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ON CONVERGENCE OF GARCH PROCESS

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Abstract. The paper deals with symmetric GARCH models. Assuming that there exists a second moment of conditional variance we will propose the necessary and sufficient conditions for convergence of the above stochastic recurrent procedure to a stationary time series. A mathematical background of proposal methods and algorithm are based on the derived by the first author covariance method for Lyapunov stability analysis of linear difference equations with Markov coefficients. That permits to write out a mean square convergence criterion for GARCH models in a convenient for application form of inequalities involving the model parameters.

Key words and phrases. GARCH(p,q) Models, Stochastic Convergence, Markov Dynamical Systems, Mean Square Stability.

1 Introduction: Symmetric GARCH(p,q) Models

The symmetric regression model GARCH(p,q)(*Generalized AutoRegressive Conditional Heteroskedasticity*) with shocks, takes the following form [2]:

$$\begin{aligned} Y_t &= b_0 + \sum_{k=1}^{n} b_k X_t^{(k)} + \xi_t,\ \mathbb{E}\{\xi_t/\Phi_{t-1}\} \equiv 0,\ \mathbb{E}\{\xi_t^2/\Phi_{t-1}\} = \sigma_t^2 \\ \sigma_t^2 &= \theta_0 + \sum_{k=1}^{p} \phi_k \sigma_{t-k}^2 + \sum_{k=1}^{q} \theta_k \sigma_{t-k}^2 \varepsilon_{t-k}^2. \end{aligned} \tag{1}$$

This process is described for time moments $t \in \mathbb{Z}$ by $q+1$ coefficients θ_k, $k = 1, ..., q$, p coefficients ϕ_k, $k = 1, 2, ..., p$, mean b_0, n linear regression coefficients b_k, $k = 1, 2, ..., n$, endogenous and exogenous variables Y_t and $X_t^{(k)}, k = 1, 2, ..., n$ respectively, conditional variance σ_t^2, white-noise type time series $\{\varepsilon_t,\ t \in \mathbb{Z}\}$ (that is, i.i.d. random variables with mean zero and variance one) and the sigma-algebra Φ_{t-1} of information up to time $t-1$. Let $\{\hat{\sigma}_t^2\}$ be a stationary time-series and $\{\sigma_t^2\}$ any other time series satisfying (1). Then a difference $x_t := \hat{\sigma}_t^2 - \sigma_t^2$ satisfies the homogeneous equation with independent random coefficients

$$x_t = \sum_{k=1}^{p} \phi_k x_{t-k} + \sum_{k=1}^{q} \theta_k x_{t-k} \varepsilon_{t-k}^2. \tag{2}$$

To achieve a stabilization property with time increasing of the above regressive model (1), it should be assumed that $\lim_{t\to\infty} x_t = 0$. The above mentioned stability property is called *Lyapunov asymptotic stability* of stationary solution of (1) or trivial solution of (2). The problem arises: to find a set of parameters $\{\phi_k, \theta_k\}$, which guarantees the above asymptotic stability property (see, for example, Hamilton (1994) and references there). This paper proposes very simple algorithm for system (2)asymptotic stability analysis, which allows to write out the necessary and sufficient asymptotic stability conditions for $q = 1$ in a form of inequality for specially constructed function of parameters $\{\phi_k, \theta_1\}$. A mathematical background of the proposal algorithm is based on devoted in (Carkova and Berdichevska (1994)) covariance method for stability analysis of difference equations with Markov coefficients.

2 Mean Square Stability of Linear Difference Equations with Markov Coefficients

Let $\{y_t,\ t \in \mathbb{N}\}$ be a homogeneous Feller Markov chain with transition probability $p(y, dz)$ on the metric space $\mathbb{Y}$. The m-dimension linear Markov difference equation is an iterative procedure in $\mathbb{R}^m$ defined by equality

$$x_t = A(y_t)x_{t-1}, \ \ t \in \mathbb{N} \tag{3}$$

where $\{A(y),\ y \in \mathbb{Y}\}$ is uniformly bounded continuous $m \times m$ matrix function. We will refer to equation (3) as *exponentially mean square stable* if there exist such constants $c > 0$ and $\lambda \in (0, 1)$ that $\mathbb{E}_{x,y}^{(k)}|x_t(k, x, y)|^2 \le c\lambda^{t-k}|x|^2$ for any $y \in \mathbb{Y}$, $x \in \mathbb{R}^n$, $k \in \mathbb{N}$ and $t \ge k$. Let $\mathbb{V}$ be the Banach space of symmetric uniformly bounded continuous $m \times m$ matrix functions $\{v(y), y \in \mathbb{Y}\}$ with norm $||v|| := \sup_{y\in\mathbb{Y},||x||=1} |(v(y)x, x)|$. Using transition probability $p(y, dz)$ of Markov chain one can define on $\mathbb{V}$ the linear continuous operator $(\mathbf{A}\, v)(y) := \int_{\mathbb{Y}} A^T(z)v(z)A(z)p(y, dz)$, where top index T denotes a transposition. It is easy to see that the above defined operator leaves as invariant the cone of positive defined $m \times m$-matrix functions and for spectral analysis one can use the methods and results of paper [3]. This property permits to derive mean square stability conditions of (3) in following form.

