Proceedings of the 11th International Conference "Reliability and Statistics in Transportation and Communication" (RelStat'11), 19–22 October 2011, Riga, Latvia, p. 249-254. ISBN 978-9984-818-46-7
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ON AN OPTIMAL INVENTORY PROBLEM FOR MULTI-ECHELON PRODUCTION SYSTEM

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A multi-echelon production system is represented as a queuing system with the purpose to estimate steady-state characteristics for total average cost using regenerative approach. Regenerative structure of multi-echelon manufacturing system is observed in this article. Stochastic processes describing the behaviour of such complex systems have very complicated correlation structure. The regeneration method could be acceptable in such situations for system simulation and estimation of characteristics. However the well-known classical regeneration is not useful in real-time simulation, because classical regeneration points (at such points a demand arrives in empty system) occur too rarely. It has been recently established that weak regeneration turns out to be very effective in simulation of complex inventory systems. Some numerical results of simulation are presented.

Keywords: weak regeneration, inventory control system, waiting time, renewal process, simulation

1. Introduction

The most important problem for any complex production system is efficient modelling and controlling the flows of inventory materials, components and final products through the various locations of the system. Any mistakes in the planning of inventory control process result in a considerable decrease of the efficiency of the system's operation and of the quality of customers' service. For decreasing costs associated with the possible shortage, one can create some safety stocks of inventory. On the other hand the supplementary stock increases the holding costs. Thus, the typical problem that often comes up at such a production system is precise determination of optimal volume of the ordered inventories as well as orders' frequency. In this context an inventory order policy can generally be divided into two parts:

- determining the order quantity, or the amount of inventory that will be purchased or produced with replenishment;
- determining the reorder point, or the inventory level at which a replenishment will be triggered.

The following well studied standard policies are used to control inventory: *EOQ*, base stock, (Q, r), order-up-to, and (S, s). Using *EOQ* policy, the order is placed when the inventory reaches zero. In case of a continuous review base stock policy, orders are placed each time a demand occurs, with a goal of returning inventory to the base-stock level. In a continuous review (Q, r), an order is placed when the inventory drops to r, with an order quantity of Q. For the periodic review case, the order-up-to policies (which are similar to base-stock) place an order each period that will return inventory back to the order-up-to level. The periodic review (S, s) policy is similar to the continuous review (Q, r) policy. It is obvious that these policies are applicable mainly in deterministic, continuous review frame work.

Different types of stochastic inventory models are considered in literature [8], [9], [10]. When analysing a stochastic production system, zero inventory level is supposed at every location/echelon, and the following challenge can arise: minimization of average total costs per time unit, and, as a consequence, minimization of waiting times for a demand at every location/echelon. For a stochastic system the following characteristics are supposed to be unknown: 1) instants of demands' arriving; 2) production time at every location/echelon; 3) transhipment time between echelons; 4) inventories delivery time, etc.

