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AUTOREGRESSIVE MODELS OF RISK PREDICTION AND ESTIMATION USING MARKOV CHAIN APPROACH

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Abstract. The possibility of identifying nonlinear time series using nonparametric estimates of the conditional mean and conditional variance is studied. Most nonlinear models satisfy the assumptions needed to apply nonparametric asymptotic theory. Sampling variations of the conditional quantities are studied by simulation and explained by asymptotic arguments for the first-order nonlinear autoregressive processes. The paper deals with the identification and prediction problems of the autoregressive models of nonlinear time series using nonparametric estimates of the conditional mean and conditional variance.

Key words. Time series, Markov chain, transition probability, regression model, statistical estimation

Mathematics Subject Classification: 60J10, 62F12, 11K45

1 Introduction

Predictive modelling has attracted significant attention from the most if not all of risk management researchers. Especially the methods and algorithms of time series analysis remain an important tool and is widely applicable in financial econometrics for assessment and prediction of risk. As it well known the time series analysis is limited by the choice of regressive model and even for the simplest Markov model requires identifying multivariate distribution function. The proposal project is devoted to analysis of semiparametric stationary Markov dynamical systems used in the contemporary applications of mathematics for risk control and prediction with the help of auto regressive models. Our identification strategy uses a nonlinear auto regressive model with no classical heteroskedastic errors. Whereas the classical approach to regression analysis assumes that the form of the relationship between collections of variables is known apart from a few unknown parameters that must be estimated from the data, our approach uses more modern techniques that employ copula based nonparametric curve fitting to produce estimators as well as to assess the validity of parametric models. Owing to coupling different marginal distributions with different copula functions, copula-based auto regressive processes help us to model not only a wide variety of marginal behaviours but also such dependence properties as clusters, positive or negative tail dependence and so on. These tools of stochastic dynamical systems analysis are characterized by nonparametric marginal distributions and parametric copula functions, while the copulas capture all the scale-free temporal dependence of the processes. Given the estimators of the marginal distribution and the copula dependence parameter, one can easily construct an estimator of the transition distribution of the time series and hence estimators of nonlinear conditional moment and conditional quintile functions. Besides we intend to combine nonparametric auto regressive process estimation with Monte Carlo simulation and an empirical analysis of stochastic dynamics. Because of this flexibility, our approach may be very relevant in the finance and insurance community, where modelling and estimating the dependence structure between several univariate time series are of great interest.

2 Preliminaries

One of main problems of modern econometrics is development of time series $\{x_t, t \in Z\}$ methods of analysis through regression models without a priori information about the form of dependence of the conditional expected value from its past values. Therefore it is necessary to deal with the estimation of unknown function in nonlinear difference equation of the first order with usual kind of information about the distribution law. In many applied problems of regression analysis for time series $\{x_t, t \in N\}$ already in simplest case [1,3,4,5]

$$x_t = f(x_{t-1}) + h_t, \tag{1}$$

where h_t are the uncorrelated tailings, on the average equal to the zero.

The problem of analysis of time series described higher got the name of nonparametric estimation of autoregression. The needed function can be defined through the conditional expected value $f(x_{t-1}) := E\{x_t \mid \mathcal{F}^{t-1}\}$. To use the sequence of sigma-algebra $\{\mathcal{F}^t, t \in \mathbb{Z}\}$ and conditional dispersion $\sigma_t^2 := E\{h_t^2 \mid \mathcal{F}^{t-1}\}$, tailings h_t can be present [7] in form work of "white noise" $\{\xi_t, t \in \mathbb{Z}\}$ in equation (1), (i.e. sequences of the independent identically distributed (i.i.d.) random values with zero mean and by single dispersion) and with conditional standard deviation: $h_t := \sigma_t \ \xi_t$.
This property of tailing's dispersion is called [7] as conditional heteroskedasticity and can be modulate through linear difference equations with coefficients, linearly depending on white noise (GARCH (p,q) processes).

3 Description of model

We will suppose that is observed random process of type

$$x_{n+1} = f(x_n) + \sigma_n \xi_{n+1} \quad , \tag{2}$$

ξ_n is a random error of observations, (i.i.d.) . $\mathrm{E}\{\xi_n\} = 0,$ $f(x_n)$ is a nonlinear function of the elements of chain .

Equation (2) can be interpreted so, that a random sequence depends on the «history». Also we can write that the conditional expected value of random variable looks like

$$\mathrm{E}\left\{x_{n+1} \mid F^{n}\right\} = \mathrm{E}\left\{x_{n+1} \mid x_{n}\right\} = \sum_{y} p(x_n, y) \cdot y = f(x_n) \qquad (3)$$

that determines non-linearity of functional dependence x_{n+1} from x_n. The purpose of our researches is to describe the dynamics of chain $\{x_n\}$. This means to find the functional dependence $f(x_n)$, due to equation (3).

