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THE IMPACT OF SERIAL CORRELATION ON RISK HEDGING

EGLE Aigars, (LV)

Abstract. There are a number of publications that documents the predictability of financial asset returns. In our paper we develop a continuous diffusion model for the case of serially correlated stock returns. We obtain European call option pricing formula written on a stock with autocorrelated returns and show that even small levels of predictability due to serial correlation can give substantial deviation from results obtain by Black-Sholes formula. Finally, we derive formulas for sensitivities of the value of European call option and show how in risk management widely used option hedging parameters depend on assumptions made about correlation in underlying asset returns.

Key words and phrases. serial correlation, diffusion approximation, option pricing

Mathematics Subject Classification. Primary 60A05, 08A72; Secondary 28E10.

1 Introduction

1.1 Empirical Evidence of the Predictability of Asset Returns

There is a wide list of financial research that documents the predictability of financial asset returns. Auto correlation in short term stock index returns has been analyzed by Lo and MacKinley in [1], Jokivuolle in [2] and Stoll and Whaley in [3]. They argue that positive autocorrelation shows up in index returns due to presence of stale prices of stocks included into the index. Above mentioned happens when the increase in the number of stocks comes from inclusion of small capitalization stocks, which are known to trade less frequently than large ones. Due to infrequent trading in small capitalization stocks the observed index value do not reflect the true market value of the underlying stock portfolio as the index value is calculated using the last observed stock transaction prices.

Conrad and Kaul in [4] avoiding the nonsynchronos trading problem analyze autocorrelation of Wednesday-Wednesday returns for size grouped portfolios and find first order autocorrelation of weekly returns varying between 0.09 to 0.30. For longer time periods Fama and French in [5] find that autocorrelation of returns of diversified portfolios of NYSE stocks becomes strongly negative.

The evidence of serial autocorrelation in stock and stock index returns contradicts assumptions made in widely accepted stock return model used by Black and Sholes in [6] and Merton in [7] to derive call option pricing formula. They assume that asset returns are distributed independently of each other.

1.2 Convergence of stochastic difference equations to stochastic differential equations

In this section we would like to present general conditions for a sequence of finite-dimensional discrete time Markov processes $\{{}_hX_t\}_{h\downarrow 0}$ to converge weakly to an Ito process. These are drawn from Nelson [10].

The formal set-up is as follows: Let $D([0,\infty),R^n)$ be the space of mappings from $[0,\infty)$ into R^n that are continuous from the right with finite left limits, and let $B(R^n)$ denote the Borel sets on R^n. D is a metric space when endowed with Skorohod metric. For each $h>0$, let M_{kh} be the σ-algebra generated by kh, ${}_hX_0$, ${}_hX_h$, ${}_hX_{2h}$, ..., ${}_hX_{kh}$, and let ν_h be a probability measure on $(R^n, B(R^n))$. For each $h>0$ and each $k=0,1,2,3,...$, let $\Pi_{h,kh}(x,\cdot)$ be a transition function on R^n, i.e.

(a) $\Pi_{h,kh}(x,\cdot)$ is a probability measure on $(R^n, B(R^n))$ for all $x\in R^n$,

(b) $\Pi_{h,kh}(\cdot,\Gamma)$ is $B(R^n)$ measurable for all $\Gamma\in B(R^n)$.

For each $h>0$, let P_h be the probability measure on $D([0,\infty),R^n)$ such that

$$P_h[{}_hX_0\in\Gamma]=\nu_h(\Gamma) \tag{1}$$

for any $\Gamma\in B(R^n)$,

$$P_h[{}_hX_t={}_h X_{kh}, kh\le t<(k+1)h]=1 \tag{2}$$

and

$$P_h[{}_hX_{(k+1)h}\in\Gamma \mid M_{kh}]=\Pi_{h,kh}({}_hXkh,\Gamma) \tag{3}$$

almost surely under P_h for all $k\ge 0$ and $\Gamma\in B(R^n)$.

For each $h>0$, (1) specifies the distribution of the random starting point and (3) the transition probabilities of n-dimentional discrete time markov process ${}_hX_{kh}$. We form the continuous time process ${}_hX_t$ from the discrete time process ${}_hX_{kh}$ by (2), making ${}_hX_t$ a step function with jumps at times h, $2h$, $3h$ and so on.

