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## RELIABILITY OF AIRCRAFT AND AIRLINE

***Yu. Paramonov, M. Hauka***

*Aviation Institute, Riga Technical University*  
*Lomonosova 1, Riga LV 1019, Latvia*  
*Tel.: +371 67255394; fax: +371 67089990*  
*E-mail yuri.paramonov@gmail.com, Maris.Hauka@gmail.com*

The reliability of aircraft (AC) and airline (AL) operation can be ensured by the implementation of a specific inspection program, which can be planned using full-scale fatigue test data and the theory of Markov Chains (MC) and a Semi-Markov process (SMP) with rewards. The process of the operation of aircraft is considered as absorbing MC with  $(n+4)$  states. The states  $E_1, E_2, \dots, E_{n+1}$  correspond to AC operation in time intervals  $[t_0, t_1), [t_1, t_2), \dots, [t_n, t_{SL})$ , where  $n$  is an inspection number,  $t_{SL}$  is specified life (SL), i.e. AC retirement time. States  $E_{n+2}$ ,  $E_{n+3}$ , and  $E_{n+4}$  are absorbing states: AC is withdrawn from service when the SL is reached or fatigue failure (FF) or fatigue crack detection (CD) takes place. In the corresponding matrix for the operation processes of AL the states  $E_{n+2}$ ,  $E_{n+3}$  and  $E_{n+4}$  are not absorbing, but correspond to the return of the MC to state  $E_1$  (AL operation returns to first interval). The problem of inspection planning is the choice of the sequence  $\{t_1, t_2, \dots, t_n, t_{SL}\}$  (in case of equal inspection intervals and fixed  $t_{SL}$ , the choice of  $n$ ) corresponding to the maximum of airline gain taking into account the limitations imposed by the intensity of AL fatigue failure (or AC failure probability).

This paper considers the case when the parameters of the fatigue crack growth exponential model are known. Numerical examples are given.

**Keywords:** inspection program, Markov chains, reliability

### 1. Introduction

This paper is devoted to the problem of the reliability of airlines and this is a development of previous publications [1-6] where the problem of the reliability of fatigue-prone aircraft has been considered. The reliability of aircraft (AC) and airline (AL) operations can be ensured by the implementation of a specific inspection program, which can be planned using full-scale fatigue test data and the theory of Markov Chains (MC) and a Semi-Markov process (SMP). The process of the operation of AC is considered as absorbing MC with  $(n+4)$  states. The states  $E_1, E_2, \dots, E_{n+1}$  correspond to AC operation in time intervals  $[t_0, t_1), [t_1, t_2), \dots, [t_n, t_{SL})$ , where  $n$  is an inspection number,  $t_{SL}$  is specified life (SL), i. e. AC retirement time. States  $E_{n+2}$ ,  $E_{n+3}$ , and  $E_{n+4}$  are absorbing states: AC is withdrawn from service when the SL is being reached or fatigue failure (FF) or fatigue crack detection (CD) takes place. In the corresponding matrix for the operation process of AL the states  $E_{n+2}$ ,  $E_{n+3}$  and  $E_{n+4}$  are not absorbing, but correspond to the return of the MC to state  $E_1$  (AL operation returns to first interval). In the matrix of transition probabilities of AC,  $P_{AC}$ , there are three units in the three last lines in the diagonal (bias), but for corresponding lines in the matrix for AL,  $P_{AL}$ , the units are in the first column, corresponding to state  $E_1$ . Using  $P_{AC}$  we can get the probability of FF of AC and cumulative distribution function, mean and variance of AC life, and the same characteristics under the condition of absorption in a specific absorbing state. Using  $P_{AL}$  we can get the stationary probabilities of AL operation  $\{\pi_1, \dots, \pi_{n+1}, \pi_{n+2}, \dots, \pi_{n+4}\}$ . Here  $\pi_{n+3}$  defines the part of MC steps when FF takes place and the MC appears in state  $E_{n+3}$ . The ratio of this value to the mean life of a new aircraft defines the intensity of FF,  $\lambda_F$ , i.e. the number of FF in one time unit. It can be calculated also using the theory of Semi-Markov Process (SMP) with rewards [7, 8]. Using this theory we can calculate also the gain of the process. The problem of inspection planning is the choice of the sequence  $\{t_1, t_2, \dots, t_n, t_{SL}\}$  (in the case of equal inspection intervals and fixed  $t_{SL}$ , choice of  $n$ ) corresponding to the maximum gain, taking into account the limitations imposed by the intensity of AL fatigue failure of AL or AC fatigue failure. A numerical example is given.

