Proceedings of the 10th International Conference "Reliability and Statistics in Transportation and Communication" (RelStat'10), 20–23 October 2010, Riga, Latvia, p. 104-111. ISBN 978-9984-818-34-4
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USING THE THEORY OF MARKOV CHAINS AND A SEMI-MARKOV PROCESS TO ESTIMATE THE RELIABILITY OF AIRCRAFT AND AIRLINES

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The reliability of aircraft (AC) and airline (AL) operation can be ensured by the implementation of a specific inspection program, which can be planned using full-scale fatigue test data and the theory of Markov Chains (MC) and a Semi-Markov process (SMP) with rewards. The process of the operation of aircraft is considered as absorbing MC with $(n+4)$ states. The states $E_1, E_2, ..., E_{n+1}$ correspond to AC operation in time intervals $[t_0,t_1),[t_1,t_2),...,[t_n,t_{SL})$, where n is an inspection number, t_{SL} is specified life (SL), i.e. AC retirement time. States E_{n+2}, E_{n+3}, and E_{n+4} are absorbing states: AC is withdrawn from service when the SL is reached or fatigue failure (FF) or fatigue crack detection (CD) takes place. In the corresponding matrix for the operation processes of AL the states E_{n+2}, E_{n+3} and E_{n+4} are not absorbing, but correspond to the return of the MC to state E_1 (AL operation returns to first interval). The problem of inspection planning is the choice of the sequence $\{t_1,t_2,...,t_n,t_{SL}\}$ (in case of equal inspection intervals and fixed t_{SL}, the choice of n) corresponding to the maximum of airline gain taking into account the limitations of intensity of AL fatigue failure (or AC failure probability). This paper considers the case when the parameters of the fatigue crack growth exponential model are known. Numerical examples are given.

Keywords: *inspection program, Markov chains, reliability*

1. Introduction

This paper is devoted to the problem of the reliability of airlines and it is a development of previous publications [1-6] where the problem of the reliability of fatigue-prone aircraft was considered. The reliability of aircraft (AC) and airline (AL) operations can be ensured by the implementation of a specific inspection program, which can be planned using full scale fatigue test data and the theory of Markov Chains (MC) and a Semi-Markov process (SMP). The process of the operation of AC is considered as absorbing MC with $(n + 4)$ states. The states E_1, E_2, ..., E_{n+1} correspond to AC operation in time intervals $[t_0,t_1),[t_1,t_2),...,[t_n,t_{SL})$, where n is an inspection number, t_{SL} is specified life (SL), i.e. AC retirement time. States E_{n+2}, E_{n+3}, and E_{n+4} are absorbing states: AC is withdrawn from service when the SL is reached or fatigue failure (FF) or fatigue crack detection (CD) takes place. In the corresponding matrix for the operation process of AL the states E_{n+2}, E_{n+3} and E_{n+4} are not absorbing , but correspond to the return of the MC to state E_1 (AL operation returns to first interval). In the matrix of transition probabilities of AC, P_{AC}, there are three units in the three last lines in the diagonal (bias), but for corresponding lines in the matrix for AL, P_{AL}, the units are in the first column, corresponding to state E_1. Using P_{AC} we can get the probability of FF of AC and cumulative distribution function, mean and variance of AC life, and the same characteristics under the condition of absorption in a specific absorbing state. Using P_{AL} we can get the stationary probabilities of AL operation $\{\pi_1,...,\pi_{n+1},\pi_{n+2},...,\pi_{n+4}\}$. Here π_{n+3} defines the part of MC steps when FF takes place and the MC appears in state E_{n+3}. The ratio of this value to the mean life of a new aircraft defines the intensity of FF, λ_F, i.e. the number of FF in one time unit. It can be calculated also using the theory of Semi-Markov Process (SMP) with rewards [7,8]. Using this theory we can calculate also the gain of the process. The problem of inspection planning is the choice of the sequence $\{t_1, t_2, ..., t_n, t_{SL}\}$ (in the case of equal inspection intervals and fixed t_{SL}, choice of n) corresponding to the maximum gain, taking into account the limitations imposed by the intensity of AL fatigue failure of AL or AC fatigue failure. A numerical example is given.

