INSPECTION PROGRAM DEVELOPMENT USING MINIMAX METHOD

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The paper describes development of the methodology of the choice of aircraft fleet’s inspection intervals. This methodology is verified, using the experimental fatigue crack growth data. Methodology includes in the economic parameters. Using this method it is possible to find safe enough and the most cost-effective inspection planning programs. In addition the paper includes testing of the methodology at different permissive withdrawal probabilities and their impact on the real probability of withdrawal and the number of inspection intervals for the model with one aircraft and with aircraft fleet. For the case when they are not known the minimax approach is offered if there are results of acceptance full-scale fatigue test and conditions of acceptance are known. Numerical examples are given.

Keywords: Engineering inspection program, Markov chains, reliability

1. Introduction

This paper deals with the problem of the reliability of airlines and it is a kind of review and development of previous publications [1-6] where the problem of the reliability of fatigue-prone aircraft has been considered. The reliability of aircraft (AC) and airline (AL) operations can be ensured by the implementation of a specific inspection program, which can be planned using full-scale fatigue test data and the theory of Markov Chains (MC) and a Semi-Markov process (SMP). The process of the operation of AC is considered as absorbing MC with \((n + 4)\) states. The states \(E_1, E_2, ..., E_{n+1}\) correspond to AC operation in time intervals \([t_0, t_1), [t_1, t_2), ..., [t_n, t_{SL}],\) where \(n\) is an inspection number, \(t_{SL}\) is specified life (SL), i.e. AC retirement time. States \(E_{n+2}, E_{n+3},\) and \(E_{n+4}\) are absorbing states: AC is withdrawn from service when the SL is reached, fatigue failure (FF) and fatigue crack detection (CD). In the corresponding matrix for the operation process of AL the states \(E_{n+2}, E_{n+3}\) and \(E_{n+4}\) are not absorbing, but correspond to the return of the MC to state \(E_1\) (AL operation returns to first interval). In the matrix of transition probabilities of AC, \(P_{AC}\), there are three units in the three last lines in the diagonal, but for corresponding lines in the matrix for AL, \(P_{AL}\), the units are in the first column, corresponding to state \(E_1\). Using \(P_{AC}\) we can get the probability of FF of AC and cumulative distribution function, mean and variance of AC life, and the same characteristics under the condition of absorption in a specific absorbing state. Using \(P_{AL}\) we can get the stationary probabilities of AL operation \(\{\pi_1, ..., \pi_{n+1}, \pi_{n+2}, ..., \pi_{n+4}\}\). Here \(\pi_{n+3}\) defines the part of MC steps when FF takes place and the MC appears in state \(E_{n+3}\). The ratio of this value to the mean life of a new aircraft defines the intensity of FF, \(\lambda_{FF}\), i.e. the number of FF in one time unit. It can be calculated also using the theory of Semi-Markov Process (SMP) with rewards [7,8]. Using this theory we can calculate also the gain of the process. The problem of inspection planning is the choice of the sequence \(\{t_1, t_2, ..., t_n, t_{SL}\}\) (in the case of equal inspection intervals and fixed \(t_{SL}\), choice of \(n\)) corresponding to the maximum gain, taking into account the limitations imposed by the intensity of AL fatigue failure of AL or AC fatigue failure. A numerical example is given.

2. Formal Setting of the Problem

For the formal setting of the problem we should define the matrix of transition probabilities of MC and the matrix of rewards for SMP. However, first the fatigue crack growth model should be defined.
2.1. Fatigue crack growth model

We suppose the following exponential approximation of fatigue crack growth function when the size of a fatigue crack is defined by the equation:

$$a_{cr}(t) = a_{cr}(0) \exp(Qt).$$  \hspace{1cm} (1)

Let $T_d$ represents the time when a crack reaches detectable size, $a_d$. It is the time when the probability of crack discovery is equal to a unit if inspection is made (we suppose that before $T_d$ this probability is equal to zero), $T_c$ is the time when the crack reaches its critical size, $a_c$, and FF takes place. The value of $a_c$ corresponds to the minimum residual strength of an aircraft component allowed by special design regulations. We have

$$T_d = (\ln a_d - \ln a_0) / Q = C_d / Q,$$  \hspace{1cm} (2)

$$T_c = (\ln a_c - \ln a_0) / Q = C_c / Q,$$  \hspace{1cm} (3)

where $a_0$ is $a(0)$. Despite its simplicity, the formula (1) in interval $[T_d, T_c]$ shows clear results.

