SIMULATION OF ECONOMIC PROCESES USING MULTIDIMENTIAL DISTRIBUTIONS

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ABSTRACT

It is often argued that financial asset returns are the cumulative outcome of a vast number of pieces of information and individual decisions arriving almost continuously in time. As such, they have been modeled by the Gaussian distribution. The strongest statistical argument for it is based on the Central Limit Theorem, which states that the sum of a large number of independent, identically distributed variables from a finite-variance distribution will tend to be normally distributed. However, financial asset returns usually have usually fat tails. Also in practice, models based on multivariate normal assumption (the so-called mean-variance approach) are widely used for their simplicity. An example of its application is the so-called parametric methods. More precisely, basic asset (stock, exchange rate, interest rate) returns are assumed to be jointly normal, i.e.: $r_t \sim N(0,\Sigma)$, where S is the covariance matrix which may be historically estimated over time. As a consequence any portfolio returns $\omega^T r_t \sim N(0,\sigma)$ is also normally distributed with mean 0 and variance $\sigma^2 = \omega^T \Sigma \omega$. Thus under this assumption many important risk measures such as Value at Risk (VaR) may have simple analytical expressions.

However, several extreme events (financial collapses, technologic catastrophes, aviaries ...) in the second half of last century have made financial practitioners worried more about the tails of their portfolio returns and the joint normal specification fails to provide a good approximation for the tails in general. Correlation coefficient is a limited measure. In the real world, there is often a non-linear dependence between different variables and correlation cannot be an appropriate measure of co-dependency. Therefore linear Spearmen's correlation coefficient is a limited measure of dependence. It is not surprising that alternative methods (the copula method) for capturing co-dependency have been considered. The concept of copulas comes from Sklar in 1959. A copula is a multivariate distribution function with uniform (0,1) marginals. Copulas are used to combine different marginal distributions. They are unique, if marginal distributions are continuous, and like dependence measures use Kendall's tau and Spearman's rho, which are invariant under strictly increasing transformation. Therefore copulas have become a powerful tool for modeling dependence between random variables. Also copula methodology is effective for modeling joint distributions with fat tails. Fat tails in financial return data have been documented in numerous real cases. Joint distribution on financial data returns is very important issue in derivative pricing, risk management and portfolio allocations.

The received theoretical and practical results of work can be used in practical financial activities of the industrial enterprises, using standard programs of modeling.

Normal assumption of asset returns. It is often argued that financial asset returns are the cumulative outcome of a vast number of pieces of information and individual decisions arriving almost continuously in time. As such, they have been modeled by the Gaussian distribution. The strongest statistical argument for it is based on the Central Limit Theorem, which states that the sum of a large number of independent, identically distributed variables from a finite-variance distribution will tend to be normally distributed. However, financial asset returns usually have usually fat tails. Also in practice, models based on multivariate normal assumption (the so-called mean-variance approach) are widely used for their simplicity. An example of its application is the so-called parametric methods. More precisely, basic asset (stock, exchange rate, interest rate) returns are assumed to be jointly normal, i.e.: $r_t \sim N(0,\Sigma)$, where Σ is the covariance matrix which may be historically estimated over time. As a consequence any portfolio returns ù ʳ r_t ~ *N(0, σ)* is also normally distributed with mean 0 and variance $\sigma^2 = \omega^T \Sigma \omega$. Thus under this assumption many important risk measures such as Value at Risk (VaR) may have simple analytical expressions.

Expected Returns are assumed to be zero since that it is found to be very hard to predict even the sign of return when horizon is more than 3 months. Σ is assumed to be constant over the period considered under the static hypothesis. In practice often use an exponentially weighted moving average (EWMA) to estimate the volatility $\sigma_t^2 = \frac{1-\lambda}{1-\lambda^{m+1}} \sum_{i=0}^{m} \lambda^i r_{t-i}^2$, or, $\sigma_t^2 = \lambda \sigma_t^2 + (1-\lambda) r_{t-1}^2$ where λ is the decay factor with an estimated value 0.94 for daily data.

However, several extreme events (financial collapses, technologic catastrophes, aviaries …) in the second half of last century have made financial practitioners worried more about the tails of their portfolio returns and the joint normal specification fails to provide a good approximation for the tails in general.