2. Formal Setting of the Problem

For the formal setting of the problem we should define the matrix of transition probabilities of MC and the matrix of rewards for SMP. However, first the fatigue crack growth model should be defined.

2.1. Fatigue Crack Growth Model

We suppose the following exponential approximation of fatigue crack growth function when the size of a fatigue crack is defined by the equation:

$$a(t) = a(0)\exp(Qt)\,. \tag{1}$$

T_d represents the time when a crack reaches detectable size, a_d. It is the time when the probability of crack discovery is equal to a unit if inspection is made (before T_d this probability is equal to zero). T_c is the time when the crack reaches its critical size, a_c, and FF takes place. The value of a_c corresponds to the minimum residual strength of an aircraft component allowed by special design regulations. We have

$$T_d = (\ln a_d - \ln a_0)/Q = C_d / Q\,, \tag{2}$$

$$T_c = (\ln a_c - \ln a_0)/Q = C_c / Q\,, \tag{3}$$

where a_0 is $a(0)$. Despite its simplicity, the formula (1) in interval $[T_d, T_c)$ shows clear results.

The values C_c, and C_d, can be derived each from another.

$$C_d = C_c - \delta\,, \tag{4}$$

$$C_c = C_d + \delta\,, \tag{5}$$

$$\delta = \ln a_c - \ln a_d = \ln\frac{a_c}{a_d}\,. \tag{6}$$

If δ and a_0 are constants then actually considering the model of fatigue crack growth, the distribution of random variables (r.v.) T_d , T_c is defined by distribution of only two r.v.: C_c and Q. Let us denote

$$X = \ln Q\,,\; Y = \ln C_c = \ln\left(\ln\left(a_c / a_d\right)\right). \tag{7}$$

From the analysis of the fatigue test data it can be assumed, that r.v. ln T_c, is distributed normally. It comes from the additive property of the normal distribution that ln T_c could be normally distributed if both ln C_c and ln Q are normally distributed or if one of these value is constant but the other has normal distribution. Here we consider the simplest case. We suppose that random variable X has normal distribution with unknown mean value θ_0 and known θ_1 but Y is constant.

2.2. Failure of AC Probability Calculation Using the Theory of Absorbing Markov Chains

Let the probability of crack detection during the inspection number i at time moment t_i be denoted as v_i; the probability of failure in service time interval (t_{i-1}, t_i) as q_i; and the probability of absence of the above-mentioned events as u_i. These three events form a complete set $u_i + v_i + q_i = 1$. In our model we also assume that there is an inspection at t_{SL}. This inspection at the end of (n+1)-th interval does not change the reliability, but it should be done in order to know the state of aircraft (to know if there is any

FC or if there are no fatigue cracks). *Effective number of inspections* is equal to *n* but in following, for short, we say that *n* is *number of inspections*. The transition probability matrix of this process, P_{AC}, can be composed as presented in Fig.1.

	E1	**E2**	**E3**	**…**	**En-1**	**En**	**En+1**	**En+2 (SL)**	**En+3 (FF)**	**En+4 (CD)**
E1	0	u_1	0	…	0	0	0	0	q_1	v_1
E2	0	0	u_2	…	0	0	0	0	q_2	v_2
E3	0	0	0	…	0	0	0	0	q_3	v_3
…	…	…	…	…	…	…	…	…	…	…
En-1	0	0	0	…	0	u_{n-1}	0	0	q_{n-1}	v_{n-1}
En	0	0	0	…	0	0	u_n	0	q_n	v_n
En+1	0	0	0	…	0	0	0	u_{n+1}	q_{n+1}	v_{n+1}
En+2 (SL)	0	0	0	…	0	0	0	1	0	0
En+3 (FF)	0	0	0	…	0	0	0	0	1	0
En+4 (CD)	0	0	0	…	0	0	0	0	0	1

