepsilon)$. Consequently this matrix can be obtained [4] from the expression

$$\mathbf{A}(\varepsilon)\mathbf{B}(\varepsilon) = \mathbf{B}(\varepsilon)\Lambda(\varepsilon) \tag{8}$$

where for the matrix $\Lambda(\varepsilon)$ also can be used the decomposition $\Lambda(\varepsilon) = \Lambda_0 + \varepsilon\Lambda_1 + \varepsilon^2\Lambda_2 + \ldots$. Therefore (8) can be rewritten into the form

$$(\mathbf{A}_0+\varepsilon\mathbf{A}_1+\varepsilon^2\mathbf{A}_2+\ldots)(\mathbf{B}^0+\varepsilon\mathbf{B}^1+\varepsilon^2\mathbf{B}^2+\ldots) = (\mathbf{B}^0+\varepsilon\mathbf{B}^1+\varepsilon^2\mathbf{B}^2+\ldots)(\Lambda_0+\varepsilon\Lambda_1+\varepsilon^2\Lambda_2+\ldots). \tag{9}$$

We can look for $\Lambda_0, \Lambda_1, \Lambda_2, \ldots$ by equating the coefficients corresponding to the same powers of ε. We start with the system of m equations what corresponds to the zero power of ε in (9) $\mathbf{A}_0\mathbf{B}^0 - \mathbf{B}^0\Lambda_0 = 0$. One can satisfy these equations with any basis $\mathbf{B}^0 = \mathbf{P}(0)\hat{M}(\mathbb{R}^n) \subset \hat{M}(\mathbb{R}^n)$ in the root subspace corresponding to eigenvalue $r(A_0)^2$ and the matrix Λ_0 of the operator $\mathbf{A}_0$

in this basis. Further we have to deal with the systems of equations which correspond to ε, ε^2 and so on in (9). These systems have solutions if and only if its right part is orthogonal to m linearly independent solutions of homogeneous adjoint equation.

4 Reducibility method for moments

The paper applies reducibility method for moments of equation (1) with near to constant Markov coefficients (7) solutions, which permits to reduce (1) to equation with constant coefficients. It is assumed that a Markov sequence $\vec{y} := \{y_t, t \in \mathbb{N}\}$ is given in a filtrated probability space $(\Omega, \mathfrak{F}, \mathfrak{F}^t, P)$, where $\{\mathfrak{F}^t\}$ is a minimal filtration adapting it. To write an operator equation for the first moments of (1) in a space of continuous n-dimensional mappings $\mathbb{C}(\mathbb{Y} \to \mathbb{R}^n) := \mathbb{C}_n(\mathbb{Y})$, a linear continuous operator is introduced:

$$y \in \mathbb{Y}, u \in \mathbb{C}_n(\mathbb{Y}) : (\mathbf{A}u)(y) = \int\limits_{\mathbb{Y}} A^T(z)u(z)p(y, dz). \tag{10}$$

Lemma 4.1 *For any $s \in \mathbb{R}, t > 0, v \in \mathbb{C}_n(\mathbb{Y}), x \in \mathbb{R}^n$*

$$\mathbb{E}\{(X_s^{s+t}x, v(y_{s+t}))/\mathfrak{F}^s\} = (x, (\mathbf{A}^t v)(y_s)).$$

It is said that the equation (1) is mean reducible, if such a continuous matrix function $\{\mathbf{B}(y), y \in \mathbb{Y}\}$ and such a matrix Λ exist, that for all $s \in \mathbb{N}$ and $t > s$ the following equality is fulfilled: $\mathbb{E}\{\mathbf{B}(y_t)x_t/\mathfrak{F}^s\} = \Lambda^{t-s}\mathbf{B}(y_s)x_s$.