Correlation coefficient is a limited measure. Recently, a lot of works have devoted to the modeling of the joint distribution [8], [9] and to see how these models could be better than the joint normal specification, for representing the tail behavior. In the real world, there is often a non-linear dependence between different variables and correlation cannot be an appropriate measure of co-dependency. Therefore linear Spearmen's correlation coefficient is a limited measure of dependence. It is not surprising that alternative methods (the copula method) for capturing co-dependency have been considered. The concept of copulas comes from Sklar [5] in 1959. In rough terms, a copula is a function

$$C : [0,1]^n \to [0,1],$$

with certain special properties. Alternatively we can say that it is a multivariate distribution function defined on the unit cube$[0,1]^n$. Copula functions are well studied object in the statistical literature. These functions have been introduced to model a joint distribution once the marginal distributions are known. When multivariate normal distribution is rejected by data, the copula may be used as an important alternative to represent the dependence in joint distributions.

Copulas. A copula is a multivariate distribution function with uniform (0,1) marginals. Copulas are used to combine different marginal distributions. They are unique, if marginal distributions are continuous, and like dependence measures use Kendall's tau and Spearman's rho, which are invariant under strictly increasing transformation. Therefore copulas have become a powerful tool for modeling dependence between random variables. Also copula methodology is effective for modeling joint distributions with fat tails. Fat tails in financial return data have been documented in numerous real cases. Joint distribution on financial data returns is very important issue in derivative pricing, risk management and portfolio allocations.

Tails dependence meaningful measure of dependence. For example, real portfolio examples clearly show that we must use heavy tailed alternatives to the Gaussian law in order to obtain acceptable estimates of market losses. But can we substitute the Gaussian distribution with other distributions in Value at Risk (Expected Shortfall) calculations for whole portfolios of assets? Remember, that the definition of VaR utilizes the quantiles of the portfolio returns distribution and not the returns distribution of individual assets in the portfolio. If all asset return distributions are assumed to be Gaussian then the portfolio distribution is multivariate normal and we can apply well known statistical tool. However, when asset returns are distributed according to different laws then the multivariate distribution may not be multivariate normal. In particular, linear correlation may no longer be a meaningful measure of dependence.

In such cases multivariate statistics offers the concept of copulas. The technical definitions of copulas that can be found in the literature often look more complicated, but to a financial modeling, this definition is enough to build an intuition from. What is important for VaR calculations is that a copula enables us to construct a multivariate distribution function from the marginal (possibly different) distribution functions of n individual asset returns in a way that takes their dependence structure into account. This dependence structure may be no longer measured by correlation, but by other adequate functions like rank correlation and, especially, tail dependence [7]. Moreover, it can be shown that for every multivariate distribution function there exists a copula which contains all information on dependence. For example, if the random variables are independent, then the independence copula (also known as the product copula) is just the product of n variables $C(u_1,\ldots,u_n)=u_1 \ldots u_n$.

Elliptical and Archimedean copulas. Copula functions do not impose any restrictions on the financial model, so in order to reach a model that is to be useful in a given risk management problem, a particular specification of the copula must be chosen. From the wide variety of copulas that exist probably the elliptical and Archimedean copulas are the ones most often used in applications. Elliptical copulas are simply the copulas of elliptically contoured (or elliptical) distributions, e.g. (multivariate) normal, t, symmetric stable.