The values $C_c$, and $C_d$, can be derived each from another.

$$C_d = C_c - \delta,$$  \hspace{1cm} (4)

$$C_c = C_d + \delta,$$  \hspace{1cm} (5)

$$\delta = \ln a_c - \ln a_d = \ln \frac{a_c}{a_d}.$$  \hspace{1cm} (6)

If $\delta$ and $a_0$ are constants then for considering the model of fatigue crack growth, the distribution of random variables (r.v.) $T_d$, $T_c$ is defined by distribution of only two r.v.: $Cc$ and $Q$.

Let us denote

$$X = \ln Q, Y = \ln C_c = \ln \left( \ln \left( \frac{a_c}{a_d} \right) \right).$$  \hspace{1cm} (7)

From the analysis of the fatigue test data it can be assumed, that r.v. $\ln T_c$, is distributed normally. It comes from the additive property of the normal distribution that $\ln T_c$ could be normally distributed if both $\ln C_c$ and $\ln Q$ are normally distributed or if one of these values is constant but the other has normal distribution. Here we consider the simplest case. We suppose that random variable $X$ has normal distribution with unknown mean value $\theta_0$ and known $\theta_1$ but $Y$ is constant.

2.2. Failure of AC probability calculation using the theory of absorbing Markov chains

Let the probability of crack detection during the inspection number $i$ at time moment $t_i$ be denoted as $v_i$; the probability of failure in service time interval $(t_{i-1}, t_i)$ as $q_i$; and the probability of absence of the above-mentioned events as $u_i$. These three events form a complete set $u_i + v_i + q_i = 1$. In our model we also assume that there is an inspection at $t_{SI}$. This inspection at the end of $(n+1)$-th interval does not change the reliability, but it should be done in order to know the state of aircraft (to know if there is any FC or if there are no fatigue cracks). Effective number of inspections is equal to $n$ but in following, for short, we say that $n$ is number of inspections. The transition probability matrix of this process, $P_{AC}$, can be composed as it is presented on Figure 1.
An example of a corresponding state transition diagram for the case of three inspections is shown on Figure 2. We suppose that if, in interval \((T_d, T_c)\), some inspection is made, then fatigue failure will be eliminated.

\[
\begin{align*}
    u_i &= P(T_d > t_i | T_d > t_{i-1}) = \frac{P(Q < C_d / t_i)}{P(Q < C_d / t_{i-1})}; \\
    q_i &= P(t_{i-1} < T_d < T_c < t_i | T_d > t_{i-1}) = \\
        & \begin{cases} 
        0, & \text{if } t_{i-1}/C_c / C_d > t_c, \\
        \frac{P(C_c / t_i < Q < C_d / t_{i-1})}{P(Q < C_d / t_{i-1})}, & \text{if } t_{i-1}/C_c / C_d \leq t_c, 
        \end{cases} \\
    v_i &= 1 - u_i - q_i
\end{align*}
\]

Unfortunately, in general the corresponding integrals are not expressed by elementary functions, but if it is assumed that \(C_c\) and \(Cd\) are some constants, then

\[
\begin{align*}
    u_i &= a_i / a_{i-1}, \\
    q_i &= \max(0, (a_{i-1} - b_i)/(1 - a_{i-1})), \\
    a_i &= \Phi(\ln(C_d / t_i - \theta_0) / \theta_1), \\
    b_i &= \Phi(\ln(C_c / t_i - \theta_0) / \theta_1)
\end{align*}
\]
Φ(.) is a distribution function of a standard normal variable.

The structure of considered matrices can be described as follows on Figure 3.

$$\begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

*Figure 3. Sub-matrices of transition probabilities matrix*

Here I is a matrix of identity corresponding to absorbing states, 0 is a matrix of zeros. Then the matrix of probabilities of absorbing in different absorbing states for different initial transient states is defined by formula (15)

$$B = (I - Q)^{-1}R.$$  \hspace{1cm} (15)

The first row of the matrix B defines the probabilities of absorption in states SL, FF, CD if the initial state is $E_1$. Particularly, $B(1,2)$ defines the failure probability of a new aircraft which begins operation during the first interval. The following rows of the matrix B define the same probabilities for different initial states (for aircraft which begin operation in different time intervals). Therefore, if, for example, aircraft begin operation in the first interval, the failure probability of aircraft in the fleet is defined by equation (16)

$$p_f = aBB',$$  \hspace{1cm} (16)

where vector row $a = (1,0,\ldots,0)$ means that all aircraft begin operation in the first interval (state E1), vector column $b = (0,1,0)'$.