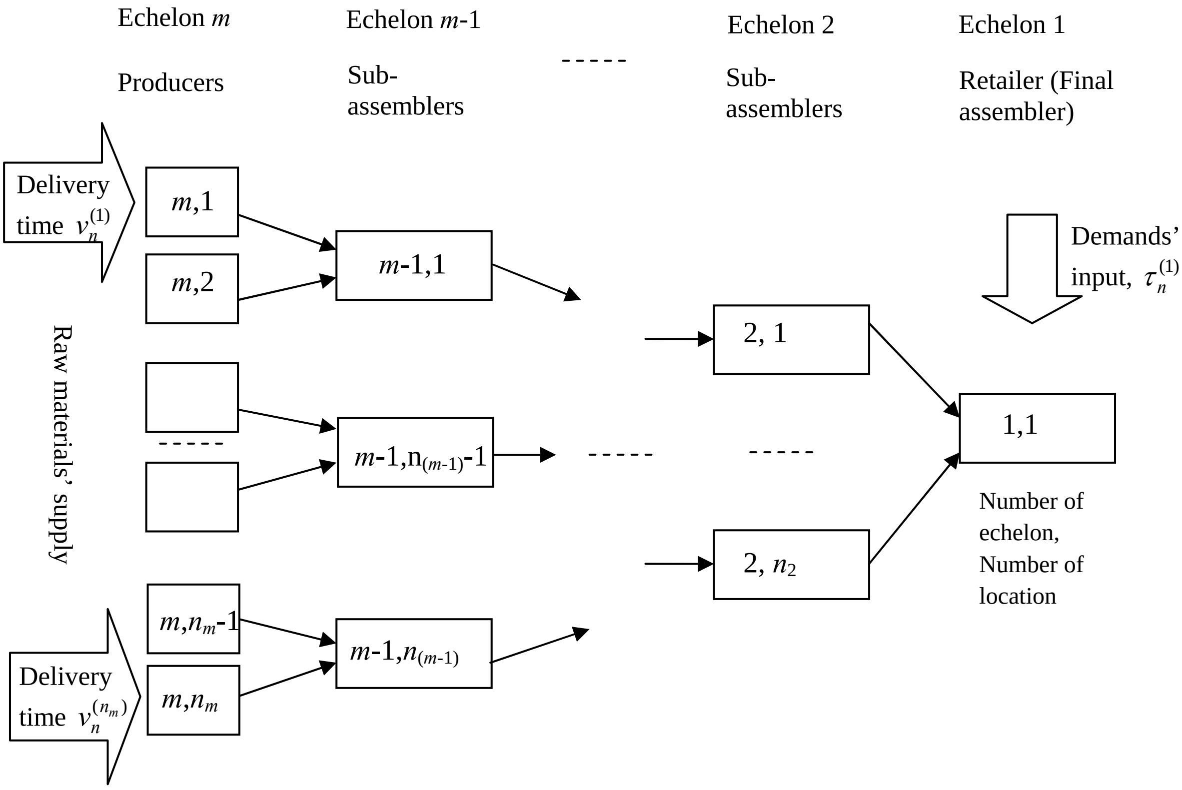

Figure 1. Multi-echelon manufacturing system

Let us pick out the task which conceptually differs from standard one "how-much-to-order-and-when": to estimate average total waiting time for a demand in the system for a regeneration cycle and condense limits for this time. As a regeneration cycle for the system a period between two consecutive regeneration points is taken. At these points a demand passes through the system without any collision with other demands, i.e. the system is idle (empty) at such points. However, this requirement is too restrictive. Moreover, efficiency of such system can be quite low, as a location cannot start to service next demand while service of current demand isn't completed. In this connection the weak regeneration points for a location/echelon will be considered, i.e. such points that a demand passes through the system without collision at every location/echelon. In other words, a location/echelon can be busy at a previous instant, but will be idle at a current instant when a demand tries to enter this echelon.
Obtained regeneration points can help to decision-maker in controlling the stochastic process of demand service. One can postpone a demand service starting time until the next regeneration point then so that total waiting time for this demand will be minimal, and average total costs per time unit will tend to minimal as well.

To be able to estimate the regenerative points, the inventory control production system is represented as a queuing system. The weak regenerative approach is used, which has been offered and tested in [1], [6], [7] for queuing networks simulation and estimation of steady-state characteristics. The regenerative approach in application to a single product inventory control model has been investigated in [5].

The paper is arranged as follows. The models of multi-echelon production system and optimisation criterion are described in the first part. Second part is devoted to regeneration points' definition. Third part deals with case study.

2. Model Description

We assume a multi-echelon manufacturing system containing m echelons, which has to satisfy a demand for a final product and represents a manufacturing and assembly process for a subassembly. Every i-th echelon consists of n_i locations, except first echelon (retailer), $i = 2, \ldots, m$; $n_1 = 1$, $\sum_{i=1}^{m} n_i = M$.

Each location produces only one component. The locations are connected by a given predecessor-successor relation as shown on Figure 1. Here first number by every location denotes number of echelon, and second is number of a location within echelon.

The described system is so called convergent multi-echelon system and is characterized by the property that a location is supplied by one or more previous locations (predecessors), and supplies exactly one location (successor). We denote locations of m-th echelon as producers, and location of first echelon as retailer.