Now if for each $h>0$ and each $\varepsilon>0$ we define

$$a_h(x,t)\equiv h^{-1}\int_{\|y-x\|\le 1}(y-x)(y-x)'\Pi_{h,h[t/h]}(x,dy), \tag{4}$$

$$b_h(x,t)\equiv h^{-1}\int_{\|y-x\|\le 1}(y-x)\Pi_{h,h[t/h]}(x,dy), \tag{5}$$

$$\Delta_{h,\varepsilon}(x,t)\equiv h^{-1}\int_{\|y-x\|\le \varepsilon}\Pi_{h,h[t/h]}(x,dy), \tag{6}$$

where $[t/h]$ is the integer part of t/h, i.e. the largest integer $k\le t/h$, it is possible under some assumtions which are in detail specified by Nelson in [10] to prove the following theorem:

Theorem 1.1 *Under some assumptions, the sequence of $_hX_t$ process defined by* (1)-(3) *converges weakly (i.e. in distribution) as $h \downarrow 0$ to the X_t process defined by the stochastic integral equation*

$$X_t = X_0 + \int_0^t b(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_{n,s}, \tag{7}$$

where $W_{n,t}$ is an n-dimentional standard Brownian motion, independent of X_0, and where for any $\Gamma \in B(R^n)$, $P(X_0 \in \Gamma) = \nu(\Gamma)$. Such an X_t process exists and is distributionally unique. This distribution does not depend on the choice of $\sigma(\cdot, \cdot)$. Finally, X_t remains finite in finite time intervals almost surely, i.e. for all $T > 0$,

$$P[\sup_{0 \le t \le T} \|X_t\| < \infty] = 1. \tag{8}$$

Carkovs in [8] follows a similar approach as Nelson [10] and analysis a discrete Markov dynamic system given in a form of stochastic difference equation

$$x_{t+1} = x_t + \varepsilon f_1(x_t, y_{t+1}) + \varepsilon^2 f_2(x_t, y_{t+1}) \tag{9}$$

where $\{y_t\}$ is an ergodic Markov chain with transition probability $p(y, dz)$, invariant measure μ and potential operator Π. Using interpolation

$$s \in [t\varepsilon^2, (t+1)\varepsilon^2] \tag{10}$$

and

$$X_{\varepsilon^2}(s) := (x_{t+1} - x_t)(s\varepsilon^{-2} - t) + x_t \tag{11}$$

Carkovs in his paper [8] is able to prove that for any $\{t_1, t_2, ..., t_n\}$ distribution of vector $\{X_{\varepsilon^2}(t_1), X_{\varepsilon^2}(t_2), ..., X_{\varepsilon^2}(t_n)\}$ for sufficiently small ε^2 may be approximated by distribution of vector $\{X(t_1), X(t_2), ..., X(t_n)\}$ defined by solution of stochastic Ito differential equation

$$dX(s) = a(X(s))ds + \sigma(X(s))dw(s), \tag{12}$$

where

$$a := \bar{f}_2 + [\mathcal{P}\Pi f_1']f_1, \tag{13}$$

$$\sigma^2 := \bar{f}_1^2 + 2\overline{f_1\mathcal{P}\Pi f_1}, \tag{14}$$

$$\mathcal{P}f(y) := \int_Y f(z)p(y, dz), \tag{15}$$

$$\bar{f} := \int_Y f(z)\mu(dz). \tag{16}$$

1.3 The Black-Scholes Option Pricing Formula

The development of option pricing models in [6] and [7] is based on existence of a dynamic investment strategy involving the underlying asset and risk free bonds that exactly replicates payoff of the option. In case when stock price $S(t)$ follows log-normal diffusion process