## 2. Formal Setting of the Problem

For the formal setting of the problem we should define the matrix of transition probabilities of MC and the matrix of rewards for SMP. However, first the fatigue crack growth model should be defined.

### 2.1. Fatigue crack growth model

We suppose the following exponential approximation of fatigue crack growth function when the size of a fatigue crack is defined by the equation:

$$a(t) = a(0)\exp(Qt) . \quad (1)$$

$T_d$  represents the time when a crack reaches detectable size,  $a_d$  is when the probability of its discovery is equal to a unit if inspection is made (before this size this probability is equal to zero).  $T_c$  is the time when the crack reaches its critical size,  $a_c$ , and FF takes place. The value of  $a_c$  corresponds to the minimum residual strength of an aircraft component allowed by special design regulations. We have

$$T_d = (\ln a_d - \ln a_0) / Q = C_d / Q , \quad (2)$$

$$T_c = (\ln a_c - \ln a_0) / Q = C_c / Q , \quad (3)$$

where  $a_0$  is  $a(0)$ . Despite its simplicity, the formula (1) in interval  $[T_d, T_c)$  shows clear results.

The values  $C_c$  and  $C_d$  can be derived each from another.

$$C_d = C_c - \delta , \quad (4)$$

$$C_c = C_d + \delta , \quad (5)$$

$$\delta = \ln a_c - \ln a_d = \ln \frac{a_c}{a_d} . \quad (6)$$

If  $\delta$  and  $a_0$  are constants then actually considering the model of fatigue crack growth, the distribution of random variables (r.v.)  $T_d, T_c$  is defined by distribution of only two r.v.:  $C_c$  and  $Q$ . Let us denote

$$X = \ln Q , Y = \ln C_c = \ln(\ln(a_c/a_d)) . \quad (7)$$

From the analysis of the fatigue test data it can be assumed, that r.v.  $\ln T_c$ , is distributed normally. It comes from the additive property of the normal distribution that  $\ln T_c$  could be normally distributed if both  $\ln C_c$  and  $\ln Q$  are normally distributed or if one of these value is constant. Here we consider the simplest case. We suppose that random variable  $X$  has normal distributions with unknown mean value  $\theta_0$  and known  $\theta_1$  but  $Y$  is constant.

### 2.2. Failure of AC probability calculation using the theory of absorbing Markov chains

Let the probability of crack detection during the inspection number  $i$  at time moment  $t_i$  be denoted as  $v_i$ ; the probability of failure in service time interval  $(t_{i-1}, t_i)$  as  $q_i$ ; and the probability of absence of the above-mentioned events as  $u_i$ . These three events form a complete set  $u_i + v_i + q_i = 1$ . In our model we also assume that there is an inspection at  $t_{SL}$ . This inspection at the end of  $(n+1)$ -th interval does not change the reliability, but it should be done in order to know the state of aircraft (to know if there is any FC or if there are no fatigue cracks). *Effective number of inspections* is equal to  $n$  but in following, for short, we say that  $n$  is *number of inspections*. The transition probability matrix of this process,  $P_{AC}$ , can be composed as presented on Figure 1.

	$E_1$	$E_2$	$E_3$	...	$E_{n-1}$	$E_n$	$E_{n+1}$	$E_{n+2}$ (SL)	$E_{n+3}$ (FF)	$E_{n+4}$ (CD)
$E_1$	0	$u_1$	0	...	0	0	0	0	$q_1$	$v_1$
$E_2$	0	0	$u_2$	...	0	0	0	0	$q_2$	$v_2$
$E_3$	0	0	0	...	0	0	0	0	$q_3$	$v_3$
...	...	...	...	...	...	...	...	...	...	...
$E_{n-1}$	0	0	0	...	0	$u_{n-1}$	0	0	$q_{n-1}$	$v_{n-1}$
$E_n$	0	0	0	...	0	0	$u_n$	0	$q_n$	$v_n$
$E_{n+1}$	0	0	0	...	0	0	0	$u_{n+1}$	$q_{n+1}$	$v_{n+1}$
$E_{n+2}$ (SL)	0	0	0	...	0	0	0	1	0	0
$E_{n+3}$ (FF)	0	0	0	...	0	0	0	0	1	0
$E_{n+4}$ (CD)	0	0	0	...	0	0	0	0	0	1