For searching for of function $f(x_n)$ we need to create separate discrete intervals of values and then on every interval we can use either least-squares or minimize specially built functional as an integral with the kernels of different form. We will consider the model of phase space discretization and presentation of him in form eventual number of no splitting areas $\{S_k, k = 1,...,r\}$ which can be examined as the states of some Markov chain.

The probabilistic behaviour of a Markov chain is determined by the transition probability matrix ***P*** and a probability distribution over the initial state X_0 , if we are given X_0 and ***P***, we may want to determine the probability distribution for each random variable X_n or possibly we may be interested in the limiting distribution of X_n as $n \to \infty$,, if such a distribution exists. Within this context, if a chain is irreducible and aperiodic and thus ergodic, then there exists a unique row vector $\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_r)$, such as

$$\lim_{m\to\infty} p_{ij}^{(m)} = \pi_j, \quad i, j = 1,2,...,r,$$

where $p_{ij}^{(m)}$ is the *(i,j)*th element of $\boldsymbol{P}^t$, $p_{ij}^{(m)} = P(X_m = j \mid X_0 = i)$

and

$$0 \le \pi_j \le 1; \qquad \sum_j \pi_j = 1, \quad j = 1,2,...,r$$

and

$$\boldsymbol{\pi} = \boldsymbol{\pi}\, P.$$

When these probabilities $p_{ij}^{(m)}$ are not depending on „m”, they are called as stationary probabilities and the Markov chain is homogeneous. She is fully determined by the matrix of transition probabilities.

4 Unbiased estimations of the transitions probabilities

Most of users wish to use the maximum likelihood estimations for the stationary transitions probabilities ([6], [8]). But maximum likelihood estimations are consistent, but not unbiased estimations. In this article we consider the possibility of constructing the consistent unbiased estimations of the transition probabilities of the Markov chain.

Let us consider the homogeneous Markov chain with a number of states $\{E_i, i = 1,2,...,s+1\}$ and with the positive matrix of transition probabilities $P = \|p_{ij}\|$ $(i, j = 1,2,...,s+1)$, p_{ij} *>0* . Then this Markov chain will be ergodic, and there is a unique set of positive final probabilities $\{p_{ij}\}$, not depending on initial vector of probabilities $p_i(1)$.

Let us denote ***m_i*** *is* a number of appearances *of state **E_i*** after *n* tests, and m_{ij} *is* a number of transitions from the state *of **E_i*** to the state *of **E_j*** after *n* tests. We will count up a number *of* different chainlets of length ***n**,* made from the ***s+1*** states, having the set number of transitions *of* ***m_ij*** and beginnings in the state ***E_i*** and endings in the state ***E_j.***

The numbers *of* ***m_i*** and ***m_ij*** meet the following conditions:

$$\sum_{j=1}^{s+1} m_{ij} = m_i , \quad \text{for} \quad i \neq j_0 ; \qquad \sum_{j=1}^{s+1} m_{j_0 j} = m_{j_0} - 1$$

$$\sum_{i=1}^{s+1} m_{ij} = m_j , \quad \text{for} \quad j \neq i_0 ; \qquad \sum_{i=1}^{s+1} m_{ii_0} = m_{i_0} - 1$$

Count of number *K* different chainlets of length ***n*** can be calculated on induction on the number of the states of Markov chain.

For ***n=2***:

$$K_{i_0 j_0}^{(2)} = c_{m_{i0}-1}^{m_{i0}-m_{i0i0}-1} . C_{m_{j0}-1}^{m_{i0}-m_{i0i0}-1}$$

For ***n>2*** ($i_0 \neq j_0$):

$$K = \frac{(m_{j_0}-1)\prod_{i=1}^{s+1} m_j !}{m_{j_0} m_{s+1} \prod_{i,j=1}^{s+1} m_{ij}!} \begin{vmatrix} 1-\frac{m_{11}}{m_1} & \dots & -\frac{m_{1i_0}}{m_1} & \dots & -\frac{m_{1j_0}}{m_1} & \dots & -\frac{m_{1s}}{m_1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{m_{i_0 1}}{m_{i_0}} & \dots & 1-\frac{m_{i_0 i_0}}{m_{i_0}} & \dots & -\frac{m_{i_0 j_0}}{m_{i_0}} & \dots & -\frac{m_{i_0 s}}{m_{i_0}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{m_{j_0 1}}{m_{j_0}-1} & & -\frac{m_{j_0 i_0}+1}{m_{j_0}-1} & & 1-\frac{m_{j_0 j_0}-1}{m_{j_0}-1} & & -\frac{m_{j_0 s}}{m_{j_0}-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{m_{s1}}{m_s} & & -\frac{m_{s i_0}}{m_s} & & -\frac{m_{s j_0}}{m_s} & & 1-\frac{m_{ss}}{m_s} \end{vmatrix}$$

For *n>2* at $i_0 = j_0$ the line of determinant with this number will assume the following:

$$-\frac{m_{j_0 1}}{m_{j_0}-1} \quad \dots \quad 1-\frac{m_{j_0 1}}{m_{j_0}-1} \quad \dots \quad -\frac{m_{j_0 s}}{m_{j_0}-1} ,$$

and other line will remain back.