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \tag{17}$$

where σ is the diffusion coefficient, μ - the drift coefficient and $W(t)$ - a standard Wiener process. It is assumed that trading is frictionless and continuous. Then the no-arbitrage condition yields the following differential equation on the call price $C(t)$

$$\frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 C(t)}{\partial S^2(t)} + \mu S(t)\frac{\partial C(t)}{\partial S(t)} + \frac{\partial C(t)}{\partial t} = \mu C(t), \tag{18}$$

where μ is the instantaneous risk-free rate of return. Given the two boundary conditions for the European call option

$$C(S(T), T) = \max(S(T) - K, 0), \tag{19}$$

$$C(0, t) = 0, \tag{20}$$

there exists a unique solution to the partial differential equation (18) and is called Black-Sholes formula

$$C_{BS}(S(t), t) = S(t)N(d_1) - K\exp(-\mu(T - t))N(d_2), \tag{21}$$

where

$$d_1 \equiv \frac{\log(S(t)/K) + (\mu + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \tag{22}$$

and

$$d_2 \equiv d_1 - \sigma\sqrt{T - t}, \tag{23}$$

where $N()$ is the standard normal cumulative distribution function.

The Black-Sholes formula (21) does not depend on drift μ, but may be an arbitrary function of $S(t)$ and other economical variables. This feature implies that the Black-Sholes formula is applicable to a different asset return processes and could reflect complex patterns of predictability and dependence on other observed and unobserved economic factors.

1.4 The Greeks

The Greeks are vital tools in risk management. Each Greek measures the sensitivity of the value of a portfolio to a small change in a given underlying parameter, so that component risks may be treated in isolation, and the portfolio rebalanced accordingly to achieve a desired exposure.

The Greeks in the BlackScholes model are relatively easy to calculate, a desirable property of financial models, and are very useful for derivatives traders, especially those who seek to hedge their portfolios from adverse changes in market conditions. For this reason, those Greeks which are particularly useful for hedging delta, gamma and vega are well-defined for measuring changes in Price, Time and Volatility. Although μ is a primary input into the BlackScholes model, the overall impact on the value of an option corresponding to changes in the risk-free interest rate is generally insignificant and therefore higher-order derivatives involving the risk-free interest rate are not common.

The most common of the Greeks are the first order derivatives: Delta, Vega, Theta and Rho as well as Gamma, a second-order derivative of the value function. The above mentioned sensitivities are common enough that they have common names, and we will list explicit formulas as they have been derived for a European call $C(S(t), t)$ by Haug in [9].

Delta, Δ, measures the rate of change of option value with respect to changes in the underlying asset's price. Delta is the first derivative of the value C of the option with respect to the underlying instrument's price S.

$$\Delta(S(t), t) \equiv \frac{\partial C}{\partial S} = N(d_1) \tag{24}$$

Theta,Θ, measures the sensitivity of the value of the derivative to the passage of time t.

$$\Theta(S(t), t) \equiv \frac{\partial C}{\partial t} = -\frac{S(t)N(d_1)\sigma}{2\sqrt{T-t}} - \mu K \exp(-\mu(T-t))N(d_2) \tag{25}$$

Vega, ν measures sensitivity to volatility σ. Vega is the derivative of the option value with respect to the volatility of the underlying.

$$\nu(S(t), t) \equiv \frac{\partial C}{\partial \sigma} = S(t)N(d_1)\sqrt{T-t} \tag{26}$$

Rho, $\mathcal{R}$, measures sensitivity to the applicable interest rate. Rho is the derivative of the option value with respect to the risk free rate. Except under extreme circumstances, the value of an option is least sensitive to changes in the risk-free-interest rates. For this reason, rho is the least used of the first-order Greeks.

$$\mathcal{R}(S(t), t) \equiv \frac{\partial C}{\partial \mu} = K(T-t)\exp(-\mu(T-t))N(d_2) \tag{27}$$

Gamma, Γ, measures the rate of change in the delta with respect to changes in the underlying asset price. Gamma is the second derivative of the value function with respect to the underlying price. Gamma is important because it corrects for the convexity of value. When a trader seeks to establish an effective delta-hedge for a portfolio, the trader may also seek to neutralize the portfolio's gamma, as this will ensure that the hedge will be effective over a wider range of underlying price movements.

$$\gamma(S(t), t) \equiv \frac{\partial \Delta}{\partial S} = \frac{N(d_1)}{S(t)\sigma\sqrt{T-t}} \tag{28}$$