Figure 1. The transition probability matrix for absorbing MC

An example of a corresponding state transition diagram for the case of three inspections is shown in Fig. 2. We suppose that if, in interval (T_d, T_c), some inspection is made, then fatigue failure will be eliminated.

$$u_i = P(T_d > t_i \mid T_d > t_{i-1}) = P(Q < C_d / t_i) / P(Q < C_d / t_{i-1}); \tag{8}$$

$$q_i = P(t_{i-1} < T_d < T_c < t_i \mid t_{i-1} < Td) = \\ = \begin{cases} 0, & if \quad t_{i-1}C_c / C_d > t_i, \\ P(C_c / t_i < Q < C_d / t_{i-1}) / P(Q < C_d / t_{i-1}), & if \quad t_{i-1}C_c / C_d > t_i, \end{cases} \tag{9}$$

$$v_i = 1 - u_i - q_i\,. \tag{10}$$

Unfortunately, in general the corresponding integrals are not expressed by elementary functions, but if it is assumed that C_c and C_d are some constants, then

$$u_i = a_i / a_{i-1}\,, \tag{11}$$

$$q_i = \max(0, (a_{i-1} - b_i) / (1 - a_{i-1}))\,, \tag{12}$$

$$a_i = \Phi\left(\ln(C_d / t_i) - \theta_0\right) / \theta_1), \quad (13)$$

$$b_i = \Phi\left(\ln(C_c / t_i) - \theta_0\right) / \theta_1), \quad (14)$$

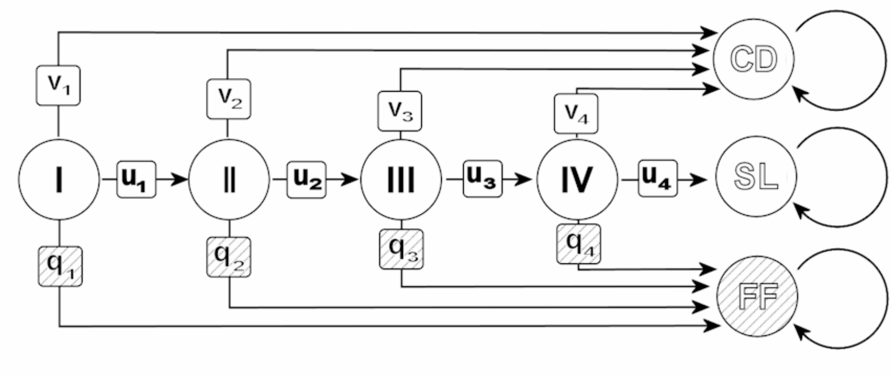

Figure 2. State transition diagram for a three-inspection program

$\Phi(.)$ is a distribution function of a standard normal variable.

The structure of considered matrices can be described as follows in Fig.3.

Q	R
0	I

Figure 3. Sub-matrices of transition probabilities matrix

Here *I* is a matrix of identity corresponding to absorbing states, 0 is a matrix of zeros. Then the matrix of probabilities of absorbing in different absorbing states for different initial transient states is defined by formula (15)

$$B = (I - Q)^{-1} R. \quad (15)$$

The first row of the matrix *B* defines the probabilities of absorption in states SL, FF, CD if the initial state is E_1. Particularly, $B(1,2)$ defines the failure probability of a new aircraft which begins operation during the first interval. The following rows of the matrix *B* define the same probabilities for different initial states (for aircraft which begin operation in different time intervals). Therefore, if, for example, aircraft begin operation in the first interval, the failure probability of aircraft in the fleet is defined in formula (16)

$$p_f = aBb, \quad (16)$$

where vector row $a = (1,0,...,0)$ means that all aircraft begin operation in the first interval (state E_1), vector column $b = (0,1,0)$ '.