Restrictions of elliptical copulas for using in risk management. Rank correlation and tail dependence coefficients can be easily calculated for elliptical copulas. There are, however, drawbacks - elliptical copulas do not have closed form expressions, are restricted to have radial symmetry and have all marginal distributions of the same type.

```
Numbers(N, μ, Cov) := | T ← cholesky(Cov)
                      | A ← μ
                      | z ← rnorm(2·N, 0, 1)
                      | for i ∈ 1 .. N
                      |   | Z ← μ + T·( z_i ; z_{i+N} )
                      |   | A ← augment(Z, A)
                      | A
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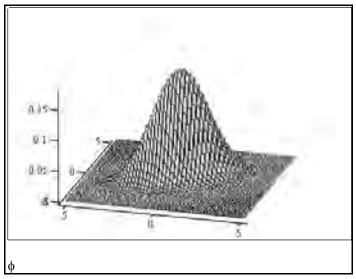
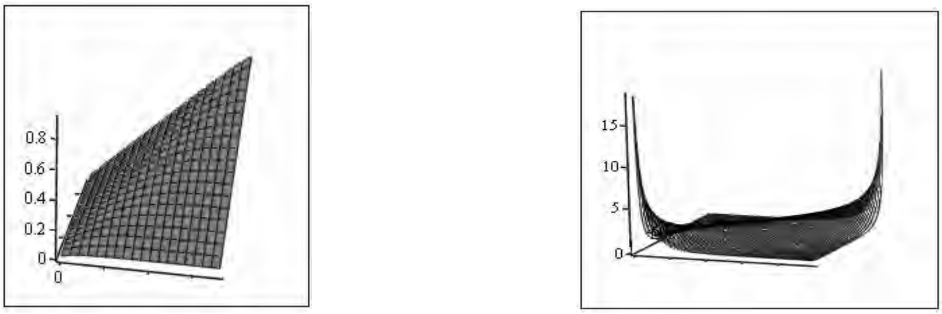


The algorithm (MathCad 2001) for calculating density of the bivariate Gaussian distribution with known covariation matrix.



The bivariate Gaussian distribution and density of the bivariate Gaussian copula with the same covariation matrix.

These restrictions may disqualify elliptical copulas from being used in some risk management problems. In particular, there is usually a stronger dependence between big losses (e.g. market crashes) than between big gains. Clearly, such asymmetries cannot be modeled with elliptical copulas. In contrast to elliptical copulas, all commonly encountered Archimedean copulas have closed form expressions. Their popularity also comes from the fact that they allow for a great variety of different dependence structures. Many interesting parametric families of copulas are Archimedean, including the well known Clayton, Frank and Gumbel copulas.

After the marginal distributions of asset returns are estimated and a particular copula type is selected, the copula parameters have to be estimated. The fit can be performed by least squares or maximum likelihood. While analytical methods for the computation of VaR exist for the multivariate normal distribution (i.e. for the Gaussian copula), in most other cases we have to use Monte Carlo simulations. A general technique for random variable generation from copulas is the conditional distributions method.

Conditional distributions methodfor copula modeling. A random vector $(u_1,\ldots,u_n)^T$ having a joint distribution function C can be generated by the following algorithm:

1) simulate $u_1 \sim U(0, 1)$;

2) for k = 2, ..., n simulate $u_k \sim C_k(\cdot \mid u_1, \ldots, u_{k-1})$.

The function $C_k(\cdot \mid u_1, \ldots, u_{k-1})$ is the conditional distribution of the variable U_k given the values of $U_1, \ldots, U_{k-1}$. Copulas allow us to construct economical (financial) models which go beyond the standard notions of correlation and multivariate Gaussian distributions.

For explanatory purpose we focus on the two variables (assets) case.

Definition of a two-dimensional copula. A two-dimensional copula is a two-dimensional distribution function C with uniformly distributed marginals U(0, 1) on [0,1]. Thus a copula is a function C: $[0,1]^2 \to [0,1]$ satisfying the following three properties:

1. For every $u, v \in [0,1]$:

$$C(u,0) = C(0,v) = 0\ ,\ C(u,1) = u$$

and $C(1,v) = v$.

2. $C(u,v)$ is increasing in u and v.

3. For every $u_1, u_2, v_1, v_2 \in [0,1]$ with $u_1 \le u_2$ and $v_1 \le v_2$ we have:

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0\ .$$

Condition 1 provides the restriction for the support of the variables and the marginal uniform distribution. Conditions 2 and 3 correspond to the existence of a nonnegative “density” function.

The most useful results of copula theory are

Sklar's theorem and Frйchet bounds [6].

Sklar's theorem - copula's first definition. Let F be a joint multivariate distribution with marginals F_1 and F_2. Then, for any x_1, x_2 there exists a copula C such that

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)) \quad (1)$$

Furthermore, if marginals F_1 and F_2 are continuous, the copula C is unique. Conversely, if F_1 and F_2 are marginal distributions and C is a copula, then the function F defined by $C(F_1(x_1), F_2(x_2))$ is a joint distribution function with marginals F_1 and F_2. If we have a random vector X = (X_1, X_2) the copula of their joint distribution function may be extracted from equation (1):