2 Derivation of a Formula for Serially Correlated Stock Return Process

The simplest mathematical model describing development of stock's price S_t and involving assumption of serial autocorrelation in stock's returns under commonly used condition on risk neutrality of probabilistic measure $\mathbb{P}$ may be written in the following way

$$S_{t+1} = S_t(1 + \varepsilon^2\mu + \varepsilon\sigma y_{t+1}), \tag{29}$$

where y_t is a Gaussian random sequence with zero mean and unit variance. When it is considered that these random numbers are independent we may write that y_t follows AR(1):

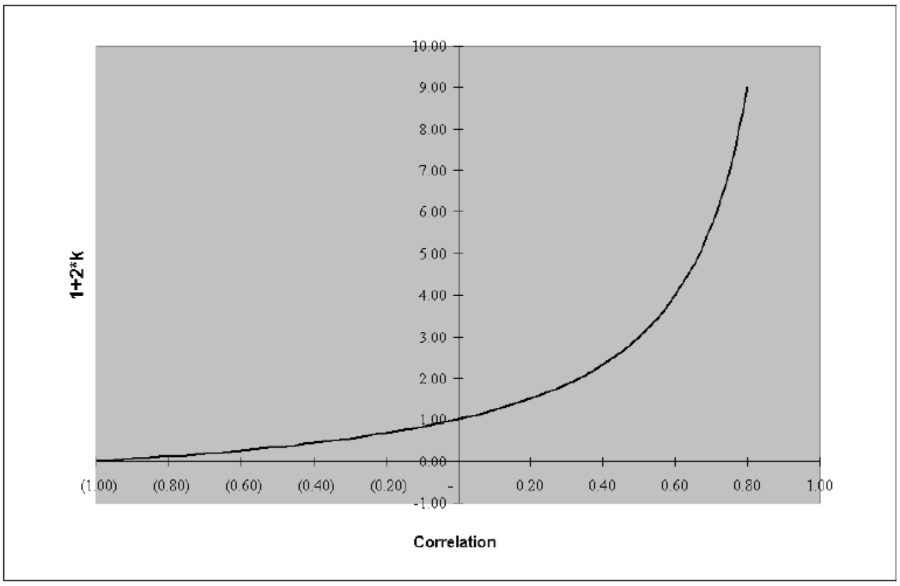

Figure 1: Ratio of $\frac{\sigma^2_{eff}}{\sigma^2}$ as a function of autocorrelation coefficient ρ

$$y_{t+1} = \rho y_t + \sqrt{1-\rho^2}\xi_{t+1}, \tag{30}$$

where $\{\xi_t\}$ is i.i.d. Gaussian sequence, $\mathbb{E}\xi_t = 0$, $\mathbb{E}\xi_t^2 = 1$.

To be able use formulas (12)-(15) derived by Carkovs in [8] we denote $x_t \equiv S_t$ and rewrite equation (29) in the following form

$$x_{t+1} = x_t + \varepsilon\sigma y_{t+1}x_t + \varepsilon^2\mu x_t \tag{31}$$

and now we can use (12) with

$$f_1(x_t, y_{t+1}) = \sigma y_{t+1}x_t \tag{32}$$

and

$$f_2(x_t, y_{t+1}) = \mu x_t. \tag{33}$$

After calculation of (13) and (14) we derive continuous time approximation of stochastic difference equation (29) in a form of diffusion process satisfying stochastic Ito differential equation

$$dS(t) = S(t)(\mu + \sigma^2 k)dt + S(t)\sqrt{1+2k}\sigma dw(t), \tag{34}$$

where

$$k := \sum_{m=1}^{\infty} Corr\{y_{t+m}, y_t\} = \frac{\rho}{1-\rho}. \tag{35}$$

Here we can introduce σ^2_{eff} as an effective volatility of the diffusion process (34)

$$\sigma^2_{eff} = \sigma^2(1+2k). \tag{36}$$

From Figure 1 we can observe that this effective volatility can be substantially greater then σ^2 if serial correlation is positive and converges to 0 as correlation approaches -1.