2.3. Calculation of the cost of airline operation using theory of stationary semi-Markov chains with rewards

In order to analyse the cost of permanent operations of AL or aircraft fleets, it is necessary to modify the transition probability matrix.

The process is not absorbed or stopped in states SL, FF or CD. If these states are reached, the process restarts from the in initial state E_1. This means that an entirely new aircraft is acquired. The modified transition probability matrix, P_{AL}, is shown in Fig.4. An example of the state transition diagram is shown in Fig.5.

	E_1	E_2	E_3	…	E_{n-1}	E_n	E_{n+1}	E_{n+2} (SL)	E_{n+3} (FF)	E_{n+4} (CD)
E_1	0	u_1	0	…	0	0	0	0	q_1	v_1
E_2	0	0	u_2	…	0	0	0	0	q_2	v_2
E_3	0	0	0	…	0	0	0	0	q_3	v_3
…	…	…	…	…	…	…	…	…	…	…
E_{n-1}	0	0	0	…	0	u_{n-1}	0	0	q_{n-1}	v_{n-1}
E_n	0	0	0	…	0	0	u_n	0	q_n	v_n
E_{n+1}	0	0	0	…	0	0	0	u_{n+1}	q_{n+1}	v_{n+1}
E_{n+2} (SL)	1	0	0	…	0	0	0	0	0	0
E_{n+3} (FF)	1	0	0	…	0	0	0	0	0	0
E_{n+4} (CD)	1	0	0	…	0	0	0	0	0	0

Figure 4. The transition probability matrix for stationary AL operation process

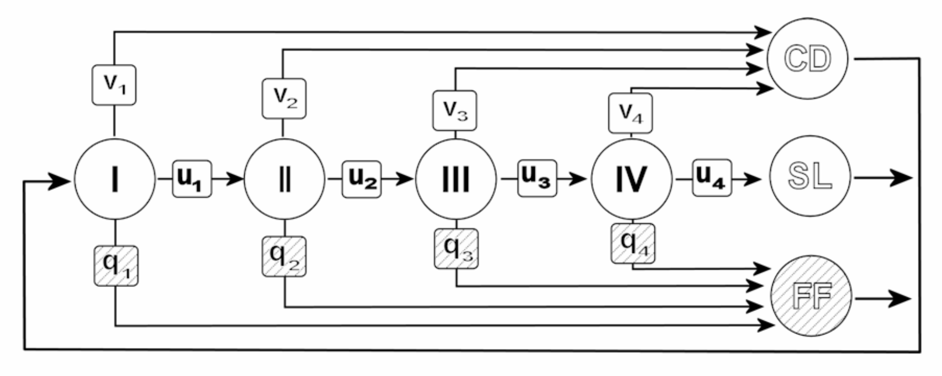

Figure 5. State transition diagram for stationary AL operation process with three inspections

Next we will consider economic analysis. The theory of Semi-Markov process with rewards is usually used to find the solution to similar problems [8]. A reward structure is described by the reward matrix R, the component of which, r_{ij}, describes the reward, connected with the transition from state i to state j; here $i, j = 1, 2, \ldots, n+4$. Let us define the reward, related to successful transitions from one operation interval to the following one by value $a(n)$; the reward related to transitions to state CD (or E_{n+4}) from any state $E_1,\ldots, E_{n+1}$ – by value b, to state FF (or E_{n+3}) – by value c; and from states SL, FF, CD to state E_1 (acquisition of new AC) – by value d.