$$C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2)),$$

where the F_1^{-1}, F_2^{-1} are the quantile functions of the margins.

Sklar's theorem provides a decomposition of the joint distribution into marginal features (that are F_1 and F_2) and dependence features (represented by copula C). The two variables X and Y are independent if and only if F(X) and G(Y) are independent. The independence condition can be written in terms of copula as C(u, v) = uv. When $C(u,v) \neq uv$, the variables X and Y (or F(X), G(Y)) are dependent and the dependence summarized in the copula depends on the variables up to (nonlinear) increasing transformation of the variables. It is important to see if they are more or less dependent, and the "sign" of the dependence. The comparison of dependence can be based on the usual first order dominance stochastic ordering applied to copula. The Fréchet bounds provide the minimal and maximal elements in this comparison.

In most financial cases we can effectively use Archimedean copulas. The Archimedean copulas provide analytical tractability and a large spectrum of different dependence measure. These copulas can be used in a wide range of applications for the following reasons:

- the simplicity with which they can be constructed;
- the many parametric families of copulas belonging to this class;
- the great variety of different dependence structures.

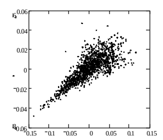
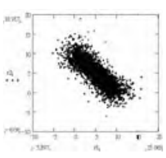
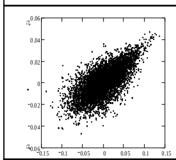
An Archimedean copula can be de?ned as follows:

Defnition of Archimedean copula's generator. Let us consider a function $\varphi:[0;1] \to [0;1]$ which is continuous, strictly decreasing, convex and for which $\varphi(0) = \infty$ and $\varphi(1) = 0$. We then define the pseudo inverse of $\varphi^{[-1]}$: [0; ∞] $\to$ [0;1] such that:

$$\varphi^{[-1]}(t) = \begin{Bmatrix} \varphi^{-1}(t) & 0 \le t \le \varphi(0) \\ 0 & \varphi(0) \le t \le \infty \end{Bmatrix}.$$

As φ is convex, the function C: $[0;1]^2 \to [0;1]$ defined as $C(u_1, u_2) = \varphi^{-1}[\varphi(u_1) + \varphi(u_2)]$ is an Archimedean copula and φ is called the generator of the copula [9].

In case of the multivariate extension for all $n \geq 2$, the function C: $[0;1]^n \to [0;1]$ defined as $C(u_1, ..., u_n) = \varphi$-1 $[\varphi(u_1) + . . . + \varphi(u_n)]$, is an n-dimensional Archimedean copula if and only if φ^{-1} is completely monotone on $[0, \infty)$.

Picture (examples)	Formula for Archimedean copula	Kendall's Tau
	**Clayton copula** $C(u_1,u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}$ $\varphi(t) = \frac{t^{-\theta} - 1}{\theta}$, $\theta \in (0, ?)$	$\tau(\theta) = \frac{\theta}{2+\theta}$
	**Frank copula** $C(u,v) = -\theta^{-1} \ln(1 + \frac{g(u) \cdot g(v)}{g(1)})$ $g(z) = e^{-\theta z} - 1$, $\varphi_\theta(t) = -\ln \frac{g(t)}{g(1)}$	$\tau(\theta) = 1 - \frac{1}{\theta}$
	**Gumbel copula** $C(u_1,u_2) = \exp\left\{-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{1/\theta}\right\}$ $\varphi(t) = (-\ln t)^\theta$, $\theta ? 1$	$\tau(\theta) = 1 - \frac{4}{\theta} + \frac{4}{\theta^2} \int_0^\theta \frac{t}{e^t - 1} dt$

Dependence

Since copulas are used to represent the dependence structure among the variables when margins are known or well estimated, it is useful to describe the basic dependence measures, which can be used to interpret the parameters appearing in parametric copula families.

Pearson's linear correlation. Pearson's linear correlation coefficient is the most frequently used measure of dependence. It is defined by:

$$cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}.$$

This measure can be applied to the basic variables X, Y themselves. In this case, it depends on both the marginal distributions and the copula. Since the copula of a multivariate distribution describe its dependence structure, it might be appropriate to use measures of dependence which are copula-based. The bivariate concordance measure Kendall's tau (t) and Spearman's rho (r), as well as the coefficient of tail dependence, can, as opposed to the linear correlation coefficient, be expressed in terms of the underlying copula alone.