After substitution of (35) into (34) we get the final equation

$$dS(t) = S(t)(\mu + \sigma^2 \frac{\rho}{1-\rho})dt + S(t)\sqrt{\frac{1+\rho}{1-\rho}}\sigma dw(t). \qquad (37)$$

3 Option Pricing on Stocks with Autocorrelated Returns

Now let's derive European call option pricing formulas if underlying stock's price process $S(t)$ satisfies the stochastic differential equation (34). The boundary conditions for the European call option is given by (19) and (20). Using well known techniques we get the following results

$$C(S(t),t) = S(t)N(d_1) - K\exp(-(\mu + \sigma^2 k)(T-t))N(d_2), \qquad (38)$$

where

$$d_1 = \frac{\log(S(t)/K) + (\mu + \sigma^2 k + \frac{1}{2}\sigma^2(1+2k))(T-t)}{\sigma\sqrt{(1+2k)(T-t)}}, \qquad (39)$$

and

$$d_2 = d_1 - \sigma\sqrt{(1+2k)(T-t)}, \qquad (40)$$

where $N()$ is the standard normal cumulative distribution function.

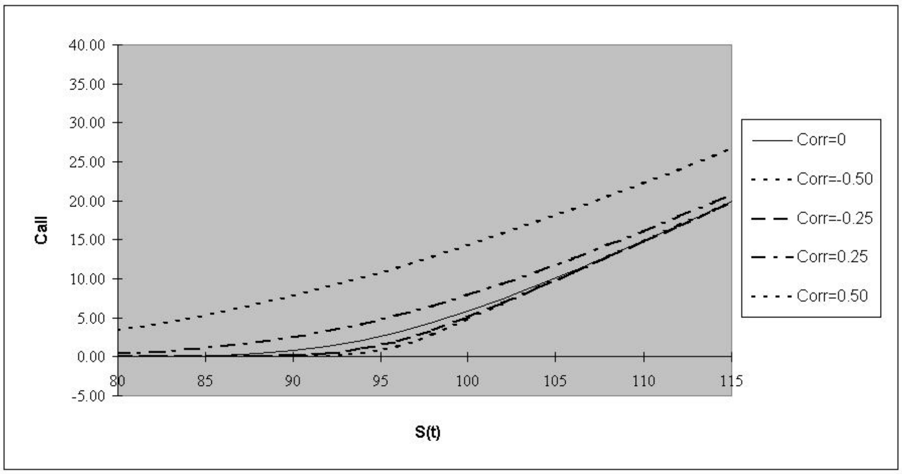

Figure 2: Value of Call Option for Different Correlation Coefficients ρ

Now we are ready to derive formulas used to calculate sensitivities of call option price to changes in underlying parameters.

Delta, Δ, the first derivative of the value C of the option with respect to the underlying instrument's price S will have the same form as in (24):

$$\Delta(S(t),t) = \frac{\partial C}{\partial S} = N(d_1). \qquad (41)$$

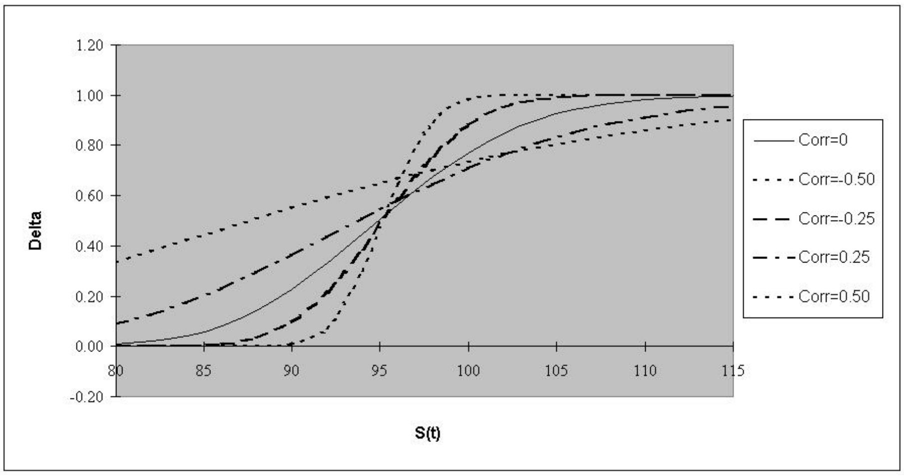

Figure 3: Value of Call Option's Delta for Different Correlation Coefficients ρ

Theta,Θ, the sensitivity of the value of the derivative to the passage of time t now will have the following form

$$\Theta(S(t),t) = \frac{\partial C}{\partial t} = -\frac{S(t)N(d_1)\sigma\sqrt{1+2k}}{2\sqrt{T-t}} - (\mu+\sigma^2 k)K\exp(-(\mu+\sigma^2 k)(T-t))N(d_2) \quad (42)$$