### 3. Calculation of the Cost of Airline Operation

In order to analyze the cost of permanent operations of AL or aircraft fleet, it is necessary to modify the transition probability matrix.

Now the process is not absorbed or stopped in states SL, FF or CD. If these states are reached, the process restarts from the initial state $E_1$. This means that an entirely new aircraft is acquired. The modified transition probability matrix, $P_{AL}$, is shown on Figure 4. An example of the state transition diagram is shown on Figure 5.

$$
\begin{array}{cccccccc}
E_1 & E_2 & E_3 & \ldots & E_{SL} & E_F & E_{SL} & E_{FF} \\
\downarrow & \downarrow & \downarrow & \ldots & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\end{array}$$

*Figure 4. The transition probability matrix for stationary AL operation process*
Next we will consider economic analysis. The theory of Semi-Markov process with rewards is usually used to find the solution of similar problems [8]. A reward structure is described by the reward matrix $R$, the component of which, $r_{ij}$, describes the reward, connected with the transition from state $E_i$ to state $E_j$; here $i,j=1,2,...,n+4$. Let us define the reward, related to successful transitions from one operation interval to the following one by value $a(n)$; the reward related to transitions to state CD (or $E_{n+4}$) from any state $E_1,E_2,...,E_{n+1}$ – by value $b$, to state FF (or $E_{n+3}$) – by value $c$; and from states SL, FF, CD to state $E_i$ (acquisition of new AC) – by value $d$.

Let us define the airline gain:

$$g(n) = \sum_{i=1}^{n+4} \pi_i g_i(n),$$

where $\pi = (\pi_1,...,\pi_{n+4})$ is the vector of stationary probabilities, which is defined by the equation system

$$\pi P = \pi, \quad \sum_{i=1}^{n+4} \pi_i = 1;$$

$$g_i(n) = \begin{cases} a(n) \cdot u_i + b \cdot q_i + c \cdot v_i, & i = 1,...,n+1, \\ d, & i = n+2,...,n+4 \end{cases};$$

$$u_i, q_i, v_i, \quad i = 1,...,n+1,$$ are probabilities of successful transitions from one to the following interval, to $E_{n+3}$ and $E_{n+4}$ states;

$$a(n) = a_0(n) + d_{\text{imp}}t_{SL}, \quad \text{where } a_0(n) = a_t t_{SL} / (n+1),$$ – is the reward, related to successful transitions from one operation interval to the following one, and the cost of one inspection (negative value), $d_{\text{imp}}t_{SL}$, which is supposed to be proportional to $t_{SL}$ if it is supposed that all intervals are equal, $a_t$ defines the reward of operation in one time unit (one hour or one flight). The dimension of $a(n)$ should coincide with dimensions of $t_{SL}$;

$$b = b_t a_0(n), \quad c = c_t a_0(n)$$ are the rewards related to transitions from any state $E_1,...,E_{n+1}$ to state $E_{n+3}$ (FF takes place) and $E_{n+4}$ (CD takes place) which are supposed to be proportional to $a_t$; $d = d_t t_{SL}$ is the negative reward, the value of which is the cost of new aircraft after events SL, FF or CD and transition to $E_i$ take place (it is supposed to be proportional to $t_{SL}$).

If $a(n) = b = c = 1, \quad d = 0$ and time transition to state $E_i$ are equal to zero, then $\pi_j = \pi_j g_j(n) / g(n)$ defines the part of time which SMP spends in state $E_j, j = 1,...,n+1$;

$L_j = g(n) / \pi_j$ defines the mean return time for state $E_j$; specifically, $L_1$ is the mean time of renewal of
AL operation in the first interval, \( L_{n+3} \) is the mean time between FF; so \( \lambda_F = 1 / L_{n+3} \) is the intensity of fatigue failure. It should be remembered that the same value can be obtained using the theory of absorbing MC. This value is equal also to the ratio of aircraft failure probability to the mean life of new aircraft.