Theorem 4.2 *Let elements of sequence $\{y_t, t \in \mathbb{N}\}$ be independent and identically distributed. Then*

(i) operator $\mathbf{A}$ leaves as invariant a subspace $\mathbb{R}^n \subset \mathbb{C}_n(\mathbb{Y})$ and restriction $\bar{\mathbf{A}}$ of operator $\mathbf{A}$ in this subspace is defined by equality

$$v \in \mathbb{R}^n : \bar{\mathbf{A}}v = \bar{A}^T v,$$

where $\bar{A} = \mathbb{E}\{A(y_0)\}$;

(ii) for each $s \in \mathbb{N}$, each $t > s$ and each $\mathfrak{F}^t$-adapted solution $\{x_t, t \geq 0\}$ of equation (1) the following equality is into force:

$$\mathbb{E}\{x_t\} = \bar{A}^{t-s}\mathbb{E}\{x_s\}.$$

To define the reduced equation of the equation with near to constant matrix coefficients dependent on Markov chain (7) the operator (10) is expressed in a form $\mathbf{A}(\varepsilon) = \mathbf{A}_0 + \varepsilon\mathbf{A}_1 + \varepsilon^2\mathbf{A}_2 + \ldots$; hereto, the operator $\mathbf{A}_0$ leaves as invariant the subspace $\mathbb{R}^n$, and it can be represented as a tensor product of operators $\mathbb{A}_0 = \mathcal{P} \otimes A_0^T$:

$$h \in \mathbb{C}(\mathbb{Y}), g \in \mathbb{R}^n : \mathbf{A}_0(h \otimes g) = \mathcal{P}h \otimes A_0^T g,$$

where $\mathcal{P}$ is a Markov operator defined by formula

$$y \in \mathbb{Y}, u \in \mathbb{C}(\mathbb{Y}) : (\mathcal{P}u)(y) = \int\limits_{\mathbb{Y}} u(z)p(y, dz).$$

The tensor representation of the operator allows to simplify the process of finding the spectrum and resolvent, using the spectrum and resolvent of operators which define them. Due to the exponential ergodicity of Markov chain, the spectrum of operator $\mathbf{A}_0$ can be expressed in a form:

$$\sigma(\mathbf{A}_0) = \{\lambda_1\lambda_2 : \lambda_1 \in \sigma(\mathcal{P}), \lambda_2 \in \sigma(A_0)\} = \sigma(A_0) \cup \sigma_\rho, \tag{11}$$

where $\sigma_\rho(A_0) := \{\lambda_1\lambda_2 : \lambda_1 \in \sigma(\mathcal{P}), \lambda_2 \in \sigma_\rho\}$. The main assumption for mean reducibility of the equation (1) is disjunction of sets in spectrum decomposition (11), that is, $\sigma(A_0) \cap \sigma_\rho = \emptyset$. It makes possible to offer the asymptotical method which is based on the decomposition of the spectral projection [4] of operator $\mathbf{A}(\varepsilon)$ by powers of a small parameter ε. The conjugated space of $\mathbb{C}_n(\mathbb{Y})$ is a space of vector-valued measures $\mathbb{C}_n^*(\mathbb{Y})$, and the scalar product of elements $v \in \mathbb{C}_n(\mathbb{Y})$ and $g \in \mathbb{C}_n^*(\mathbb{Y})$ is defined by the equality $\langle g, v\rangle := \int\limits_{\mathbb{Y}} (g(dy), v(y))$.

Lemma 4.3 *If all the above mentioned assumptions are into force, then for sufficiently small $\bar{\varepsilon} > 0$ and for all $|\varepsilon| < \bar{\varepsilon}$, a difference equation is mean reducible; hereto, the matrix function $\{\mathbf{B}(y, \varepsilon), y \in \mathbb{Y}\}$ is a basis in operator $\mathbf{A}(\varepsilon)$ root subspace that corresponds to the part of the spectrum $\sigma_0(\varepsilon)$ that is defined by equality $\lim\limits_{\varepsilon\to 0} \sigma_0(\varepsilon) = \sigma_0$, but the matrix $\Lambda(\varepsilon)$ is the operator's $\mathbf{A}(\varepsilon)$ restriction matrix to this root subspace. For each $|\varepsilon| < \bar{\varepsilon}$, $n \times n$-matrix function of the basis $\{\mathbf{B}(y, \varepsilon), y \in \mathbb{Y}\}$ and constant $n \times n$-matrix $\Lambda(\varepsilon)$ unambiguously are defined by the equality*

$$y \in \mathbb{Y}, |\varepsilon| < \bar{\varepsilon} : (\mathbf{A}(\varepsilon)\mathbf{B})(y, \varepsilon) = \mathbf{B}(y, \varepsilon)\Lambda^T(\varepsilon). \tag{12}$$