Vega, ν, the sensitivity to volatility σ will be

$$\nu(S(t),t) = \frac{\partial C}{\partial \sigma} = S(t)N(d_1)\sqrt{(1+2k)(T-t)} \quad (43)$$

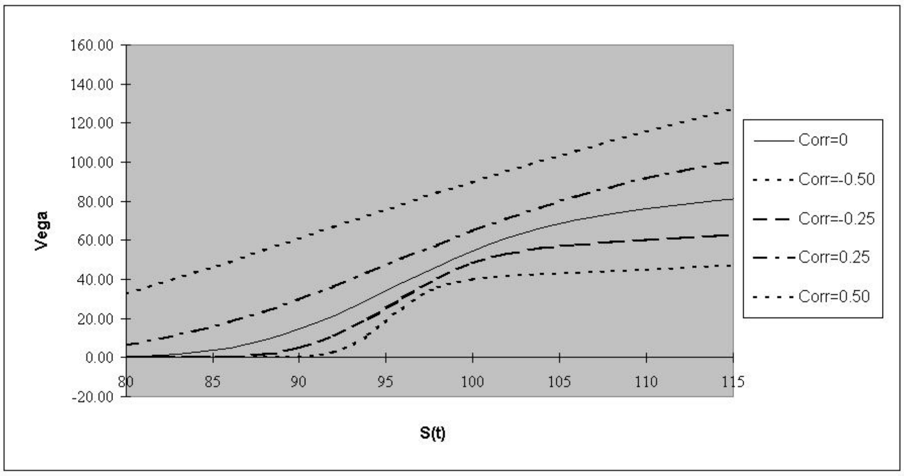

Figure 4: Value of Call Option's Vega for Different Correlation Coefficients ρ

Rho, $\mathcal{R}$, the sensitivity to the applicable interest rate

$$\mathcal{R}(S(t),t) \equiv \frac{\partial C}{\partial \mu} = K(T-t)\exp(-(\mu+\sigma^2 k)(T-t))N(d_2) \quad (44)$$

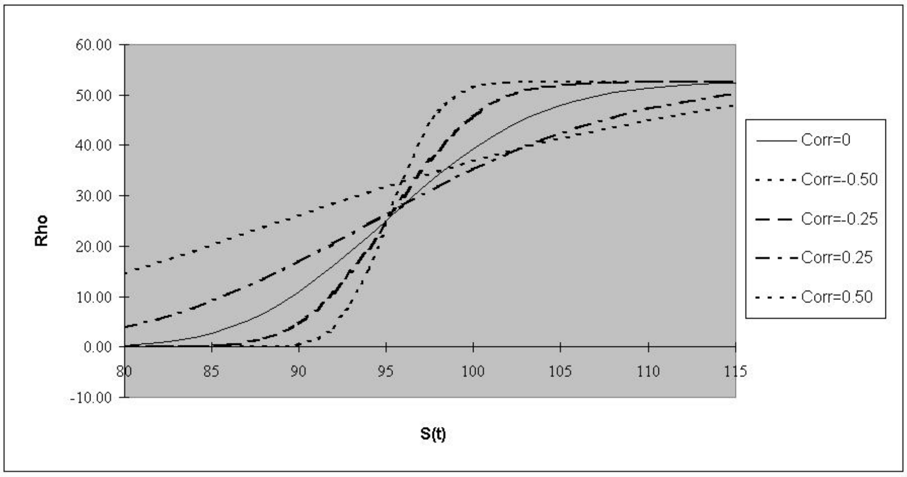

Figure 5: Value of Call Option's Rho for Different Correlation Coefficients ρ

Gamma, Γ, that measures the rate of change in the delta with respect to changes in the underlying asset price will be

$$\gamma(S(t), t) = \frac{\partial \Delta}{\partial S} = \frac{N(d_1)}{S(t)\sigma\sqrt{(1+2k)(T-t)}} \tag{45}$$

