Integral Representation of the Solution to the Vector Helmholtz Equation in the System of Arbitrary Orthogonal Curvilinear Coordinates

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Abstract: In the integral representation of the solution to the vector Helmholtz equation known in literature the electromagnetic field vector potential is expressed in terms of a triple integral of multiplication of current density vector and fundamental solution of the scalar Helmholtz equation. This representation has the simplest form in the rectangular coordinate system, in which the unit vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$ do not depend on coordinates. In the present paper the integral representation of the solution to the vector Helmholtz equation is obtained for the system of arbitrary orthogonal curvilinear coordinates, in which the unit vectors $\vec{e}_{q1}, \vec{e}_{q2}, \vec{e}_{q3}$ are prescribed functions of coordinates. As particular cases of the representation obtained, the integral representations of the solution to the vector Helmholtz equation are found for the systems of cylindrical and spherical coordinates.

Key-Words: Vector potential, Helmholtz equation, Integral representation, Lame coefficient

1 Introduction

The Helmholtz equation for the vector potential used in electrodynamics has the form

$$\Delta \vec{A} + k^2 \vec{A} = -\mu_0 \mu \vec{I}^{\,e}, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \qquad (1)$$

where $k^2 = \mu_0 \varepsilon_0 \mu \varepsilon \omega^2$, $\vec{I}^{\,e} = \vec{I}^{\,e}(M)$ is the external current vector density; ε_0 and μ_0 are the electric and magnetic constants, respectively; ε and μ are the relative permittivity and relative magnetic permeability of the medium, respectively.

Vectors $\vec{A}(M)$ and $\vec{I}^{\,e}(M)$ in the system of Cartesian coordinates has the form

$$\vec{A}(M) = A_x(M)\vec{e}_x + A_y(M)\vec{e}_y + A_z(M)\vec{e}_z, \qquad (2)$$

$$\vec{I}^{\,e}(M) = I_x^e(M)\vec{e}_x + I_y^e(M)\vec{e}_y + I_z^e(M)\vec{e}_z. \qquad (3)$$

Consider the integral representation of Helmholtz equation in the vector form for a point $M(x, y, z)$ situated in the region, where the external current vector $\vec{I}^{\,e} = 0$ (see [1], page 322):

$$\vec{A}(M) = \frac{\mu_0 \mu}{4\pi} \iiint_V \vec{I}^{\,e}(\tilde{M}) \frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}} d\tilde{V}, \qquad (4)$$

where the integration is performed over the points $\tilde{M}(\tilde{x}, \tilde{y}, \tilde{z}) \in V$, in which $\vec{I}^{\,e} \neq 0$; $\mu = 1$, $\varepsilon = 1$ since the wire is situated in free space, $r_{M\tilde{M}}$ is the distance between the points $M(x, y, z)$ and $M(\tilde{x}, \tilde{y}, \tilde{z})$:

$$r_{M\tilde{M}} = \sqrt{(x - \tilde{x})^2 + (y - \tilde{y})^2 + (z - \tilde{z})^2}. \qquad (5)$$

Conditions for all the components, A_x, A_y, A_z, of the vector potential at infinity are so-called Sommerfeld's conditions of radiation (see [2], page 509):

$$R^2 = r^2 + z^2 \to \infty : A = O\left(\frac{1}{R}\right), \ \frac{\partial A}{\partial R} + jkA = o\left(\frac{1}{R}\right), \qquad (6)$$

where symbol $O(1/R)$ denotes that A and $1/R$ are infinitesimal of the same order at $R \to \infty$, but symbol $o(1/R)$ denotes that $\partial A / \partial R + jkA$ is infinitesimal of higher order than $1/R$ at $R \to \infty$.

It can be easy verified that if the functions $I_x^e(M)$, $I_y^e(M)$, $I_z^e(M)$ are continuous in some closed region V and, consequently, they are bounded in this region, then the vector function $\vec{A}(M)$ in formula (4) satisfy Sommerfeld's conditions (5). Consequently, in this case, formula (4) gives the solution to the problem (1), (6) under the condition that the vector function $\vec{I}^{\,e}(M)$ is prescribed.