The problem is to maximize the gain, \( g(n) \), under limitations of probability of aircraft FF or AL intensity of fatigue failure \( \lambda_F \). In the following numerical example we consider the last version of the problem.

4. Choice of Inspection Number If Parameter of Fatigue Crack Growth Trajectory Is Known

Usually in Aircraft Design Bureau documents the sequence \( (t_1, \ldots, t_n) \) is defined by equation \( t_i = t_1 + (i-1)(t_{SL} - t_1) / n, \quad i = 1, 2, \ldots, n + 1 \), with specific choice of \( t_1 \) and \( n \). To simplify the numerical example we suppose that all intervals of operation are equal. On Figures 6 and 7 we see numerical examples of calculations of gain, \( g \), and of intensity of fatigue failure, \( \lambda_F \), as functions of \( n \).

The procedure of choice of \( n \) is the following. Firstly, we should calculate \( g(n) \) and choose the number of \( n \), corresponding to the maximum of gain \( g \). \( n_g = \arg \max g(n) \). Then, we should calculate expected value of intensity of fatigue failure, \( \lambda_F(n) \), which is a function of \( n \), and choose \( n_\lambda \) – it is such value of \( n \) that for any large inspection number the function \( \lambda_F(n) \) will be equal or less than \( \lambda_F^* \), which the value intensity of fatigue failure, allowed by specific aviation regulation

\[
\lambda_F^* = \min \{ n : \lambda_F(n) \leq \lambda_F^*, \quad \text{for all } n \geq n_\lambda \}.
\]

And finally, we should choose \( n = \max(n_g, n_\lambda) \).

Consider the numerical example of calculation of gain, \( g \), and intensity of fatigue failure, \( \lambda_F \), for \( n=2 \) and following initial data: \( t_{SL}=40 \) 000, \( \theta_0 = -8.5885, \theta_1 = 0.34600, a_0 = 0.2181; a_c = 102, a_d = 20 \), (these values have been used already, for example, in [1]); \( a_1 = 1, b_1 = 0.05, c_1 = -0.05, d_1 = -0.3, d_{\text{indep}} = -0.0001 \).

Using equation (10-14) we have (using the MATLAB notation for the matrix and vector)

\[
P_{UL} = \begin{bmatrix}
0 & 9.40e-001 & 0 & 0 & 6.38e-003 & 5.40e-002 \\
0 & 0 & 3.47e-001 & 0 & 2.69e-001 & 3.85e-001 \\
0 & 0 & 0 & 1.60e-001 & 2.42e-001 & 5.97e-001 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}
\]

In matrix \( P_{UL} \) the first three lines are the same, but in the last three lines the units are in the first column.

Using \( P_{UL} \) and equations (18) we have the stationary distribution for the case when time unit is one step in MC: \( \pi = [3.06e-001, 2.88e-001, 9.97e-002, 1.60e-002, 1.04e-001, 1.87e-001] \).

Let us define the matrix \( P_{uv} \) with \( u_i, q_j, v_j \) in every line for \( i = 1, \ldots, n + 1 \). In the example

\[
P_{uv} = \begin{bmatrix}
9.3958e-001 & 6.3833e-003 & 5.4037e-002 \\
3.4655e-001 & 2.6900e-001 & 3.8445e-001 \\
1.6030e-001 & 2.4240e-001 & 5.9730e-001 
\end{bmatrix}
\]
Then \( g = \pi[P_{eq}(a; b; c; d; d; d)] \). In considered case \( g = 1896 \).

If the time dimension unit is the inspection interval then the vector of return times for every state
\[ L = \{g_1; \ldots; g_6\} = [2.27, 2.41, 6.96, 4.34, 6.70, 3.72] \].

Here \( g_i = \pi[1; 1; 0; 0; 0] \) is the mean airline gain if income is the time and the time unit is inspection interval. It is useful to note that \( g_i = g \) if \( a = b = c = 1, d = 0 \). In considered case \( g_i = 0.694 \).

The return time of failure (to state FF), \( T_L \), is equal to corresponding \((n+3)\)-th component of vector \( L : L = L(n+3) = 6.7 \). The time length of one interval \( t_i = t_{SL} / (n + 1) = 13.333 \).

The intensity of failure in t-time unit (flight or flight hour) \( \lambda_f = (1 / L) / t_i = 0.0000112 \).