So, we are continued to consider the homogeneous Markov chain with a number of states $\{E_i , i = 1,2,..., s+1\}$ and with the matrix of transition probabilities

$P = \|p_{ij}\| \; (i, j = 1, 2, ..., s+1)$. Indeed, we assume that Markov chain is situated in state E_i at initial moment of time. Looking after the homogeneous Markov chain during ***n*** steps, will get some sequence of events $E_{i_0} \to E_{i_1} \to \cdots \to E_{j_0}$, so called as transitions trajectories. In the case of Markov chain with the fixed number of steps *of* ***n,*** the results of observations can be written down as a matrix of observations $M = \|m_{ij}\| \left(i, j = \overline{1, s+1} \right)$, where m_{ij} is a number of transitions of Markov process from the state E_i in the state E_j. Thus it should be noted that the matrix of observations and initial state of Markov chain fully determines the final state of the observed process. On results of observations it is required to estimate an unknown matrix $P = \|p_{ij}\|$.

Transition probabilities $P^{(n)}_{i_0 j_0}(M)$ from the state E_{i_0} in the state E_{j_0} after ***n*** steps with the matrix of observations *of M* can be calculated by the formula.

$$P^{(n)}_{i_0 j_0}(M) = K_{i_0 j_0}(M) \prod_{k,j=1}^{s+1} p_{kj}^{m_{kj}}, \tag{4}$$

where $K_{i_0 j_0}(M)$ is number of trajectories, which come from the state E_{i_0} in the state E_{j_0} The formula (4) is true due the following probabilities..

$$P(x_0, x_1, ..., x_n) = P(x_0) \prod_{t=1}^{n} P(x_t \mid x_{t-1}) \tag{5}$$

$$P(x_0, x_1, ..., x_n \mid M) = P(x_0) \prod_{i,j} p_{ij}^{m_{ij}} \tag{6}$$

The next formula allow to calculate the number of trajectories by the following kind,

$$K_{kj}(N) = A_{jk}(S) \cdot \frac{\prod_{k=1}^{m} \omega_k \,!}{\prod_{k,j=1}^{m} n_{kj} \,!}, \tag{7}$$

where $A_{jk}(S)$ is algebraic addition of element with indexes *(j,k)* in a matrix

$S = \|s_{ij}\| \; (i, j = 1, ..., m)$, where

$$s_{ij} = \begin{cases} \delta_{ij}, & if \;\; \omega_i = 0, \\ \delta_{ij} - \dfrac{n_{ij}}{\omega_i}, & if \;\; \omega_i > 0. \end{cases} \tag{8}$$

In the formula (8) δ_{ij} is the "kroneker" character, and $\omega_i = \sum_{j=1}^{n} n_{ij}$ − number of exits from the state of E_i.

Another kind of formulas for estimation of matrix $P = \|p_{ij}\|$ can be concluded by the following. We will denote $R(l)$ for the set of matrix of observation $L = \|l_{kj}\| \ (k, j = 1,...,m)$, with the property $\sum_{k,j=1}^{m} l_{kj} = l$, which present realization of Markov chain with the initial state $\boldsymbol{E_i}$.

The unbiased estimation of transition probability takes place from the state *of* $\boldsymbol{E_k}$ in the state *of* $\boldsymbol{E_j}$ after $\boldsymbol{l}$ steps for the Markov chain:

$$\hat{P}_{kj}^{(l)} = \frac{\sum_{L \in R(l)} P_{kj}^{(l)}(L) \cdot P_{kj_n}^{(n-l)}(N-L)}{P_{kj_n}^{(n)}}, \ N \in R(n),$$

where $\boldsymbol{j_n}$ - is the index of final state of the Markov chain.

5 Conclusions

And so, we can show two following conclusions.

Conclusion 1. The unbiased estimation of transition probability $\hat{P}_{kj}^{(l)}$ can be also calculated by the following formula:

$$\widehat{P}_{kj}^{(l)} = \frac{\sum_{L \in R(l)} K_{kj}(L).K_{kj_n}(N-L)}{K_{kj}(N)}, \qquad N \in R(n).$$

Conclusion 2. The unbiased estimation of the element $\hat{p}_{kj}$ of the transition probability matrix is the following.

$$\hat{\Theta}_{kj} = \frac{K_{kj_n}(N-L)}{K_{kj_n}(N)}, \ N \in R(n),$$

Where the matrix $\boldsymbol{L}$ has the dimension m , and the element $l_{kj} = 1$, and all other elements are equal to zero.

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