In the Cartesian coordinate system, the unit vectors $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ are constant. Therefore, in this case, according to formula (4), the each component of the vector $\vec{A}$ is expressed in terms of a triple integral of a corresponding component of the vector $\vec{I}^{\,e}$ (i.e. A_x in terms of I_x^e, A_y in terms of I_y^e and so on). For example,

$$A_x(M)=\frac{\mu_0\mu}{4\pi}\iiint\limits_V I_x^e(\tilde M)\frac{\exp(-jkr_{M\tilde M})}{r_{M\tilde M}}d\tilde V\,, \qquad (7)$$

and so on. However, in the all other orthogonal curvilinear coordinate systems, the unit vectors are already the functions of coordinates. For example, in the system of cylindrical polar coordinates (r,φ,z), vector $\vec I^{\,e}(\tilde M)$ has the form

$$\vec I^{\,e}(\tilde M)=I_r^e(\tilde M)\vec e_r+I_\varphi^e(\tilde M)\vec e_\varphi+I_z^e(\tilde M)\vec e_z\,, \qquad (8)$$

and the only unit vector $\vec e_z$ is constant. In this case, the equality

$$\iiint\limits_V I_r^e(\tilde M)\vec e_r(\tilde M)\frac{\exp(-jkr_{M\tilde M})}{r_{M\tilde M}}d\tilde V=$$

$$=\iiint\limits_V I_r^e(\tilde M)\frac{\exp(-jkr_{M\tilde M})}{r_{M\tilde M}}d\tilde V\cdot\vec e_r(\tilde M) \qquad (9)$$

and the similar equality for $I_\varphi^e(\tilde M)\cdot\vec e_\varphi(\tilde M)$ are wrong. Consequently, in this case, the components $A_r(M)$ and $A_\varphi(M)$ has the form of a triple integral of some linear combination of components $I_r^e(\tilde M)$ and $I_\varphi^e(\tilde M)$. In authors' papers [3] and [4] problem (1), (6) has been solved by using the cylindrical polar coordinates as follows. At first, triple integral (7) is transformed into the single integral in the Cartesian coordinates and only, at second, the passing to the cylindrical polar coordinates is performed in the solution obtained.

In this paper, the integral representation of the solution to the vector Helmholtz equation (4) is obtained in the system of cylindrical polar, spherical and also arbitrary orthogonal curvilinear coordinates. Components of the vector $\vec A(M)$ are expressed in terms of triple integrals of the linear combinations of the components of the current vector density $\vec I^{\,e}(\tilde M)$.

2 Integral Representation in the System of Cylindrical Polar Coordinates

In the system of cylindrical polar coordinates (r,φ,z), vectors $\vec A(M)=\vec A(r,\varphi,z)$ and $\vec I^{\,e}(\tilde M)=\vec I^{\,e}(\tilde r,\tilde\varphi,\tilde z)$ have the form

$$\vec A(M)=A_r(M)\vec e_r(M)+A_\varphi(M)\vec e_\varphi(M)+A_z(M)\vec e_z\,, \qquad (10)$$

$$\vec I^{\,e}(\tilde M)=I_r^e(\tilde M)\vec e_r(\tilde M)+I_\varphi^e(\tilde M)\vec e_\varphi(\tilde M)+I_z^e(\tilde M)\vec e_z\,, \qquad (11)$$

the components $A_r(M)$, $A_\varphi(M)$ can be expressed in terms of the components $A_x(M)$, $A_y(M)$ as

$$A_r=A_x\cos\varphi+A_y\sin\varphi\,, \qquad (12)$$

$$A_\varphi=-A_x\sin\varphi+A_y\cos\varphi\,. \qquad (13)$$

The components $I_x^e(\tilde M)$, $I_y^e(\tilde M)$ can be expressed in terms of the components $I_r^e(\tilde M)$ and $I_\varphi^e(\tilde M)$:

$$I_x^e(\tilde M)=I_r^e(\tilde M)\cos\tilde\varphi-I_\varphi^e(\tilde M)\sin\tilde\varphi\,, \qquad (14)$$

$$I_y^e(\tilde M)=I_r^e(\tilde M)\sin\tilde\varphi+I_\varphi^e(\tilde M)\cos\tilde\varphi\,. \qquad (15)$$