The described system is able to produce K types of production, each type of production demands is a set of components which are produced at corresponding locations. For every type of production the retailer defines the route containing required components (locations) and amount of inventory materials supplied for every producer and retailer immediately forwards a demand to the producers with respect to a given route. In other words, there are K routes in the system.

The order lead time consists of the production times at each locations and transhipment times between locations plus possible delays (waiting times and delays of materials supply). The interarrival times of demand orders to retailer are assumed to be independent and identically distributed random variables (i.i.d.). The same we assume for production times at locations, delays of materials supply and transhipment times between locations.

To estimate the performance of a given system we assume the following cost factors: cost function $C_z^{(i)}$ for transhipment from i-th echelon to (i-1)-th echelon per time unit, waiting time cost $C_w^{(i,j)}$ per time unit at location j of echelon i, j = 1, …, n_i, i = 2, …, m and shortage cost $C_v^{(j)}$, j = 1, …, n_m. We remark that waiting time cost means cost for delays at locations due to collisions with other demands. Shortage cost is cost for delay of demands at retailer because of inventory materials deficit.

For steady state expected cost TAC we have now:

$$TAC = \sum_{i=1}^{m}\sum_{j=1}^{n_i} C_w^{(i,j)}\overline{W} + \sum_{i=2}^{m} C_z^{(i)}\overline{Z} + \sum_{j=1}^{n_m} C_v^{(i,j)}\overline{V},$$

where $\overline{Z}$ is the average transhipment time per location, $\overline{V}$ is the average delay time per location, and $\overline{W}$ is the average waiting time per location.

Next part of the present paper is devoted to defining regenerative structure of the investigated system.

3. Regenerative Structure

To estimate the average values used in expression for expected cost TAC we propose to apply weak regeneration approach.

Let $t_n^{(i)}$ be arrival time of n-th demand at echelon i, $\tau_n^{(i)} = t_{n+1}^{(i)} - t_n^{(i)}$, i = 1, …, m, $n \geq 1 -$ i.i.d. interarrival times of demands at echelon i with rate λ_i.

Let $S_n^{(i,j)}$ be the production time of n-th demand at location j of echelon i, i = 1, …, m; j = 1, …, m_i, $S_n^{(1)}$ the production time of n-th demand at first echelon. We assume that j-th location of i-th echelon has production time with rate $\mu_{ij} = 1/ES_1^{(i,j)}$.

Let $z_n^{(i)}$ be the transhipment time of n-th demand between echelon i and echelon (i-1), i = 2,…, m. Let $v_n^{(j)}$ be the delay time of n-th demand due to materials delivery to j-th producer, j = 1, …, n_m.

We denote $I_n = \left(\delta_n^{(i)}, \ldots, \delta_n^{(m)}\right)$ the route of n-th demand, $I_n \in I_k$, where I_k is a set of routes, $\delta_n^{(i)} = \left(\delta_n^{(i,1)}, \ldots, \delta_n^{(i,n_i)}\right)$, i = 2, …, m, and $\delta_n^{(i,j)} = 1$, if j-th location of i-th echelon is part of n-th demand route, and 0, otherwise.

If the known condition

$$\rho_i = \frac{\lambda_i}{\max\limits_{j=1,\ldots,n_i} \mu_{ij}} \leq 1, i = 1, \ldots, m\,, \tag{1}$$

holds and moreover

$$\mathrm{P}\left(\tau_1^{(1)} > \sum_{i=2}^{m}\left(\left(\max_{j=1,\dots,n_i} S_1^{(i,j)} + \sum_{j=1}^{n_m} v_1^{(j)}\right)\delta_n^{(i,j)} + z_1^{(i)}\right) + S_1^{(1)}\right) > 0, \tag{2}$$

then a positive recurrent renewal process of classical regeneration points of the system exists, and the system is empty at such instants. However, such points are generally too rare in real inventory systems, and the following weaker assumption

$$\mathrm{P}\left(\tau_1^{(1)} > S_1^{(1)}\right) > 0,\ \mathrm{P}\left(\tau_1^{(1)} > \max_{j=1,\dots,n_m}\left(\left(S_1^{(m,j)} + v_1^{(j)}\right)\delta_n^{(m,j)}\right)\right) > 0,$$

$$\mathrm{P}\left(\tau_1^{(1)} > \max_{j=1,\dots,n_i}\left(S_1^{(i,j)}\delta_n^{(i,j)}\right)\right) > 0,\ i = 2,\dots,m-1 \tag{3}$$

seems more suitable. Under assumptions (1), (3), one can construct weak regeneration points in which case a demand can cross the system without collisions.