Theorem[1] *The equation (2) is exponentially mean square stable if and only if the real part of the spectrum $\sigma(\mathbf{A})$ of operator $\mathbf{A}$ is situated in the set $\{z \in \mathbb{R} : -1 < z < 1\}$.*

If the sequence $\{y_t, t \in \mathbb{N}\}$ consists of independent random variables with the same distribution $p(dy)$ one can consider a contraction $\hat{\mathbf{A}}$ of the above defined operator $\mathbf{A}$ on the space of symmetric $m \times m$ real matrices

$$\hat{\mathbf{A}}v := \mathbb{E}\{A^T(y_t)vA(y_t)\} = \int_{\mathbb{Y}} A^T(y)vA(y)p(dy) \tag{4}$$

with cone of the positive defined matrices. In this case the equation (3) is exponentially mean square stable if and only if the real part of the spectrum $\sigma(\hat{\mathbf{A}})$ of operator $\hat{\mathbf{A}}$ is situated in the inteval $\{z \in \mathbb{R} : |z| < 1\}$.

3 GARCH(p,1) Models

The late assertion of second section is very convenient for convergence analysis of GARCH(p,1) model given by equation (1). As it has been mention in Introduction, for that one can analyze convergence to zero of any solution of scalar difference equation (2) of order p. Introducing normalized i.i.d. random variables $y_k := (\varepsilon_k^2 - 1)/s^2$, where $s^2 := \sqrt{\mathbf{E}\{\varepsilon_k^4\} - 1}$, one can rewrite this equation as a first order difference equation in $\mathbb{R}^p$ with matrix $A(\vec{y}_t) := A + s^2 y_{t-1} H$, where

$$A = \begin{pmatrix} 0 & 1 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ . & . & ... & . & . \\ 0 & 0 & ... & 0 & 1 \\ \varphi_p & \varphi_{p-1} & ... & \varphi_2 & \varphi_1 + \theta_1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & ... & 0 & 0 & 0 & ... & 0 \\ 0 & ... & 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & 0 & 0 & ... & 0 \\ h & ... & 0 & 0 & 0 & ... & 0 \end{pmatrix}$$

The second moment of any solution of (2) exponentially decreases to zero if and only if there are no positive defined solutions of the matrix equation $\hat{\mathbf{A}}v = \rho v$ for any real $\rho \geq 1$. Therefore, if the eigenvalues of matrix A are situated in the half-plane $\{|z| < 1\}$, then there exists such a positive number $\mathbf{r}^2$ that for any $s^4 < \mathbf{r}^2$ second moment of any solution (2) $\mathbb{E}|x_n|^2 := \mathbb{E}|\hat{\sigma}_n - \sigma_n|^2$ tends to zero as $n \to \infty$, but for $s^4 > \mathbf{r}^2$ – exponential increases to infinity. Analyzing equation $A^T v A + \mathbf{r}^2 H^T v H = v$ one can find this positive number $\mathbf{r}^2$ and having used a special form of above defined matrices A, H write out the system of p linear homogeneous equations involving variables $\{v_{1p}, v_{2p}, ..., v_{pp}\}$:

$$v_{ip} - \sum_{l=1}^{i} a_{l+1} v_{(p-i-1+l)p} - \sum_{k=i+1}^{p} a_{k-i} v_{kp} = 0, \; i = 1, 2, ..., p-1 \tag{5}$$

$$v_{pp}\big(1 - \sum_{s=1}^{p} a_s^2 - \mathbf{r}^2 h^2\big) - 2 \sum_{l=1}^{p-1} \sum_{s=1}^{l} a_{p-l+1} a_{l-s+1} v_{lp} = 0. \tag{6}$$

Because number $\mathbf{r}^2$ has been chosen in such a way that the system has nontrivial solution, a determinant of linear system (5)–(6) has to be equal to zero. By construction this determinant is linear function of parameter $\mathbf{r}^2$ and equating to zero the above determinant one can easy find the critical number $\mathbf{r}^2$ as a function of parameters a_j, h, $j = 1, 2, ..., p$. The necessary and sufficient condition for convergence of GARCH(p,1) (1) to stationary time series $\{\hat{\sigma}_t^2\}$ with second moment has a form of inequality $s^4 < \mathbf{r}^2$.

Example. Let we should deal with GARCH(2,1) regressive model and $\varepsilon_k \sim N(0,1)$. Then $s^4 = \mathbb{E}\{\varepsilon_t^4\} - 1 = 2$, $p = 2$ and system (5)-(6) has the following form

$$\begin{aligned} v_{12}(\phi_2 - 1) + (\phi_1 + \theta_1) v_{22} &= 0, \\ 2(\phi_1 + \theta_1)\phi_2 v_{12} - (1 - (\phi_1 + \theta_1)^2 - \phi_2^2 - \mathbf{r}^2 \theta_1^2) v_{22} &= 0. \end{aligned}$$

Equating to zero determinant of this system one can find a critical value $\mathbf{r}^2 = (1 + \phi_2)[(1 - \phi_2)^2 - (\phi_1 + \theta_1)^2]/\theta_1^2(1 - \phi_2)$. Therefore, GARCH(2,1) regressive model of type (1) with Gaussian noice $\{\varepsilon_k\}$ convergence to stationary time series $\{\hat{\sigma}_t\}$ with $\mathbb{E}\sigma_t^4 < \infty$ if and only if $2\theta_1^2(1 - \phi_2) < (1 + \phi_2)[(1 - \phi_2)^2 - (\phi_1 + \theta_1)^2]$.

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