Kendall's tau – dependency measure. Kendall's tau of two variables X and Y is (in terms of copulas fuctions):

$$\tau(X,Y) = 4\int_{[0,1]^2} C(u,v)dC(u,v) - 1,$$

where C(u, v) is the copula of the bivariate distribution function of X and Y. For the Gaussian and Student's t-copulas and also all other elliptical copulas, the relationship between the linear correlation coefficient cor(X, Y) and Kendall's tau is given by

$$cor(X,Y) = \sin\left(\frac{\pi}{2}\tau\right) \qquad (2)$$

For an Archimedean copula ф(X, Y) can be evaluated directly from the generator of the copula:

$$\tau(X,Y) = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt$$

Also for Archimedean copulas, Kendall's tau can be related to the dependence parameter. For the Clayton copula it is given by

$\tau(X,Y) = \frac{\delta}{\delta+2}$, and for the Gumbel copula it is $\tau(X,Y) = 1 - \frac{1}{\delta}$.

For the Gaussian and Student's t-copulas, Kendall's tau must be estimated empirically.

Spearman's correlation. The concept of linear correlation can also be applied to transformation of the basic variables. For instance, we can consider the correlation between the ranking associated with X and Y, that are U = F(X), V = G(Y). The correlation between the ranks, so-called Spearman's rho, is $S_\rho = Cor(U,V) = Cor(F(X),G(Y))$. It depends on the copula only and is given by:

$$S_\rho(X,Y) = 12\int_{[0,1]^2} C(u,v)dudv - 3$$

where C(u, v) is the copula of the bivariate distribution function of X and Y. Let X and Y have distribution functions F and G, respectively. Then, we have the following relationship between Spearman's rho and the linear correlation coefficient

$$S_\rho(X,Y) = cor(F(X),F(Y)).$$

For the Gaussian and Student's t-copulas, we have that the relationship between the linear correlation coefficient and Spearman's rho is

$$cor(X,Y) = 2\sin\left(\frac{\pi}{6}S_\rho\right).$$

Both $\tau(X,Y)$ and $S_\rho(X,Y)$ may be considered as measures of the degree of the monotonic dependence between X and Y, whereas linear correlation measures the degree of linear dependence only. Moreover, these measures are invariant under monotone transformations, while the linear correlation generally isn't.

Tail dependence. There is saying in finance that in times of stress, correlations will increase. Bivariate tail dependence measures the amount of dependence in the upper and lower quadrant tail of a bivariate distribution. This is of great for risk manager trying to guard against concurrent bad events. Let $X \sim F_X$ and $Y \sim F_Y$. By definition, the upper tail dependence coefficient is [7],

$$\lambda_u(X,Y) = \lim_{\alpha \to 1} P\ Y > F_Y^{-1}\ \alpha\ |X > F_X^{-1}\ \alpha$$

and quantifies the probability to observe a large Y, assuming that X is large. Analogously, the coefficient of lower tail dependence is

$$\lambda_l(X,Y) = \lim_{\alpha \to 0} P\left(Y \le F_Y^{-1}(\alpha) | X \le F_X^{-1}(\alpha)\right).$$

These measures are independent of the marginal distributions of the asset returns. Moreover, they are invariant under strictly increasing transformations of the X and Y. For elliptical distributions, $\lambda_u(X,Y) = \lambda_l(X,Y)$. If $\lambda_u(X,Y) > 0$, large events tend to occur simultaneously. On the contrary, when $\lambda_u(X,Y) = 0$, the distribution has no tail dependence, and the variables X and Y are said to be asymptotically independent. It is important to note that while independence of X and Y implies $\lambda_u(X,Y)=\lambda_l(X,Y)=0$, the converse is not true in general. That is $\lambda_u(X,Y)=\lambda_l(X,Y)=0$, does not necessarily imply that X and Y are statistically independent. Thus asymptotic independent should be considered as the "weakest dependence which can be quantified by the coefficient of the tail dependence" [7].