The decompositions of the basis matrix $\mathbf{B}(y, \varepsilon)$ and the matrix $\Lambda(\varepsilon)$ are used in a form of uniformly converged sequences by powers of a small parameter ε: $\Lambda(\varepsilon) := \Lambda_0 + \varepsilon\Lambda_1 + \varepsilon^2\Lambda_2 + \ldots$ and $\mathbf{B}(y, \varepsilon) := \mathbf{B}_0 + \varepsilon\mathbf{B}_1(y) + \varepsilon^2\mathbf{B}_2(y) + \ldots$. For each sufficiently small ε these decompositions can be substituted in the expression (12). Equating the coefficients of equal powers of ε, the equations can be obtained for finding the unknown elements of series $\Lambda(\varepsilon)$ and $\mathbf{B}(y, \varepsilon)$:

$$\mathbf{A}_0\mathbf{B}_0 = \mathbf{B}_0\Lambda_0^T \tag{13}$$

$$\mathbf{A}_0\mathbf{B}_1 - \mathbf{B}_1\Lambda_0^T = \mathbf{B}_0\Lambda_1^T - \mathbf{A}_1\mathbf{B}_0 \tag{14}$$

$$\mathbf{A}_0\mathbf{B}_2 - \mathbf{B}_2\Lambda_0^T = \mathbf{B}_0\Lambda_2^T + \mathbf{B}_1\Lambda_1^T - \mathbf{A}_0\mathbf{B}_2 - \mathbf{A}_1\mathbf{B}_1 \tag{15}$$

$$\ldots$$

Let us define an operator

$$\begin{aligned} y \in \mathbb{Y}, v \in \hat{\mathbb{C}} : (\mathbb{L}v)(y) :=& (\mathbf{A}_0 v)(y) - v(y)A_0^T := \\ :=& \int\limits_{\mathbb{Y}} A_0^T(v(z) - v(y))p(y, dz) + A_0^T v(y) - v(y)A_0^T := \\ :=& (\mathbb{H}v)(y) + (\mathbb{G}v)(y) \end{aligned} \tag{16}$$

for the elements of continuous matrix functions space $\hat{\mathbb{C}}$. Looking at $\hat{\mathbb{C}}$ as at $\mathbb{R}^{n^2}$, similarly as in the case with $\mathbb{C}_n(\mathbb{Y})$, count additive matrix-valued measure in $\hat{\mathbb{C}}^*$ can be found, which will be a conjugated space; and the scalar product of elements $g \in \hat{\mathbb{C}}^*$ and $v \in \hat{\mathbb{C}}$ can be defined by formula $\langle g, v\rangle := Tr\{\int\limits_{\mathbb{Y}} v^T(y)g(dy)\}$, where $Tr\{\}$ is a matrix trace. Taking unit matrix $\mathbf{B}_0 := I$ as basis in $\mathbb{R}^n$ and substituting it in the equation (13), $\Lambda_0^T = A_0^T$ can be found, that is, $\Lambda_0 = A_0$.

Using Fredholm alternative about normal solvability, the necessary and sufficient conditions can be verified to ensure that a solution exists. The matrix Λ_2 can be found from the equation (14), afterwards also $\mathbf{B}_2(y)$ can be found. Then the next equations can be written for finding $\Lambda_3, \mathbf{B}_3(y)$ until the necessary accuracy of decomposition of the matrix $\Lambda(\varepsilon)$ is obtained. Since $\mathbb{Y}$ is compact and matrices $\{\mathbf{B}_j(y), j = 1, 2, \ldots\}$ are continuous, the elements of the obtained basis $\mathbf{B} := I + \varepsilon\mathbf{B}_1 + \varepsilon^2\mathbf{B}_2 + \ldots$ are linearly independent for sufficiently small ε.

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Current address

Carkova Viktorija, Dr. Math., Assoc. Prof.
University of Latvia, 8 Zellu street, Riga, LV-1002, Latvia, tel. number +371 67033727 and e-mail tsarkova@latnet.lv

Goldšteine Jolanta, Dr. Math.
University of Latvia, 8 Zellu street, Riga, LV-1002, Latvia, tel. number +371 29498160 and e-mail jolanta.goldsteine@gmail.com

Swerdan Myhailo, Dr. Phys.-Math. Sci., Prof.
The State Financial Academy of Bukovina, 1 Shterna street, Chernivtsi, 58000, Ukraine, tel. number +380 372 235152