Figure 1. The transition probability matrix for absorbing MC

An example of a corresponding state transition diagram for the case of three inspections is shown on Figure 2. We suppose that if, in interval  $(T_d, T_c)$ , some inspection is made, then fatigue failure will be eliminated.

$$u_i = P(T_d > t_i | T_d > t_{i-1}) = P(Q < C_d / t_i) / P(Q < C_d / t_{i-1}); \tag{8}$$

$$q_i = P(t_{i-1} < T_d < T_c < t_i | t_{i-1} < Td) = \begin{cases} 0, & \text{if } t_{i-1}C_c / C_d > t_i, \\ P(C_c / t_i < Q < C_d / t_{i-1}) / P(Q < C_d / t_{i-1}), & \text{if } t_{i-1}C_c / C_d > t_i, \end{cases} \tag{9}$$

$$v_i = 1 - u_i - q_i. \tag{10}$$

Unfortunately, in general the corresponding integrals are not expressed by elementary functions, but if it is assumed that  $C_c$  and  $C_d$  are some constants, then

$$u_i = a_i / a_{i-1}, \tag{11}$$

$$q_i = \max(0, (a_{i-1} - b_i) / (1 - a_{i-1})), \tag{12}$$

$$a_i = \Phi(\ln(C_d / t_i) - \theta_0) / \theta_1, \tag{13}$$

$$b_i = \Phi(\ln(C_c / t_i) - \theta_0) / \theta_1, \tag{14}$$

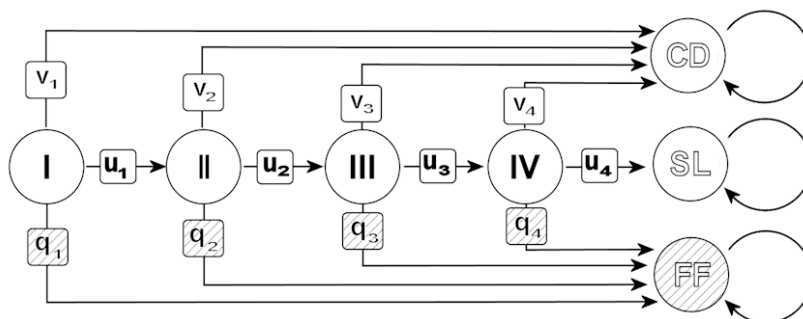


Figure 2. State transition diagram for a three-inspection program

$\Phi(\cdot)$  is a distribution function of a standard normal variable.

The structure of considered matrices can be described as follows on Figure 3.

Q	R
0	I

Figure 3. Sub-matrices of transition probabilities matrix

Here  $I$  is a matrix of identity corresponding to absorbing states,  $0$  is a matrix of zeros. Then the matrix of probabilities of absorbing in different absorbing states for different initial transient states is defined by formula (18)

$$B = (I - Q)^{-1} R. \tag{15}$$

The first row of the matrix  $B$  defines the probabilities of absorption in states SL, FF, CD if the initial state is  $E_1$ . Particularly,  $B(1,2)$  defines the failure probability of a new aircraft which begins operation during the first interval. The following rows of the matrix  $B$  define the same probabilities for different initial states (for aircraft which begin operation in different time intervals). Therefore, if, for example, aircraft begins operation in the first interval, the failure probability of aircraft in the fleet is defined in formula (19)

$$p_f = aBb, \tag{16}$$

where vector row  $a = (1, 0, \dots, 0)$  means that all aircraft begin operation in the first interval (state  $E_1$ ), vector column  $b = (0, 1, 0)'$ .

**2.3. Calculation of the cost of airline operation using theory of stationary semi-Markov chains with rewards**

In order to analyse the cost of permanent operations of AL or aircraft fleets, it is necessary to modify the transition probability matrix.

