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PROJECTOR SYSTEMS

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Systems of evolution differential equations are applied in various problems of imitation modelling. But any computational solution needs the analytical evaluations. In this connection the projector technique is very effective. The solution of the system of the nonlinear differential equations considerably can be simplified, if the system is transformed to Jordan form, i.e. to such type, when the system consists of blocks of independent system of functions.

Keywords: *nonlinear differential equations, projector technique, Jordan matrix*

1. Introduction

Both in the theory and in many practical questions exclusively important role is played with problems of the decision of systems of the ordinary equations. One of such problems consists in splitting the given system of the equations on blocks, i.e. in such subsystems, each of which contains smaller number of the unknown functions entering simultaneously and under a sign of a derivative, and in the right parts.

Abundantly clear, that integration of such subsystems is a simpler problem, than integration of initial system. Therefore there is a question on a finding of ways of transformation of such systems to the split kind.

Such problem has been solved for a case of linear systems of the differential equations. It has appeared that splitting of such systems to equivalently reduction of a square matrix to Jordan to the form. To each cell Jordan there corresponds the certain block of the split system.

One of such ways of splitting has been offered to K. G. Valeev [1]. It has been entered into consideration special matrixes – linear projectors [2]. Also the iterative way of the approached finding of two supplementing each other up to nonsingular matrixes of projectors from factors of the right parts of the set system of the equations has been found.

That it is available there has been a way approached dichotomy systems. The given way easily enough gave in to machining. However, the geometry of occurring processes at such splitting remained accomplished obscure. Obscure there was also a fact of transferring of such splitting on nonlinear systems.

2. The Projector of the Ordinary Differential Equations

Let the system of the linear differential equations with constant factors (1) is given

$$\frac{dy^i}{dt} = a_j^i \, y^j \,. \tag{1}$$

Having found two projectors of a matrix, system (1) it is possible to split on two blocks then to each of the received blocks to apply the same operation (dichotomy). The further splitting will appear impossible as soon as the spectrum of any matrix will be reduced to one own to number [3].

Let the splitting transformation, leading system to a block kind, translates all Euclid space E_n of dimension n which coordinates of points are values of unknown functions of system (1) in some-dimensional plane which is passing through the beginning of coordinates. Designing occur some projecting plane E_{n-k} .

The equations of such plane are easy for receiving from the formulas defining splitting of system (1). It has appeared, that each projector generates in Euclid space E_n system of the projecting planes forming projecting network of carry at which it is easy to define the first square-law form.

If splitting of system (1) to carry out by means of linear transformations, that each of such transformations is broken into two singular transformations and generates singular matrixes which are projectors of system (1). It is natural to set the task of transferring these statements on a nonlinear case.

By analogy to a linear case it can be made as follows. It has been noticed, that elements of linear projectors are private derivatives of new coordinates on old. We shall consider nonlinear independent system of the differential equations

$$\frac{dy^i}{dt} = f^i(y^j), \quad f^i(0,...,0) = 0. \tag{2}$$

Let's consider nonlinear transformations:

$$y^i = \sum_{k=1}^{n} \Psi_k{}^i(z^k), \quad \Psi_i^i(z^i) = z^i, \quad \Psi_k{}^i = 0 . \tag{3}$$

Making matrixes (already functional) which elements are private derivatives of new coordinates on old and demanding that these matrixes satisfied to conditions of projectors, we shall come to system of the differential equations in private derivatives of the first order on function costing in the right parts of equality (2).

Let's make it so. The split system is presented by the equations.

$$\frac{dz^i}{dt} = \mathrm{X}^i(z^j) \quad \mathrm{X}^i(0) = 0 . \tag{4}$$

Differentiating (3) on and considering (4), we receive a general view of functions f^i in variables z^j:

$$f^i = \alpha_k{}^i \mathrm{X}^k, \quad \alpha^i{}_k = \Psi''^i{}_k(z^k), \quad \alpha^I{}_I = 1. \tag{5}$$

Equality (5) shows that functions f^i depend on n^2 functions of one argument $(n-1)n$ – functions (have considered equality) and functions. Having solved the equations (3) rather and having brought the found values in the right parts of equality (5). Let's find a general view of functions in variables. Differentiating (5) on and accepting considering that according to (4), we shall receive the following

$$\frac{\partial y^i}{\partial z^k} = \alpha_k{}^i ,$$

let's receive

$$\frac{\partial f^i}{\partial y^j} \alpha^j{}_k = \alpha'_k{}^i \mathrm{X}^k + \alpha^i{}_k \mathrm{X}'^k . \tag{6}$$