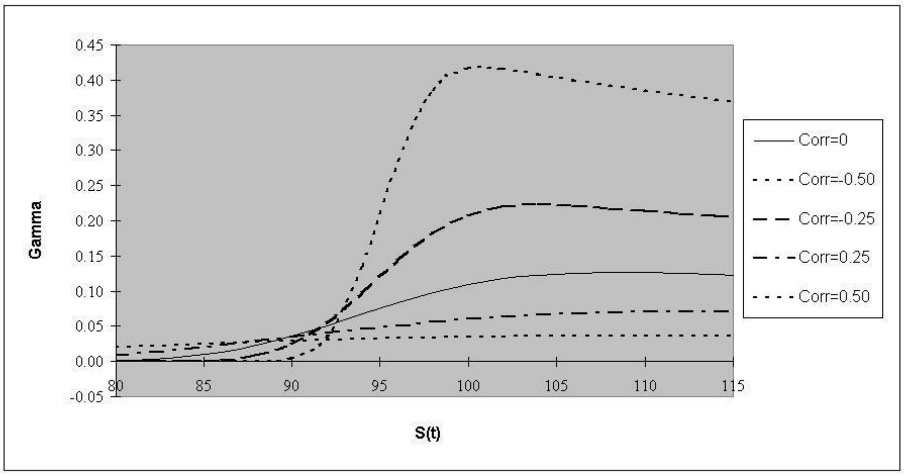

Figure 6: Value of Call Option's Gamma for Different Correlation Coefficients ρ

4 Conclusions and Further Research

In our paper we have derived an analytical formulas for calculation of the value of European call option and its sensitivities to underlying parameters. The formulas demonstrate an important relationship between the value of the option, its risk parameters and underlying asset return autocorrelation coefficient. We have been able to show that autocorrelation can have substantial impact on obtained results and it should be considered when someone tries to estimate correct historical volatility of the underlying return process. In a case of no autocorrelation, our result reduces to a well known Black-Sholes option pricing formula.

The approach used in our paper can be further applied to discrete time stochastic difference equation systems where volatility is stochastic or it is modeled by generalized autoregressive conditional heteroscedasticity process.

Another topic which deserves further analysis is how the convergence of discrete time stochastic difference equation to its continuous time approximation depends on autocorrelation coefficient.

References

[1] LO, A., MACKINLEY, A. C.: *An Econometric Analysis of Non-Synchronous Trading.* Journal of Econometrics, No. 45, pp. 181-211, 1990.

[2] JOKIVUOLLE, E.: *Measuring True Stock Index Value in the Presence of Infrequent Trading.* Journal of Financial and Quantitative Analysis, September, 1995.

[3] STOLL, H. R., WHALEY, R. E.: *The Dynamics of Stock Index and Stock Index Futures Returns.* Journal of Financial and Quantitative Analysis, No. 25, pp. 441-468, 1990.

[4] COUNRAD, J., KAUL, G.: *Time-variation in Expected Returns.* Journal of Business, No. 61, pp. 409-425, 1988.

[5] FAMA, E. F., FRENCH, K. R.: *Permanent and Temporary Components of Stock Market Prices.* Journal of Political Economy, No. 96, pp. 246-273, 1988.

[6] BLACK, F., SCHOLES, M.: *The pricing of options and corporate liabilities.* Journal of Political Economy, No. 81, pp. 637-659, 1973.

[7] MERTON, R.: *Theory of Rational Option Pricing* . Bell Journal of Economics, No. 4, pp. 141-183, 1973.

[8] CARKOVS, J.: *On Diffusion Approximation of Discrete Markov Dynamical Systems.* In Computational Geometry, Proceedings of World Academy of Science, Engineering and Technology, Vol. 30, July, pp. 1-6, 2008.

[9] HAUG, E. D.: *The Complete Guide to Option Pricing Formulas.* McGraw-Hill Professional, 2007.

[10] NELSON, D. B.: *ARCH models as diffusion approximations.* Journal of Econometrics, Vol. 45, Issues 1-2, Pages 7-38, July-August 1990

Current address

Aigars Egle, PhD st.
Riga Technical University, Kaļķu 1, LV-1050, Riga, LATVIA, phone: +371 67010847,
e-mail: aigars.egle@citadele.lv