It follows from (4) that

$$A_x(M)=\frac{\mu_0\mu}{4\pi}\iiint\limits_V I_x^e(\tilde M)\Phi(M,\tilde M)\,d\tilde V\,, \qquad (16)$$

$$A_y(M)=\frac{\mu_0\mu}{4\pi}\iiint\limits_V I_y^e(\tilde M)\Phi(M,\tilde M)\,d\tilde V\,, \qquad (17)$$

where

$$\Phi(M,\tilde M)=\frac{\exp(-jkr_{M\tilde M})}{r_{M\tilde M}}\,, \qquad (18)$$

and the distance, $r_{M\tilde M}$, between the points M and $\tilde M$ is defined by formula (5). It follows from (5) by substituting

$$x=r\cos\varphi,\ y=r\sin\varphi,\ z=z;$$
$$\tilde x=\tilde r\cos\varphi,\ \tilde y=\tilde r\sin\tilde\varphi,\ \tilde z=\tilde z \qquad (19)$$

that

$$r_{M\tilde M}=\sqrt{r^2+\tilde r^2-2r\tilde r\cos(\varphi-\tilde\varphi)+(z-\tilde z)^2}\,. \qquad (20)$$

It follows from (12) by substituting expression (16) for $A_x(M)$ and expression (17) for $A_y(M)$ that

$$A_r(M)=\frac{\mu_0\mu}{4\pi}\iiint\limits_V I_x^e(\tilde M)\Phi(M,\tilde M)\,d\tilde V\cdot\cos\varphi+$$
$$+\frac{\mu_0\mu}{4\pi}\iiint\limits_V I_y^e(\tilde M)\Phi(M,\tilde M)\,d\tilde V\cdot\sin\varphi\,. \qquad (21)$$

Substituting the expression (14) for $I_x^e(\tilde M)$ and (15) for $I_y^e(\tilde M)$ into (21) yields

$$A_r(M)=\frac{\mu_0\mu}{4\pi}\iiint\limits_V [I_r^e(\tilde M)\cos\tilde\varphi-I_\varphi^e(\tilde M)\sin\tilde\varphi]\times$$
$$\times\Phi(M,\tilde M)\,d\tilde V\cdot\cos\varphi+$$
$$+\frac{\mu_0\mu}{4\pi}\iiint\limits_V [I_r^e(\tilde M)\sin\tilde\varphi+I_\varphi^e(\tilde M)\cos\tilde\varphi]\times$$
$$\times\Phi(M,\tilde M)\,d\tilde V\cdot\sin\varphi\,. \qquad (22)$$

The final expression for the component $A_r(M)$ can be easily obtained from (22) by performing some elementary transforms, and it has the form

$$A_r(M)=\frac{\mu_0\mu}{4\pi}\iiint\limits_V [I_r^e(\tilde M)\cos(\varphi-\tilde\varphi)+$$

$$+ I_\varphi^e(\tilde{M})\sin(\varphi-\tilde{\varphi})]\Phi(M,\tilde{M})\,d\tilde{V}\,, \qquad (23)$$

where

$$d\tilde{V} = \tilde{r}\,d\tilde{r}\,d\tilde{\varphi}\,d\tilde{z}\,. \qquad (24)$$

The expression for the component $A_\varphi(M)$ is obtained by performing the similar transforms and by using formulae (13) for $A_\varphi(M)$, (16) for $A_x(M)$, (17) for $A_y(M)$, (14) for $I_x^e(\tilde{M})$ and (15) for $I_y^e(\tilde{M})$. It has the form

$$A_\varphi(M) = \frac{\mu_0\mu}{4\pi}\iiint_V [I_r^e(\tilde{M})\sin(\varphi-\tilde{\varphi}) + + I_\varphi^e(\tilde{M})\cos(\varphi-\tilde{\varphi})]\Phi(M,\tilde{M})\,d\tilde{V}\,. \qquad (25)$$

The component $A_z(M)$ has the same form as in the Cartesian coordinates:

$$A_z(M) = \frac{\mu_0\mu}{4\pi}\iiint_V I_z^e(\tilde{M})\,\Phi(M,\tilde{M})\,d\tilde{V}\,. \qquad (26)$$

Thus, formulae (10), (23), (25) and (26) give the integral representation of the solution to the vector Helmholtz equation (4) in the cylindrical polar coordinates.