Gaussian copula. For the Gaussian copula, the coefficients of lower tail and upper tail dependence are $\lambda_u(X,Y)=\lambda_l(X,Y)=2\lim_{x\to-\infty}\Phi\left(x\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)=0$.

Where Φ is Laplase function. This means, that regardless of high correlation r we choose, if we go far enough into the tail, extreme events appear to occur independently in X and Y.

The Student's t-copula. For the Student's t-copula, the coefficients of lower tail and upper tail dependence are

$\lambda_u(X,Y)=\lambda_l(X,Y)=2t_{v+1}\left(-\sqrt{v+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$, where t_{v+1} denotes the distribution function of a univariate Student's t-distribution with v + 1 degrees of freedom. The stronger the linear correlation r and the lower the degrees of freedom v, the stronger is the tail dependence.

The Clayton copula. The Clayton copula is lower tail dependent. That is, the coefficient of the upper tail dependence $\lambda_u(X,Y)=0$, and the coefficient of the lower tail dependence is $\lambda_l(X,Y)=2^{1/\delta}$.

The Gambel copula. The Gambel copula is upper tail dependence . That is, the coefficient of the lower tail dependence $\lambda_l(X,Y)=0$ and the coefficient of the upper tail dependence is $\lambda_u(X,Y)=2-2^{1/\delta}$.

The coefficient of tail dependence seems to provide a useful measure of the extreme dependence between two random variables. However, a difficult problem remains unsolved, namely to estimate the tail dependence for an empirical data. One alternative is to use a parametric approach. For instance, choose to model dependence with the Gumbel copula and determine the associated tail dependence.

In most cases

Example of copula (Gaussian) modeling (using program MathCad 2001).

The first step is empirical estimation of Kendall's tau using formula:

$$\hat{\tau}(X,Y)=\binom{n}{2}^{-1}\cdot\sum_{i<j} sign[(X_i-X_j)\cdot(Y_i-Y_j)].$$

For Gaussian copula from known linear correlation coefficient $\hat{\tau}$, we can define linear correlation between X and Y from equation (2):

$$cor(X,Y)=\sin\left(\frac{\pi}{2}\hat{\tau}\right).$$

After then we can construct covariance matrix Cov and algorithm for Gaussian copula modeling is:

- simulate $V = N(0,Cov)$;
- set $U = (\Phi^{-1}(V_1),\Phi^{-1}(V_2))$.

For generating random variables *(u,v)* whose joint distribution is an Archimedean copula with generator φ in Ω we can use such algorithm [6]:

Algorithm:

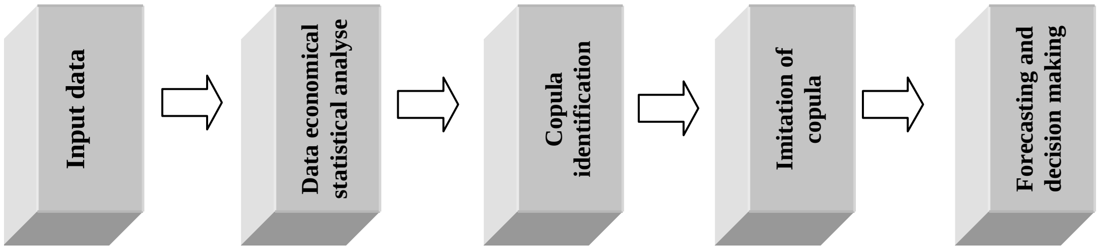
- Generate two independent uniform (0,1) variables *s* and *q*;
- Set $t = K_C^{-1}(q)$, where $K_C^{-1}(q)$ denotes the quasi-inverse of the distribution function K_C;
- Set $u = \varphi^{-1}(s\cdot\varphi(t))$ and $v=\varphi^{-1}[(1-s)\cdot\varphi(t)]$;
- The desired pair is *(u,v)*.

In this algorithm $K_C(t)=t-\frac{\varphi(t)}{\varphi'(t^+)}$.

For example see this algorithm realization for Gambel copula (for n pairs (u,v) and copula's parameter q) in programm MathCad 2001:

```
Gambel(n,θ) := | for i ∈ 1..n
               |   s ← rnd(1)
               |   q ← rnd(1)
               |   t ← 0.5
               |   t_i ← root(t − t·ln(t)/θ − q, t)
               |   u_i ← (t_i)^(s^(1/θ))
               |   v_i ← (t_i)^[(1−s)^(1/θ)]
               | M ← augment(u,v)
```

Fragments of the algorithm for Gambel copula in MathCad.



The common scheme of economical systems analysis with copulas.

Conclusion

The modern economic analysis basing on the using of information technologies, shows that in the real systems the parameters describing the economic objects, not always have the Gauss distribution. The unlinear dependence exists between various factors. In this cases it is impossible to use the linear correlation coefficient for evaluation of measure of dependences between factors. It requires to use another methods for evaluation the measure of dependences between factors.

In our days, designing real economic systems [8], [9], very much popular is becoming the use of copulas, which fully characterizes the unlinear connection between main factors of the model and allows to unite margin functions into multivariate distribution function.

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