The process is not absorbed or stopped in states SL, FF or CD. If these states are reached, the process restarts from the in initial state  $E_1$ . This means that an entirely new aircraft is acquired. The modified transition probability matrix,  $P_{AL}$ , is shown on Figure 4. An example of the state transition diagram is shown on Figure 5.

	$E_1$	$E_2$	$E_3$	...	$E_{n-1}$	$E_n$	$E_{n+1}$	$E_{n+2}$ (SL)	$E_{n+3}$ (FF)	$E_{n+4}$ (CD)
$E_1$	0	$u_1$	0	...	0	0	0	0	$q_1$	$v_1$
$E_2$	0	0	$u_2$	...	0	0	0	0	$q_2$	$v_2$
$E_3$	0	0	0	...	0	0	0	0	$q_3$	$v_3$
...	...	...	...	...	...	...	...	...	...	...
$E_{n-1}$	0	0	0	...	0	$u_{n-1}$	0	0	$q_{n-1}$	$v_{n-1}$
$E_n$	0	0	0	...	0	0	$u_n$	0	$q_n$	$v_n$
$E_{n+1}$	0	0	0	...	0	0	0	$u_{n+1}$	$q_{n+1}$	$v_{n+1}$
$E_{n+2}$ (SL)	1	0	0	...	0	0	0	0	0	0
$E_{n+3}$ (FF)	1	0	0	...	0	0	0	0	0	0
$E_{n+4}$ (CD)	1	0	0	...	0	0	0	0	0	0

Figure 4. The transition probability matrix for stationary AL operation process

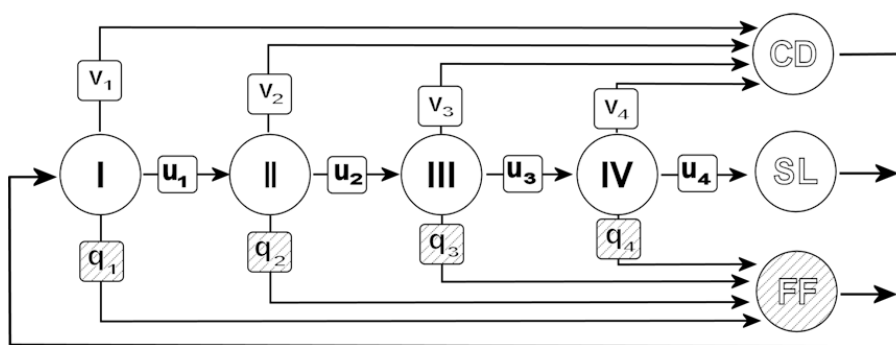


Figure 5. State transition diagram for stationary AL operation process with three inspections

Next we will consider economic analysis. The theory of Semi-Markov process with rewards is usually used to find the solution to similar problems [8]. A reward structure is described by the reward matrix  $R$ , the component of which,  $r_{ij}$ , describes the reward, connected with the transition from state  $i$  to state  $j$ ; here  $i, j = 1, 2, \dots, n+4$ . Let us define the reward, related to successful transitions from one operation interval to the following one by value  $a(n)$ ; the reward related to transitions to state CD (or  $E_{n+4}$ ) from any state  $E_j, \dots, E_{n+1}$  – by value  $b$ , to state FF (or  $E_{n+3}$ ) – by value  $c$ ; and from states SL, FF, CD to state  $E_1$  (acquisition of new AC) – by value  $d$ .

Let us define the gain:

$$g(n) = \sum_{i=1}^{n+4} \pi_i g_i(n), \tag{17}$$

where  $\pi = (\pi_1, \dots, \pi_{n+4})$  is the vector of stationary probabilities, which is defined by the equation system

$$\pi P = \pi, \quad \sum_{i=1}^{n+4} \pi_i = 1; \tag{18}$$

$$g_i(n) = \begin{cases} a(n) \cdot u_i + b \cdot q_i + c \cdot v_i, & i = 1, \dots, n+1, \\ d, & i = n+2, \dots, n+4 \end{cases}; \tag{19}$$

$u_i, q_i, v_i, i = 1, \dots, n+1$ , are probabilities of successful transitions from one to the following interval, to  $E_{n+3}$  and  $E_{n+4}$  states;