3 Integral Representation in the System of Spherical Coordinates

In the system of spherical coordinates (ρ, θ, φ) vectors $\vec{A}(M) = \vec{A}(\rho,\theta,\varphi)$ and $\vec{I}^{\,e}(\tilde{M}) = \vec{I}^{\,e}(\tilde{\rho},\tilde{\theta},\tilde{\varphi})$ have the form

$$\vec{A}(M) = A_\rho(M)\vec{e}_\rho(M) + A_\theta(M)\vec{e}_\theta(M) + + A_\varphi(M)\vec{e}_\varphi(M)\,, \qquad (27)$$

$$\vec{I}^{\,e}(\tilde{M}) = I_\rho^e(\tilde{M})\vec{e}_\rho(\tilde{M}) + I_\theta^e(\tilde{M})\vec{e}_\theta(\tilde{M}) + + I_\varphi^e(\tilde{M})\vec{e}_\varphi(\tilde{M})\,. \qquad (28)$$

The components $A_\rho(M)$, $A_\varphi(M)$ and $A_\theta(M)$ can be expressed in terms of the components $A_x(M)$, $A_y(M)$ and $A_z(M)$:

$$A_\rho(M) = [A_x(M)\cos\varphi + A_y(M)\sin\varphi]\sin\theta + + A_z(M)\cos\theta\,, \qquad (29)$$

$$A_\theta(M) = [A_x(M)\cos\varphi + A_y(M)\sin\varphi]\cos\theta - - A_z(M)\sin\theta\,, \qquad (30)$$

$$A_\varphi(M) = -A_x(M)\sin\varphi + A_y(M)\cos\varphi\,, \qquad (31)$$

and the components $I_x^e(\tilde{M})$, $I_y^e(\tilde{M})$ and $I_z^e(\tilde{M})$ can be expressed in terms of the components $I_\rho^e(\tilde{M})$, $I_\varphi^e(\tilde{M})$ and $I_\theta^e(\tilde{M})$ (see [5], page 582):

$$I_x^e(\tilde{M}) = I_\rho^e(\tilde{M})\sin\tilde{\theta}\cos\tilde{\varphi} + I_\theta^e(\tilde{M})\cos\tilde{\theta}\cos\tilde{\varphi} - - I_\varphi^e(\tilde{M})\sin\tilde{\varphi}\,, \qquad (32)$$

$$I_y^e(\tilde{M}) = I_\rho^e(\tilde{M})\sin\tilde{\theta}\sin\tilde{\varphi} + I_\theta^e(\tilde{M})\cos\tilde{\theta}\sin\tilde{\varphi} + + I_\varphi^e(\tilde{M})\cos\tilde{\varphi}\,, \qquad (33)$$

$$I_z^e(\tilde{M}) = I_\rho^e(\tilde{M})\cos\tilde{\theta} - I_\theta^e(\tilde{M})\sin\tilde{\theta}\,. \qquad (34)$$

It follows from formula (4) that

$$\begin{pmatrix} A_x(M) \\ A_y(M) \\ A_z(M) \end{pmatrix} = \frac{\mu_0\mu}{4\pi}\iiint_V \begin{pmatrix} I_x^e(\tilde{M}) \\ I_y^e(\tilde{M}) \\ I_z^e(\tilde{M}) \end{pmatrix} F(M,\tilde{M})\,d\tilde{V}\,, \qquad (35)$$

where

$$F(M,\tilde{M}) = \frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}\,, \qquad (36)$$

and the distance, $r_{M\tilde{M}}$, between the points M and $\tilde{M}$ is defined by formula (5). It follows from (5) by substituting

$$x = \rho\sin\theta\cos\varphi,\quad y = \rho\sin\theta\sin\varphi,\quad z = \rho\cos\theta\,,$$
$$\tilde{x} = \tilde{\rho}\sin\tilde{\theta}\cos\tilde{\varphi},\quad \tilde{y} = \tilde{\rho}\sin\tilde{\theta}\sin\tilde{\varphi},\quad \tilde{z} = \tilde{\rho}\cos\tilde{\theta} \qquad (37)$$

that

$$r_{M\tilde{M}} = \sqrt{\rho^2 + \tilde{\rho}^2 - 2\rho\tilde{\rho}[\sin\theta\sin\tilde{\theta}\cos(\varphi-\tilde{\varphi}) + + \cos\theta\cos\tilde{\theta}]}\,. \qquad (38)$$