Let us define the gain:

$$g(n) = \sum_{i=1}^{n+4} \pi_i g_i(n), \tag{17}$$

where $\pi = (\pi_1,...,\pi_{n+4})$ is the vector of stationary probabilities, which is defined by the equation system

$$\pi P = \pi, \quad \sum_{i=1}^{n+4} \pi_i = 1; \tag{18}$$

$$g_i(n) = \begin{cases} a(n)\cdot u_i + b\cdot q_i + c\cdot v_i, & i = 1,...n+1, \\ d, & i = n+2,...,n+4 \end{cases} \tag{19}$$

u_i, q_i, v_i, $i = 1,...,n+1$, are probabilities of successful transitions from one to the following interval, to E_{n+3} and E_{n+4} states;
$a(n) = a_0(n) + d_{insp}t_{SL}$, where $a_0(n) = a_1 t_{SL}/(n+1)$ –is the reward, related to successful transitions from one operation interval to the following one, and the cost of one inspection (negative value), $d_{insp}t_{SL}$, which is supposed to be proportional to t_{SL} if it is supposed that all intervals are equal, a_1 defines the reward of operation in one time unit (one hour or one flight). The dimension of $a(n)$ should coincide with dimensions of t_{SL});
$b = b_1 a_0$, $c = c_1 a_0$ are the rewards related to transitions from any state E_1, ..., E_{n+1} to state E_{n+3} (FF takes place) and E_{n+4} (CD takes place) which are supposed to be proportional to a_0; $d = d_1 t_{SL}$ is the negative reward, the value of which is the cost of new aircraft after events SL, FF or CD and transition to E_1 take place (it is supposed to be proportional to t_{SL}).

If $a(n) = b = c = 1$, $d = 0$ and time transition to state E_1 are equal to zero. Then $\pi_{tj} = \pi_j g_j / g$ defines the part of time which SMP spends in state $E_j, j = 1,...,n+1$; $L_j = g/\pi_j$ defines the mean return time for state E_j; specifically, L_1 is the mean time of renewal of AL operation in the first interval, L_{n+3} is the mean time between FF; $\lambda_F = 1/L_{n+3}$ is the intensity of fatigue failure. It should be remembered that the same value can be obtained using the theory of absorbing MC. This value is equal also to the ratio of aircraft failure probability to the mean life of new aircraft.

The problem is to maximise the gain, *g*, under limitations of probability of aircraft FF or AL intensity of fatigue failure λ_F. In the following numerical example we consider the last version of the problem.

3. Choice of Inspection Number. Numerical Example

To simplify the numerical example we suppose that all intervals of operation are equal. In Fig. 6 and 7 we see numerical examples of calculations of gain, g, and of intensity of fatigue failure, λ_F, as functions of *n*.

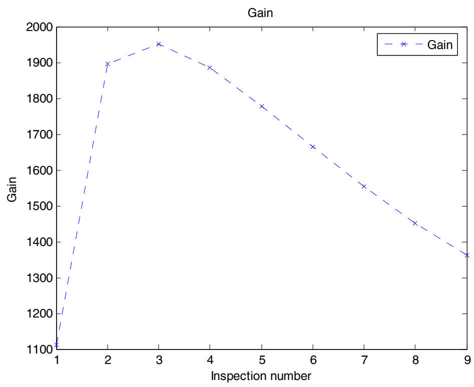

Figure 6. Gain as function of inspection number

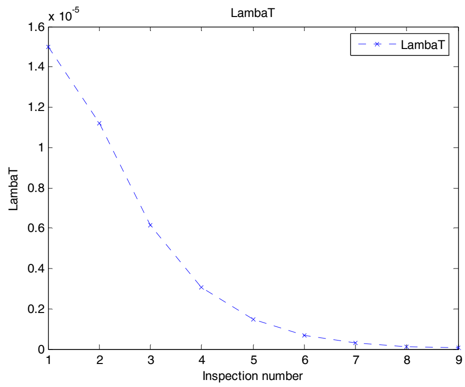

Figure 7. AL intensity of AC fatigue failure as function of inspection number

First we should choose the number of *n*, corresponding to the maximum of gain *g*: $n_g = \arg\max g(n)$.Then we should choose the minimum value of *n* under condition; that of intensity of fatigue failure λ_F, which is a function of n, $\lambda_F(n)$, will be equal or less than some design allowed value of intensity of fatigue failure, λ_F^*: $n_\lambda = \min\{n : \lambda_F(n) \le \lambda_F^*, \text{ for all } n \ge n_\lambda\}$.