Consider a regenerative stochastic process $X = \{X_n\}_{n\ge 1}$ with an increasing sequence of weak regeneration points $\beta = (\beta_1, \beta_2, \dots)$, and let $\beta_0 = 0$. The process *X* (or the pair (*X*, *β*)) is called weakly regenerative if the distribution of the post-process $\left((X_{\beta_{k+n}})_{n\ge 0}, (\beta_l - \beta_k)_{l\ge k}\right)$ does not depend on $k \ge 1$ and $\{\beta_1, \dots, \beta_{k-1}\}$. In this case, the embedded process *β* is still the renewal one that is regeneration cycle lengths $\{\beta_{k+1} - \beta_k = \alpha_k\}, k \ge 1$ are i.i.d. (as in classical case) while regeneration cycles $\{X_k, \beta_l \le k < \beta_{l+1}\}, l \ge 0$, are generally one-dependent.

Assume that a weak limit X_∞ of the basic regenerative process *X* exists. To estimate a stationary characteristic $r = Ef(X_\infty)$, we introduce the i.i.d. variables

$$Y_l = \sum_{k=\beta_l}^{\beta_{l+1}-1} f(X_k),\ U_l = Y_l - r\alpha_l,\ l \ge 1, \tag{4}$$

assuming $E(Y_1 + \alpha_1)^2 < \infty$. It is the weakest condition for confidence estimation based on regenerative approach to be held, [4]. Then the following well-known ratio $r = EY_1 / E\alpha_1$ holds.

In a classical regeneration case, the Central Limit Theorem (CLT) gives the following asymptotic $(1-2\gamma)\%$ confidence interval for *r*

$$\left[r_N \pm z_\gamma \frac{S(N)}{N^{1/2}\overline{\alpha}_N}\right] \tag{5}$$

where $r_N = \overline{Y}_N / \overline{\alpha}_N$, $\overline{\alpha}_N$ and $\overline{Y}_N$ are sample means of $\{\alpha_l\}$ and $\{Y_l\}$ respectively, z_γ is selected so that $\mathrm{P}(N(0,1) \le z_\gamma) = 1-\gamma$, and the sample variance $S^2(N) = \overline{Y}_N^2 + r_N^2\overline{\alpha}_N^2 - 2r_N\overline{S}_N^2 \to \sigma^2$, with probability 1 as $N \to \infty$, where

$$\sigma^2 = \mathrm{E}U_1^2 = \mathrm{Var}Y_1 + r^2\mathrm{Var}\alpha_1 - 2r\cdot\mathrm{cov}(Y_1, \alpha_1). \tag{6}$$

Weak regeneration in a system is usually reduced to the one - dependence, in which case,

$$\mathrm{E}(U_1U_2) = \mathrm{cov}(Y_1 - r\alpha_1, Y_2 - r\alpha_2) = \mathrm{cov}(Y_1, Y_2) - r\mathrm{cov}(\alpha_2, Y_1). \tag{7}$$

Thus, the CLT for the one-dependent variables gives us the following $(1-2\gamma)\%$ confidence interval for *r*

$$r_N \pm \frac{z_\gamma\sqrt{S^2(N) + 2(t_1(N) - r_N t_2(N))}}{\sqrt{N}\overline{\alpha}_N}, \tag{8}$$

where $t_1(N)$, $t_2(N)$ are standard estimates of $cov(Y_1, Y_2)$, and $cov(\alpha_2, Y_1)$, respectively, [3], [1]. Also with probability 1 as $N \to \infty$,

$$S^2(N) + 2\left(t_1(N) - r_N t_2(N)\right) \to \sigma^2, \tag{9}$$

where now $\sigma^2 = \mathrm{Var}U_1 + 2\mathrm{E}(U_1 U_2)$, [2].