It follows from (29) by substituting (35) that

$$A_\rho(M) = \frac{\mu_0\mu}{4\pi}\iiint_V I_x^e(\tilde{M})F(M,\tilde{M})\,d\tilde{V}\cdot\cos\varphi\sin\theta + + \frac{\mu_0\mu}{4\pi}\iiint_V I_y^e(\tilde{M})F(M,\tilde{M})\,d\tilde{V}\cdot\sin\varphi\sin\theta + + \frac{\mu_0\mu}{4\pi}\iiint_V I_z^e(\tilde{M})F(M,\tilde{M})\,d\tilde{V}\cdot\cos\theta\,. \qquad (39)$$

The final expression for the component $A_\rho(M)$ is obtained by substituting expressions (32) - (34) of the components of $\vec{I}^{\,e}(\tilde{M})$ into (39) and by performing some elementary transforms, and it has the form

$$A_\rho(M) = \frac{\mu_0\mu}{4\pi}\iiint_V \{I_\rho^e(\tilde{M})[\sin\theta\sin\tilde{\theta}\cos(\varphi-\tilde{\varphi}) + + \cos\theta\cos\tilde{\theta}] + + I_\theta^e(\tilde{M})[\sin\theta\cos\tilde{\theta}\cos(\varphi-\tilde{\varphi}) - \cos\theta\sin\tilde{\theta}] + + I_\varphi^e(\tilde{M})\sin\theta\sin(\varphi-\tilde{\varphi})\}\,F(M,\tilde{M})\,d\tilde{V}\,, \qquad (40)$$

where

$$d\tilde{V} = \tilde{\rho}^2 \sin^2 \tilde{\theta}\, d\tilde{\rho}\, d\tilde{\theta}\, d\tilde{\varphi} \ . \quad (41)$$

By performing the similar transforms for the components $A_\theta(M)$ and $A_\varphi(M)$, the final expression for these components has the form

$$A_\theta(M) = \frac{\mu_0\mu}{4\pi}\iiint_V \{I^e_\rho(\tilde{M})[\sin\tilde{\theta}\cos\theta\cos(\varphi-\tilde{\varphi}) -$$
$$-\cos\tilde{\theta}\sin\theta] +$$
$$+ I^e_\theta(\tilde{M})[\cos\tilde{\theta}\cos\theta\cos(\varphi-\tilde{\varphi}) + \sin\tilde{\theta}\sin\theta] +$$
$$+ I^e_\varphi(\tilde{M})\cos\theta\sin(\varphi-\tilde{\varphi})\}\, F(M,\tilde{M})\, d\tilde{V}\ , \quad (42)$$

$$A_\varphi(M) = -\frac{\mu_0\mu}{4\pi}\iiint_V \{I^e_\rho(\tilde{M})\sin\tilde{\theta}\sin(\varphi-\tilde{\varphi}) +$$
$$+ I^e_\theta(\tilde{M})\cos\tilde{\theta}\sin(\varphi-\tilde{\varphi}) -$$
$$- I^e_\varphi(\tilde{M})\cos(\varphi-\tilde{\varphi})\}\, F(M,\tilde{M})\, d\tilde{V}\ . \quad (43)$$

Thus, formulae (27), (40), (42) and (43) give the integral representation of the solution to the vector Helmholtz equation (4) in the spherical coordinates.