We differentiate (7) on (in this case $\alpha^j{}_k, \alpha'_k{}^i, \mathrm{X}^k, \mathrm{X}'^k$ behave as constants):

$$\frac{\partial^2 f^i}{\partial y^j \partial y^l} \alpha^j{}_k \alpha^l{}_p = 0 . \tag{7}$$

Let $\dfrac{\partial^2 f^i}{\partial y^j \partial y^l} = a^i{}_{jl} = a^i{}_{lj}$.

At fixed y^i the equations $a^i{}_{jl} x^j x^l = 0$ define n hyper-quadric in $n(n-1)$ measured projective space P_{n-1}. Equation (7), i.e. $a^i{}_{jl} \alpha^j{}_k \alpha^l{}_p = 0$ means a polar interlinking of points $M_k(\alpha^j{}_k), M_p(\alpha^l{}_p)$ concerning mentioned hyper-quadric. To find the solution of system (7) is means to find the general auto-polar reference point concerning data hyper-quadric.

This reference point should be non-degenerative, as it is offered transformation (4), $\left| \dfrac{\partial y^i}{\partial z^k} \right| = \left| \alpha^i{}_k \right| \neq 0$. If to carry space P_{n-1} to this reference point all the equations hyper-quadric will accept an initial kind $\alpha'_{ii}{}^i (x'^j) = 0$.

Hence, the problem (the decision of the equations (7)) is reduced to simultaneous reduction of n square-law forms $a^i{}_{jl} x^j x^l$ an initial kind. It is known, however, that this problem is possible only for two square-law forms, and one of these two forms should be positively certain. Let the form $\varphi^1 = a_{jl}{}^1 x^j x^l$ is positively certain (we can assume, that those it is in a point (0,0..., 0). Then by virtue of a prospective continuity of functions $\dfrac{\partial^2 f^i}{\partial y^i \partial y^l}$ it will be positively certain and in a vicinity of the specified point).

Let's take any second form, for example $\varphi^2 = a_{jl}{}^2 x^j x^l$. Let $\alpha_1{}^i$ – one of tops of a required general auto-polar reference point hyper-quadric $\varphi^1 = 0, \quad \varphi^2 = 0.$ Then polar hyper-planes of this point concerning specified hyper-quadric $a_{jl}{}^1 \alpha_1{}^j x^l = 0, \quad a_{jl}{}^2 \alpha_1{}^j x^l = 0$ should coincide

$$\frac{a_{jl}{}^1 \alpha_1{}^j}{a_{j2}{}^2 \alpha_1{}^j} = \frac{a_{j2}{}^1 \alpha_1{}^j}{a_{j2}{}^2 \alpha_1{}^j} = \ldots = \frac{a_{jn}{}^1 \alpha_1{}^j}{a_{jn}{}^2 \alpha_1{}^j} = S .$$

(Through the S size of the general attitude) is designated. From here $(a_{jl}{}^1 - s a_{jl}{}^2)\alpha_1{}^j = 0$. As all $\alpha_1{}^j$ simultaneously are not equalled to zero,

$$\left| a_{jl}{}^1 - S a_{jl}{}^2 \right| = 0 . \tag{8}$$

Let's name this equation the characterize equation of two linear operators $x'_j = a_{jl}{}^1 x^l, \quad x'' = a_{jl}{}^2 x^l$. If $x'_j = S x''_j$, x^l refers to as a double vector of linear operators. For definition of double operators we have system of the linear equations

$$(a_{jl}{}^1 - S a_{jl}{}^2)\, x^l = 0. \tag{9}$$

From here, excepting from consideration zero-vectors, we come to the characteristic equation (8). Generally set of such vectors forms so-called Jacoby variety of two hyper-quadric $\varphi^1 = \varphi^2 = 0$. In a case considered by us Jacoby the variety represents - (reference point). Obviously, each simple S_i root of the characteristic equation (8) there corresponds only one point $\alpha^l{}_i$ defined by the equations $(a^1{}_{jl} - S_i a^2{}_{jl})\alpha_i{}^l = 0$.