It is useful to note that the same values can be obtained using the theory of absorbing MC. We calculate the matrix of absorption
\[
B = \begin{bmatrix}
5.2196e-002 & 3.3806e-001 & 6.0975e-001 \\
5.5552e-002 & 3.5300e-001 & 5.9145e-001 \\
1.6030e-001 & 2.4240e-001 & 5.9730e-001 
\end{bmatrix}.
\]

Then we take from it the probability of absorption of new aircraft in state FF, \( p_f = 0.338 \), calculate the mean time to absorption of new aircraft, \( T_f = t_i \) (where \( t_i = 2.27 \) is the mean time to absorption if the time unit is inspection interval; its value coincides with \( L(1) \)) and calculate the ratio \( p_f / T_f = 0.0000112 \).

We see on Figures 8 and 9 results of similar calculations for \( n = 1, \ldots, 9 \) after specific choice of designed failure intensity \( \lambda_{FD} \). Using this calculation we can make the choice of \( n \).

### 5. Minimax Choice of Inspection Number

For the considered strategy for the choice of inspection number the real intensity of failure is a function of \( \theta \) and designed allowed value of intensity of fatigue failure, \( \lambda_{FD} \), which, as it will be shown, should not be equal to \( \lambda_f^* \). But really we do not know \( \theta \) and we can only, using test results, to get some estimate of this parameter, \( \hat{\theta} \). So real intensity will be a random variable, \( \lambda_f(\hat{\theta}, \lambda_{FD}) \). We can limit the mean value of this function if we take into account that really full-scale fatigue test is approval test and redesign of airframe will be made (service of aircraft of tested design version will not be allowed) if some requirement are not met (for example, if it appears, that estimate of mean time to failure, \( \hat{\lambda}_c = C_c / \exp(\hat{\theta}) \), is too small or calculated required inspection number, based on using \( \hat{\theta} \) instead of \( \theta \) is too great. If full-scale fatigue test is approved and there is not necessity to make airframe redesign let us write \( \hat{\theta} \in \Theta_0 \), where \( \Theta_0 \) is some part of parameter space. Let us denote corresponding fatigue intensity function by

\[
\lambda_f(\hat{\theta}, \lambda_{FD}, \Theta_0) = \begin{cases} 
\lambda_f(\hat{\theta}, \lambda_{FD}) & \text{if } \hat{\theta} \in \Theta_0, \\
0 & \text{if } \hat{\theta} \notin \Theta_0.
\end{cases}
\]  

On Figure 6 we see example of calculation of expected value of this function for different \( \theta_0 \).

Additionally, we define, that there is event \( \hat{\theta} \in \Theta_0 \) if \( \text{mean } t_c = C_c / \exp(\theta_0) \geq k_rt_{SL} \). For Fig. 8 \( k_r = 1, \lambda_{FD} = 0.000001 \).
Figure 6. Mean intensity of fatigue failure as function of mean $t_C$

Figure 7. Example of fatigue crack size as function of flight number (result of full scale fatigue test)

We see that there is maximum value of $\lambda_F$ because if design is too bad and test is not approved, then we redesign the airframe. But if it is too good then fatigue failure does not take place before the specified life will be reached. Maximum value of $\lambda_F$ is equal to $1.6 \times 10^{-6}$, it is in 1.6 time more than designed $\lambda_{FD} = 0.000001$. So we can use this value of $\lambda_{FD}$ for calculation of inspection number only if allowed failure intensity $\lambda^* \geq 0.0000016$. Suppose that just it is required reliability.

On Figure 7 we see example of fatigue crack, which was observed in full-scale fatigue test of some airframe. Using this crack data and regression analysis we can get estimates $\hat{\theta} = \ln Q = -8.5885$. For these data we can make calculations of estimates of inspection intensity $\lambda_F$ and gain $g$ as function of inspection number and, finally, can find required inspection number (see Fig. 8). In this example $n_L = 6$; $n_g = 3$; $n = \max(n_g, n_L) = 6$
6. Conclusions

This paper presents, (based on MC and SMP with rewards theory), a method to solve the problem of inspection planning corresponding to maximum airline gain while airline fatigue failure intensity is limited. If the parameters of fatigue crack growth exponential models are unknown, but there are results of approval full-scale fatigue test of corresponding airframe the minimax approach (see [1]) should be used. Numerical examples are given.

References