4 Integral Representation in the System of Arbitrary Orthogonal Curvilinear Coordinates

Let the system of arbitrary orthogonal curvilinear coordinates (q_1, q_2, q_3) be given by the functions

$$x = x(q_1,q_2,q_3)\,,\ \ y = y(q_1,q_2,q_3)\,,$$
$$z = z(q_1,q_2,q_3) \quad (44)$$

and, respectively,

$$\tilde{x} = x(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3)\,,\ \ \tilde{y} = y(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3)\,,$$
$$\tilde{z} = z(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3)\ . \quad (45)$$

Let $\vec{e}_{q_1}$, $\vec{e}_{q_2}$, $\vec{e}_{q_3}$ be the units vectors of this coordinate system. Then vectors $\vec{A}(M)$ and $\vec{I}^{\,e}(\tilde{M})$ have the form

$$\vec{A}(M) = A_{q_1}(M)\vec{e}_{q_1}(M) + A_{q_2}(M)\vec{e}_{q_2}(M) +$$
$$+ A_{q_3}(M)\vec{e}_{q_3}(M)\,, \quad (46)$$

$$\vec{I}^{\,e}(\tilde{M}) = I^e_{q_1}(\tilde{M})\vec{e}_{q_1}(\tilde{M}) + I^e_{q_2}(\tilde{M})\vec{e}_{q_2}(\tilde{M}) +$$
$$+ I^e_{q_3}(\tilde{M})\vec{e}_{q_3}(\tilde{M})\,. \quad (47)$$

The components $A_{q_1}(M)$, $A_{q_2}(M)$, $A_{q_3}(M)$ can be expressed in terms of the components $A_x(M)$, $A_y(M)$, $A_z(M)$:

$$A_{q_j}(M) = \frac{1}{H_j(M)}\left[A_x(M)\frac{\partial x}{\partial q_j} + A_y(M)\frac{\partial y}{\partial q_j} + \right.$$
$$\left. + A_z(M)\frac{\partial z}{\partial q_j}\right],\ \ j = 1,2,3, \quad (48)$$

and the components $I^e_x(\tilde{M})$, $I^e_y(\tilde{M})$, $I^e_z(\tilde{M})$ can be expressed in terms of the components $I^e_{q_1}(\tilde{M})$, $I^e_{q_2}(\tilde{M})$, $I^e_{q_3}(\tilde{M})$ (see [5], page 561):

$$I^e_x(\tilde{M}) = \sum_{k=1}^{3} I^e_{q_k}(\tilde{M})\frac{1}{\tilde{H}_k}\frac{\partial\tilde{x}}{\partial\tilde{q}_k}\,, \quad (49)$$

$$I^e_y(\tilde{M}) = \sum_{k=1}^{3} I^e_{q_k}(\tilde{M})\frac{1}{\tilde{H}_k}\frac{\partial\tilde{y}}{\partial\tilde{q}_k}\,, \quad (50)$$

$$I^e_z(\tilde{M}) = \sum_{k=1}^{3} I^e_{q_k}(\tilde{M})\frac{1}{\tilde{H}_k}\frac{\partial\tilde{z}}{\partial\tilde{q}_k}\,, \quad (51)$$

where $H_k = H_k(q_1,q_2,q_3)$, $\tilde{H}_k = H_k(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3)$ are the Lame coefficients of the prescribed coordinate system (see [5] with the notation that $H_k = h_k^{-1}$).

It follows from formula (4) that

$$\begin{pmatrix} A_x(M) \\ A_y(M) \\ A_z(M) \end{pmatrix} = \frac{\mu_0\mu}{4\pi}\iiint_V \begin{pmatrix} I^e_x(\tilde{M}) \\ I^e_y(\tilde{M}) \\ I^e_z(\tilde{M}) \end{pmatrix} G(M,\tilde{M})\, d\tilde{V}\,, \quad (52)$$

where

$$G(M,\tilde{M}) = \frac{\exp(-jkr_{M\tilde{M}})}{r_{M\tilde{M}}}\,, \quad (53)$$

and the distance, $r_{M\tilde{M}}$, between the points M and $\tilde{M}$ is defined by formula (5), where x,y,z and $\tilde{x},\tilde{y},\tilde{z}$ are the functions of q_1,q_2,q_3 and $\tilde{q}_1,\tilde{q}_2,\tilde{q}_3$, respectively, and they are given by formulae (44), (45).