$a(n) = a_0(n) + d_i t_{SL}$ ,  $a_0(n) = a_1 t_{SL} / (n+1)$  – is the reward, related to successful transitions from one operation interval to the following one, and the cost of one inspection (negative value),  $d_i$ , which is supposed to be proportional to  $t_{SL}$  if it is supposed that all intervals are equal,  $a_1$  defines the reward of operation in one time unit (one hour or one flight). This dimension should coincide with dimensions of  $t_{SL}$ ;  $b = b_1 a_0$ ,  $c = c_1 a_0$  are the rewards related to transitions from any state  $E_j, \dots, E_{n+1}$  to state  $E_{n+3}$  (FF takes place) and  $E_{n+4}$  (CD takes place) which are supposed to be proportional to  $a_0$ ;  $d = d_1 t_{SL}$  is the negative reward, the value of which is the cost of new aircraft after SL, FF or CD and transition to  $E_1$  takes place (it is supposed to be proportional to  $t_{SL}$ ).

If  $a(n) = b = c = 1$ ,  $d = 0$  and time transition to state  $E_1$  are equal to zero. Then  $\pi_{ij} = \pi_j g_j / g$  defines the part of time which SMP spends in state  $E_j, j = 1, \dots, n+1$ ;  $L_j = g / \pi_j$  defines the mean return time for state  $E_j$ ; specifically,  $L_1$  is the mean time of renewal of AL operation in the first interval,  $L_{n+3}$  is the mean time between FF;  $\lambda_F = 1 / L_{n+3}$  is the intensity of fatigue failure. It should be remembered that the same value can be obtained using the theory of absorbing MC. This value is equal also to the ratio of aircraft failure probability to the mean life of new aircraft.

The problem is to maximise gain  $g$  under limitations of intensity of fatigue failure  $\lambda_F$ .

### 3. Choice of Inspection Number. Numerical Example

To simplify the numerical example we suppose that all intervals of operation are equal. On Figures 6 and 7 we see numerical examples of calculations of gain,  $g$ , and of intensity of fatigue failure,  $\lambda_F$ , as functions of  $n$ .

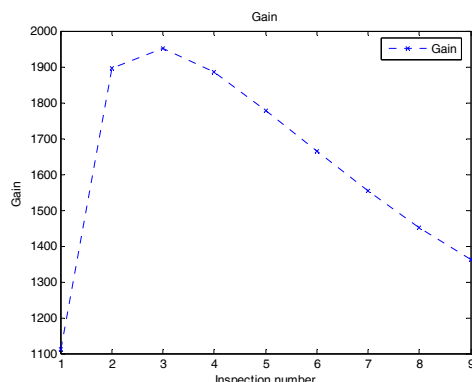


Figure 6. Gain as function of inspection number

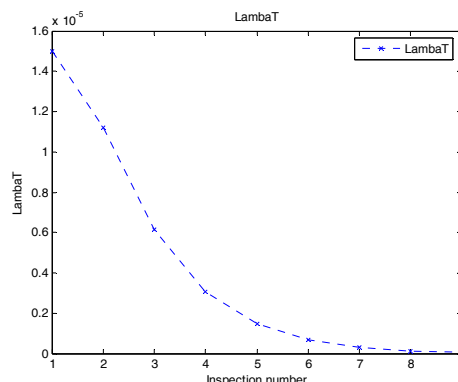


Figure 7. AL intensity of AC fatigue failure as function of inspection number

First we should choose the number of  $n$ , corresponding to the maximum of gain  $g$ :  $n_g = \arg \max g(n)$ . Then we should choose the minimum value of  $n$  under condition; that of intensity of fatigue failure  $\lambda_F$ , which is a function of  $n$ ,  $\lambda_F(n)$ , will be equal or less than some design allowed value of intensity of fatigue failure,  $\lambda_F^*$ :  $n_\lambda = \min \{n : \lambda_F(n) \leq \lambda_F^*, \text{ for all } n \geq n_\lambda\}$ .

And finally we should choose  $n = \max(n_g, n_\lambda)$ .