And finally we should choose $n = \max(n_g, n_\lambda)$.

Consider the numerical example of calculation of gain, g, and intensity of fatigue failure, λ_F, for n = 2 and following initial data: t_{SL} = 40 000, θ_0 = -8.5885, θ_1 = 0,34600, a_c = 102, a_d = 20 (these values have been used already, for example, in [1]); $a_1 = 1$, $b_1 = 0.05$, $c_1 = -0.05$, $d_1 = -0.3$, $d_i = -0.0001$.

Using equation (10-14) we have (using the MATLAB notation for the matrix)

P_{AC} = [0 9.3958e-001 0 0 6.3833e-003 5.4037e-002
0 0 3.4655e-001 0 2.6900e-001 3.8445e-001
0 0 0 1.6030e-001 2.4240e-001 5.9730e-001
0 0 0 1.0000e+000 0 0
0 0 0 0 1.0000e+000 0
0 0 0 0 0 1.0000e+000]

In matix P_{AL} the first three lines are the same, but in the last three lines the units are in the first column. Using P_{AL} and equations (18) we have a stationary distribution for the case when time unit is one steps in MC:

π = [3.06e-001, 2.88e-001, 9.97e-002, 1.60e-002, 1.04e-001, 1.87e-001].

Let us define the matrix P_{uqv} with u_i, q_i, v_i in every line for $i = 1,\ldots,n+1$. In the example

P_{uqv} =[9.3958e-001 6.3833e-003 5.4037e-002
3.4655e-001 2.6900e-001 3.8445e-001
1.6030e-001 2.4240e-001 5.9730e-001].

Then $g = \pi[P_{uqv}[a;b;c];d;d;d]$. In considered case g = 1 896.

If the time unit is time interval then the vector of return times for every state

$L = [g_t / \pi_1, \ldots, g_t / \pi_6]$ = [2.27, 2.41, 6.96, 4.34, 6.70, 3.72].

Here $g_t = \pi[1;1;1;0;0;0]$ is the mean gain if income is time and the time unit is interval. It is useful to note that $g_t = g$ if $a = b = c = 1$, $d = 0$. In considered case g_t = 0.694.

The return time of failure (state FF) is equal to corresponding (n + 3)-th component of vector L: $L_F = L(n+3)$ = 6.7. The time lenth of one interval $t_1 = t_{SL} / (n+1)$ = 13 333.

The intensity of falure in t-time unit (flight or flight hour) $\lambda_F = (1/L_F)/t_1$ = 0.0000112.

It is useful to note that the same values can be obtained using the theory of absorbing MC. We calculate the matrix of absorption

B = [5.2196e-002 3.3806e-001 6.0975e-001
5.5552e-002 3.5300e-001 5.9145e-001
1.6030e-001 2.4240e-001 5.9730e-001].

Then we take from it the probability of absorption of new aircrft in state FF, p_f = 0.338, calculate the mean time to absorption of new aircraft, $T_1 = \tau_1 t_1$ (where τ_1 = 2.27 is the mean time to absorption if the time unit is interval-which coincides with $L(1)$!) and calculate the ratio p_f / T_1 = 0.0000112.

In Fig. 6 and 7 we see similar calculations for n = 1, …, 9. And we see that the maximum of g we have for n = 3. We can choose this number of inspections if allowed failure intensity is higher or equal to (approximately) 0.000006, because λ_F is equal just to this value at this number of inspections.

4. Conclusions

This paper presents, (based on MC and SMP with rewards theory), a method to solve the problem of inspection planning corresponding to maximum airline gain while airline fatigue failure intensity is limited. In this paper the case is consdered when parameters of fatigue crack growth exponential models are known. If a parameter is unknown, the minimax approach (see [1]) should be used. Numerical examples are given.

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