An important problem which arises is simulation practice is how to identify efficiently weak regeneration points during simulation. Fortunately, one can indicate a wide class of the renovating events that allow us to do it for given inventory system.

Consider the process $\{W_n = (W_n^{(1)}, \ldots, W_n^{(m)})\}$, where $W_n^{(i)}$ is the current unfinished workload when n-th demand arrives at echelon i, $i = 1, \ldots, m$, $n \geq 1$.

Define the following events

$\Omega_n =$ {Demands n, n+1, … cross the inventory system along the given route without collision with demands 1, 2, …, n-1.} (10)

Define recursively the points

$$\beta_0 = 0;$$

$$\beta_{n+1} = \min\{k : t_k > \beta_n, \text{event } \Omega_{k-1} \text{ occured}\}, n \geq 0. \tag{11}$$

Then points $\{\beta_n^{(\alpha)}\}$ form a positive recurrent renewal process of weak regeneration points for the current workload process $W = \{W_n, n \geq 0\}$.

Unfinished current workload $W_n^{(i)}$ at each echelon i can be easily computed through well-known Kiefer-Wolfowitz recursion which for our case has the following form:

$$W_{n+1}^{(i)} = \left(W_n^{(i)} + \max_{j=1,\ldots,n_i}\left(S_n^{(i,j)}\delta_n^{(i,j)}\right) - \tau_n^{(i)}\right)^+, n \geq 0.$$

Numerical example is stated in the next part of the paper.

4. Numerical Results

Let us consider the example of simulation of inventory system with 3 echelons (m = 3, n_2 = 4, n_3 = 3) and 2 types of production. First route consists of retailer, locations 1, 2, 3 of second echelon and locations 1, 2 of third echelon. Second route consists of retailer, locations 3, 4 of second echelon and locations 2, 3 of third echelon. Times of production are the following:

- at the retailer $S_1^{(1)} = 3$ with probability 0.94 and 10 otherwise;
- for first type route: $S_1^{(2,1)} \sim U[6,8]$, $S_1^{(2,2)} \sim U[2,5]$, $S_1^{(2,3)} \sim U[4,8]$, $S_1^{(3,1)} \sim U[6,8]$, $S_1^{(3,2)} \sim U[2,5]$;
- for second type route: $S_1^{(2,3)} \sim U[4,8]$, $S_1^{(2,4)} \sim U[2,7]$, $S_1^{(3,2)} \sim U[4,5]$, $S_1^{(3,3)} \sim U[3,5]$.

In all simulations we use 900 demands. In Table 1 we present the frequency of classical and weak regeneration points and 95% confidence intervals for average waiting time of demands in system based on weak regeneration. This examples show that weak regeneration is more effective for estimation of waiting time of demands in the system. For example, for interarrival times with normal distribution N(8; 1) we have just 1 classical regeneration point and 575 weak regeneration points, it means that 64% of demands cross system without collisions with others.

Table 1. Cycle length and waiting time.

Interarrival times, τ_n	**Number of classical regeneration points**	**Number of weak regeneration points**	**Cycle length**	**95% confidence interval for $\overline{W}$**
N(6, 1)	1	11	254.34	[1.06; 2.51]
N(7, 1)	1	243	28.82	[0.3; 0.4]
N(8, 1)	1	575	12.4	[0.08; 0.12]
N(11.5, 1)	4	845	12.3	[0.008; 0.015]
N(12, 1)	21	853	12.6	[0.006; 0.012]

This example shows that weak regeneration gives reliable estimates for average values of *TAC* in real-time simulation.

5. Conclusions

In the present paper authors represent a multi-echelon production system as a queuing system. Stochastic processes describing the behaviour of this system and its regenerative structure are observed in this article. The weak regeneration approach for the inventory control system simulation and estimation of characteristics is used. The weak regeneration points are described (such points a demand arrives in empty echelon). Weak regeneration cycle length and 95% confidence intervals for average waiting times of demands are estimated by using the proposed approach.

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