Consider the numerical example of calculation of gain,  $g$ , and intensity of fatigue failure,  $\lambda_F$ , for  $n=2$  and following initial data:  $t_{SL}=40\,000$ ,  $\theta_0 = -8.5885$ ,  $\theta_1 = 0,34600$ ,  $a_c = 102$ ,  $a_d = 20$  (these values have been used already, for example, in [1]);  $a_1 = 1$ ,  $b_1 = 0.05$ ,  $c_1 = -0.05$ ,  $d_1 = -0.3$ ,  $d_i = -0.0001$ .

Using equation (10-14) we have (using the MATLAB notation for the matrix)

$$P_{AC} = \begin{bmatrix} 0 & 9.3958e-001 & 0 & 0 & 6.3833e-003 & 5.4037e-002 \\ 0 & 0 & 3.4655e-001 & 0 & 2.6900e-001 & 3.8445e-001 \\ 0 & 0 & 0 & 1.6030e-001 & 2.4240e-001 & 5.9730e-001 \\ 0 & 0 & 0 & 1.0000e+000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000e+000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000e+000 \end{bmatrix}$$

In matrix  $P_{AL}$  the first three lines are the same, but in the last three lines the units are in the first column.

Using  $P_{AL}$  and equations (18) we have a stationary distribution

$$\pi = [3.06e-001, 2.88e-001, 9.97e-002, 1.60e-002, 1.04e-001, 1.87e-001].$$

Let us define the matrix  $P_{uqv}$  with  $u_i, q_i, v_i$  in every line for  $i = 1, \dots, n+1$ . In the example

$$P_{uqv} = \begin{bmatrix} 9.3958e-001 & 6.3833e-003 & 5.4037e-002 \\ 3.4655e-001 & 2.6900e-001 & 3.8445e-001 \\ 1.6030e-001 & 2.4240e-001 & 5.9730e-001 \end{bmatrix}$$

Then  $g = \pi [P_{uqv} [a; b; c]; d; d; d]$ . In considered case  $g = 1\,896$ .

If the time unit is interval then the vector of return times for every state

$$L = [g_t / \pi_1, \dots, g_t / \pi_6] = [2.27, 2.41, 6.96, 4.34, 6.70, 3.72].$$

Here  $g_t = \pi[1;1;1;0;0;0]$  is the mean gain if income is time and the time unit is interval. It is useful to note that  $g_t = g$  if  $a = b = c = 1$ ,  $d = 0$ . In considered case  $g_t = 0.694$ .

The return time of failure (state FF) is equal to corresponding  $(n+3)$ -th component of vector  $L$ :  $L_F = L(n+3) = 6.7$ . The time length of one interval  $t_1 = t_{SL} / (n+1) = 13.333$ .

The intensity of failure in t-time unit (flight or flight hour)  $\lambda_F = (1 / L_F) / t_1 = 0.0000112$ .

It is useful to note that the same values can be obtained using the theory of absorbing MC. We calculate the matrix of absorption

$$B = \begin{bmatrix} 5.2196e-002 & 3.3806e-001 & 6.0975e-001 \\ 5.5552e-002 & 3.5300e-001 & 5.9145e-001 \\ 1.6030e-001 & 2.4240e-001 & 5.9730e-001 \end{bmatrix}.$$

Then we take from it the probability of absorption of new aircraft in state FF,  $p_f = 0.338$ , calculate the mean time to absorption of new aircraft,  $T_1 = \tau_1 t_1$  (where  $\tau_1 = 2.27$  is the mean time to absorption if the time unit is interval-which coincides with  $L(1)!$ ) and calculate the ratio  $p_f / T_1 = 0.0000112$ .

On Figures 6 and 7 we see similar calculations for  $n=1, \dots, 9$ . And we see that the maximum of  $g$  we have for  $n=3$ . We can choose this number of inspections if allowed failure intensity is higher or equal to (approximately) 0.000006, because  $\lambda_F$  is equal just to this value at this number of inspections.

#### 4. Conclusions

This paper presents (based on MC and SMP with rewards theory) a method to solve the problem of inspection planning corresponding to maximum airline gain while respecting limitations due to the intensity of fatigue failure. In this paper the case is considered when parameters of fatigue crack growth exponential models are known. If a parameter is unknown, the minimax approach (see [1]) should be used. Numerical examples are given.

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