Assuming, that all roots of the characteristic equation are various, we shall receive a unique general auto-polar reference point with tops $\alpha_p{}^l$ concerning two quadric $\alpha_p{}^l = \alpha_p{}^l(y^j)$. At presence of multiple roots of the characteristic equation last decision will be not the only thing.

In case of when any of square-law forms $a_{jl}{}^i x^j x^l$ is not positively certain, presence of the general auto-polar reference point at them also is possible, but at performance of some additional conditions. Let (13) – some general reference point of hyper-surfaces $\varphi^1 = \varphi^2 = 0$ (unique if roots of the characteristic equation are various). As coordinates $\alpha_p{}^l$ it turns out to within the general multipliers we shall choose these multipliers so that $\alpha_i{}^i = 1$.

Each of points $\alpha_p{}^l$ should depend only from z^p that is why differentiation $\alpha_p{}^l = \alpha_p{}^l(y^j)$ from z^q on gives a zero:

$$\frac{\partial \alpha_p{}^l}{\partial y^j} \alpha_q{}^j = 0. \tag{10}$$

To such conditions should satisfy functions f^i that the system (1) supposed projector splitting. However these conditions generally are not sufficient. Additional conditions we shall find from (6). Let $\left(\beta_i{}^k\right)$ – a matrix, return to a matrix $\left(\alpha_k{}^i\right)$: $\beta_l{}^i \alpha_i{}^k = \delta_l{}^k, \quad (\beta_l{}^i = \frac{\partial z^i}{\partial y^l})$.

Then from (6) it is received

$$\mathrm{X}^k = \beta_i{}^k f^i .$$

Replacing here an index on an index and differentiating on z^q $(q \neq p)$, we shall receive

$$\left(\frac{\partial\beta_i{}^p}{\partial y^l}+\beta_i{}^p\frac{\partial f^i}{\partial y^l}\right)\alpha_q{}^l=0. \tag{11}$$

This equation should be attached to equation (10).

Theorem 1. Let the system of the differential equations is given

$$\frac{dy^i}{dt}=f^i(y^j)\ . \tag{12}$$

Let for functions conditions (10) and (11) where $\alpha_k{}^i$ are under formulas (5), and $(\beta_i{}^k)$ – a matrix, return to a matrix $(\alpha_k{}^i)$ are satisfied. Then splitting of system (12) can be found by means of quadratures.

The proof.

Really

$$\alpha_p{}^l=\psi'_p{}^l=\frac{\partial\psi_p{}^l}{\partial y^i}\alpha_p{}^i,\quad (\psi'_p{}^l=\frac{d\psi_p{}^l}{dz^q}), \tag{13}$$

$$0=\frac{d\psi_p{}^l}{dz^q}=\frac{\partial\psi_p{}^l}{\partial y^i}\alpha_q{}^i,\quad (\frac{d\psi_p{}^l}{dz^q}=0,\ p\neq q)\ .$$

They are the n algebraic equations from unknown $\dfrac{\partial\psi_p{}^l}{\partial y^i}$ $(p,l$ – are fixed). The system can be written down in the form of:

$$\alpha_q{}^i\frac{\partial\psi_p{}^l}{\partial y^i}=\delta_p{}^q\alpha_q{}^l\ . \tag{14}$$

Then, $\dfrac{\partial\psi_p{}^l}{\partial y^i}=\delta_p{}^q\alpha_q{}^l\beta_i{}^q$,

or

$$\frac{\partial\psi_p{}^l}{\partial y^i}=\alpha_p{}^l\beta_i{}^p\ . \tag{15}$$

It is easy to see, that the system (21) is quite integrated.

Really

$$I\equiv\frac{\partial^2\psi_p{}^l}{\partial y^i\partial y^j}-\frac{\partial^2\psi_p{}^l}{\partial y^j\partial y^i}=\frac{\partial\alpha_p{}^l}{\partial y^j}\beta_i{}^p+\alpha_p{}^l\frac{\partial\beta_i{}^p}{\partial y^j}-\frac{\partial\alpha_p{}^l}{\partial y^i}\beta_j{}^p-\alpha_p{}^l\frac{\partial\beta_j{}^p}{\partial y^i}\ . \tag{16}$$

Differentiating (10), we receive $\dfrac{\partial\beta_l{}^i}{\partial y^i}\alpha_i{}^k+\beta_l{}^i\dfrac{\partial\alpha_i{}^k}{\partial y^j}+0$, whence $\dfrac{\partial\beta_l{}^i}{\partial y^j}=-\beta_k{}^i\beta_l{}^h\dfrac{\partial\alpha_h{}^k}{\partial y^j}$.