It follows from (48), at first, by substituting (49)-(51) into (52), and, at second, by substituting the new obtained form of (52) into (48), that

$$A_{q_j}(M) = \frac{\mu_0\mu}{4\pi}\frac{1}{H_j(M)}\iiint_V \sum_{k=1}^{3} \frac{1}{H_k(\tilde{M})} I^e_{q_k}(\tilde{M})\times$$
$$\times\left[\frac{\partial\tilde{x}}{\partial\tilde{q}_k}\frac{\partial x}{\partial q_j} + \frac{\partial\tilde{y}}{\partial\tilde{q}_k}\frac{\partial y}{\partial q_j} + \frac{\partial\tilde{z}}{\partial\tilde{q}_k}\frac{\partial z}{\partial q_j}\right]\times$$
$$\times G(M,\tilde{M})\, d\tilde{V}\ ,\quad j = 1,2,3, \quad (54)$$

where

$$d\tilde{V} = H_1(\tilde{M})H_2(\tilde{M})H_3(\tilde{M})d\tilde{q}_1 d\tilde{q}_2 d\tilde{q}_3\,. \quad (55)$$

Formulae (46) and (54) give the integral representation of the solution to the vector Helmholtz equation (4) in the system of arbitrary orthogonal curvilinear coordinates.

The integral representation of the Helmholtz equation (4) can be obtained for any orthogonal coordinate system by substituting the Lame coefficients of these coordinate system into equations (54) and (55). For example, in the system of cylindrical polar coordinates (r,φ,z), it follows from (54) by substituting

$$q_1 = r\,,\quad q_2 = \varphi\,,\quad q_3 = z\,,\quad \tilde{q}_1 = \tilde{r}\,,\quad \tilde{q}_2 = \tilde{\varphi}\,,\quad \tilde{q}_3 = \tilde{z}\,,$$
$$H_1 = 1\,,\; H_2 = r\,,\; H_3 = 1\,,\; \tilde{H}_1 = 1\,,\; \tilde{H}_2 = \tilde{r}\,,\; \tilde{H}_3 = 1\,,$$
$$G(M,\tilde{M}) = \Phi(M,\tilde{M})\,,$$

and by using formula (19) that at $j = 1$

$$A_r(M) = \frac{\mu_0\mu}{4\pi}\iiint_V [I_r^e(\tilde{M})(\cos\tilde{\varphi}\cos\varphi + \sin\tilde{\varphi}\sin\varphi) + \\ + I_\varphi^e(\tilde{M})(-\sin\tilde{\varphi}\cos\varphi + \cos\tilde{\varphi}\sin\varphi)]\times \\ \times\Phi(M,\tilde{M})\,d\tilde{V}\,. \qquad (56)$$

Formula (56) coincides with previously obtained formula (23). By similar way, formulae (25) of $A_\varphi(M)$ and (26) of $A_z(M)$ can be obtained from (54) by substituting $j = 2$ and $j = 3$, respectively.

In the system of spherical coordinates (ρ,θ,φ), formulae (40), (42) and (43) can be obtained from (54) by substituting

$$q_1 = \rho\,,\quad q_2 = \theta\,,\quad q_3 = \varphi\,,\quad \tilde{q}_1 = \tilde{\rho}\,,\quad \tilde{q}_2 = \tilde{\theta}\,,\quad \tilde{q}_3 = \tilde{\varphi}\,,$$
$$H_1 = 1\,,\quad H_2 = \rho\,,\quad H_3 = \rho\sin\theta\,,\quad \tilde{H}_1 = 1\,,\quad \tilde{H}_2 = \tilde{\rho}\,,$$
$$\tilde{H}_3 = \tilde{\rho}\sin\tilde{\theta}\,,\; G(M,\tilde{M}) = F(M,\tilde{M})$$

and by using formula (37).

5 Conclusion

The integral representation of the solution to the vector Helmholtz equation for the vector potential $\vec{A} = \vec{A}(M)$ has been obtained in the system of cylindrical polar and spherical coordinates. In the cylindrical polar coordinate system the radial and axial components of the vector potential, $A_\rho(M)$ and $A_\varphi(M)$, are expressed in terms of a triple integral of linear combination of the radial and axial components of current vector density. The similar results have been obtained for the system of spherical coordinates.

The integral representation of the solution to the vector Helmholtz equation has been also obtained for the system of arbitrary orthogonal curvilinear coordinates. The integral representation of the solution to the vector Helmholtz equation can be easily found for any orthogonal curvilinear coordinate system by only substituting the Lame coefficients of this coordinate system into the representation obtained.

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