Let's bring it in (16):

$$I\equiv\frac{\partial\alpha_p{}^l}{\partial y^j}\beta_i{}^p-\alpha_p{}^l\beta_k{}^p\beta_i{}^h\frac{\partial\alpha_h{}^k}{\partial y^j}-\frac{\partial\alpha_p{}^l}{\partial y^i}\beta_j{}^p+\alpha_p{}^l\beta_k{}^p\beta_j{}^h\frac{\partial\alpha_h{}^k}{\partial y^i}. \tag{17}$$

Let's add to equality (10) equality

$$\frac{\partial\alpha_p{}^l}{\partial y^j}\alpha_p{}^j=\frac{d\alpha_p{}^l}{dz^p}=\alpha'_p{}^l\ . \tag{18}$$

Equations (10) and (18) can be united $\alpha_q{}^j \dfrac{\partial \alpha_p{}^l}{\partial y^j} = \delta_p{}^q \alpha'_p{}^l$.

From here

$$\frac{\partial \alpha_p{}^l}{\partial y^j} = \delta_p{}^q \alpha'_q{}^l \beta_j{}^q = \alpha'_p{}^l \beta_j{}^p . \tag{19}$$

Let's bring (19) in (17):

$$I \equiv \alpha'_p{}^l \beta_j{}^p \beta_i{}^p - \alpha_p{}^l \beta_k{}^p \beta_i{}^h \alpha'_h{}^k \beta_j{}^h - \alpha'_p{}^l \beta_i{}^p \beta_j{}^p - \alpha_p{}^l \beta_k{}^p \beta_j{}^h \alpha'_h{}^k \ \beta_i{}^h \equiv 0 ,$$

and it also proves full that the system (21) is quite integrated systems (21). The right parts of this system depend only from $y^1,..., y^n$ and do not depend from $\psi_p{}^l$ the decision can be found by means of quadratures. According to usual procedure we shall take the equation $\dfrac{\partial \psi}{\partial y^1} = a_1$. Integrating, we receive

$$\psi = \int_0^{y^1} a_1 dy^1 + A_2(y^2,..., y^n) , \tag{20}$$

where A_2 is while unknown function of the specified arguments. We differentiate (20) on y^2, having considered, that $\dfrac{\partial \psi}{\partial y^2} = a_2$, $\quad \dfrac{\partial a_1}{\partial y^2} = \dfrac{\partial a_2}{\partial y^1}$,

we receive

$$\frac{\partial A_2}{\partial y^2} = a_2 - \int_0^{y^1} \frac{\partial a_2}{\partial y^1} dy^1 \tag{21}$$

The right part (21) depends only from $y^2,..., y^n$. Integrating, we find

$$A_2 = \int_0^{y^2} \left[a_2 - \int_0^{y^1} \frac{\partial a_2}{\partial y^1} dy^1 \right] dy^2 + A_3(y^3,..., y^n) . \tag{22}$$

We differentiate (22) on, having considered, that

$$\frac{\partial A_2}{\partial y^3} = \frac{\partial \psi}{\partial y^3} - \int_0^{y^1} \frac{\partial a_1}{\partial y^3} dy^1 = a_3 - \int_o^{y^1} \frac{\partial a_3}{\partial y^1} dy^1 ,$$

we receive $a_3 - \displaystyle\int_0^{y^1} \frac{\partial a_3}{\partial y^1} dy^1 = \int_0^{y^2} \left[\frac{\partial a_2}{\partial y^3} - \int_0^{y^1} \frac{\partial^2 a_2}{\partial y^1 \partial y^3} dy^1 \right] dy^2 + \frac{\partial A_3}{\partial y^3}$, whence

$$\frac{\partial A_3}{\partial y^3} = a_3 - \int_0^{y^1} \frac{\partial a_3}{\partial y^1} dy^1 - \int_0^{y^2} \left[\frac{\partial a_2}{\partial y^3} - \int_0^{y^1} \frac{\partial^2 a_2}{\partial y^1 \partial y^3} dy^1 \right] dy^2 .$$

Integrating, we find

$$A_3 = \int_0^{y^3} \left\{ a_3 - \int_0^{y^1} \frac{\partial a_3}{\partial y^1} dy^1 - \int_0^{y^2} \left[\frac{\partial a_2}{\partial y^3} - \int_0^{y^1} \frac{\partial^2 a_2}{\partial y^1 \partial y^3} dy^1 \right] dy^2 \right\} dy^3 + A^4(y^4,..., y^n) .$$

And so on. The theorem is proved. Equality (4) become

$$y^i = \sum_{k=1}^{n} \psi_k{}^i (y^1,..., y^n) . \tag{23}$$

But on construction $\alpha_i^{\,i} = \dfrac{d\psi_i^{\,i}}{dz^i} = 1$ (not summarize). Hence, $\psi_i^{\,i} = z^i$, and then, from (23), it is found

$$
\begin{aligned}
z^1 &= y^1 - \psi_2^{\,1}(y^k) - \psi_3^{\,1}(y^k) - \ldots - \psi_n^{\,1}(y^k),\\
z^2 &= y^2 - \psi_1^{\,2}(y^k) - \psi_3^{\,2}(y^k) - \ldots - \psi_n^{\,2}(y^k),\\
&-----------------------\\
z^n &= y^n - \psi_1^{\,n}(y^k) - \psi_2^{\,n}(y^k) - \ldots - \psi_{n-1}^{\,n}(y^k).
\end{aligned}
\tag{24}
$$

We have found formulas of the direct transformation, giving full project splitting. To receive formulas of return transformation (4) (namely, these formulas enable us to receive from (3) the equations (5)), it is necessary to solve the equations (24) y^k rather. Technically, it can represent, certainly, significant difficulties.

The second theorem project splitting is proved as follows.

Theorem 2. Formulas direct project splitting are by means of quadratures.

The note. It is of interest to write down to write down the characteristic equation (8) in variables z^i. With this purpose we shall write $\dfrac{\partial^2 f^i}{\partial y^j \partial y^l}$ out values of private derivatives in the specified variables. We differentiate (5):

$$
\frac{\partial f^i}{\partial y^j} = (\alpha_k^{\,i} \mathrm{X}^k)' \frac{\partial z^k}{\partial y^j} = (\alpha_k^{\,i} \mathrm{X}^k)' \beta_j^{\,k},
$$

$$
\frac{\partial^2 f^i}{\partial y^j \partial y^l} = (\alpha_k^{\,i} \mathrm{X}^k)'' \beta_j^{\,k} \beta_l^{\;\;k} + (\alpha_k^{\,i} \mathrm{X}^k)' \frac{\partial \beta_j^{\,k}}{\partial y^l}.
$$

But, differentiating (14), we receive $\dfrac{\partial \beta_j^{\,h}}{\partial y^l} \alpha_h^{\,t} + \beta_i^{\,a} \alpha_a'^{\,t} \dfrac{\partial z^a}{\partial y^l} = 0$, whence $\dfrac{\partial \beta_j^{\,h}}{\partial y^l} = -\alpha_a'^{\,t} \beta_t^{\,h} \beta_j^{\,a} \beta_l^{\,a}$.

Hence, $\dfrac{\partial^2 f^i}{\partial y^l \partial y^j} = (\alpha_k^{\,i} \mathrm{X}^k)'' \beta_j^{\,k} \beta_l^{\,k} - (\alpha_a^{\,i} \mathrm{X}^a)' \beta_t^{\,a} \;\; \alpha_k'^{\,t} \beta_j^{\,k} \beta_l^{\,k}$.

Thus,

$$
\frac{\partial^2 f^1}{\partial y^j \partial y^l} - S \frac{\partial^2 f^1}{\partial y^j \partial y^l} = \lambda_k \beta_j^{\,k} \beta_l^{\,k},
$$

where

$$
\lambda_k = (\alpha_k^{\,1} X^k)'' - (\alpha_a^{\,1} X^a)' \beta_t^{\,a} \alpha_k'^{\,t} - S\left[(\alpha_k^{\,2} X^k)'' - (\alpha_a^{\,2} X^a)' \beta_t^{\,a} \alpha_k'^{\,t}\right].
$$

Hence

$\left| \dfrac{\partial^2 f^1}{\partial y^j \partial y^l} - S \dfrac{\partial^2 f^1}{\partial y^j \partial y^l} \right| = \lambda_{1\ldots}\lambda_n \left|\beta_j^{\,k}\right|^2$, and the characteristic equation becomes $\lambda_1 \quad \lambda_2 \ldots \lambda_n = 0.$

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