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ON STABILITY ANALYSIS OF QUASILINEAR DIFFERENCE EQUATIONS IN BANACH SPACE (SPECTRAL THEORY APPROACH)

CARKOVS Jevgenijs, (LV), SLYUSARCHUK Vasyl, (UA)

Abstract. The paper deals with the mappings of Banach space $\mathcal{E}$ given in a form of quasilinear difference equation

$$x_{n+1} = \mathbf{A}x_n + \mathbf{F}_n(x_n),\ n \geq 0 \tag{1}$$

where $\mathbf{A}$ is linear continuous operator, $\{\mathbf{F}_n : \mathcal{E} \to \mathcal{E}\}$ are nonlinear bounded operators satisfying identity $\mathbf{F}_n(0) \equiv 0$. Side by side with the above equation we consider an equation of the first approximation, that is, the linear difference equation

$$y_{n+1} = \mathbf{A}y_n,\ n \geq 0 \tag{2}$$

We will discuss the assertions which guarantee local stability or instability for the trivial solution of (1) if (2) to be of this specificity. The proposal paper not only generalizes well known finite dimensional stability analysis results for quasilinear difference equations. Using spectral properties of operator $\mathbf{A}$ as a basis, our research shows that the infinite dimension of the space $\mathcal{E}$ not only strongle complicates computations and proofs of relevant theorems on stability analysis by the first approximation but also can have significant influence to statement of these results.

Key words and phrases. Quasilinear difference equations; Lyapunov stability; Instability..

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1 Notations, main definitions, and auxiliary assertions

We will follow the giving below classical notations of linear opertor theory [12]:

$\mathbb{L}(\mathcal{E})$ – Banach algebra of linear continuous operators with unit $\mathbf{I}$;

$\mathbb{K}(\mathcal{E})$ – subset of compact operators in $\mathbb{L}(\mathcal{E})$;

$Ker(\mathbf{A})$ – kernel of operator $\mathbf{A} \in \mathbb{L}(\mathcal{E})$;

$Im(\mathbf{A})$ – image of operator $\mathbf{A} \in \mathbb{L}(\mathcal{E})$;

$\sigma(\mathbf{A})$ – spectrum of operator $\mathbf{A} \in \mathbb{L}(\mathcal{E})$, that is,
$\lambda \in \sigma(\mathbf{A}) \Leftrightarrow Im(\mathbf{A} - \lambda\mathbf{I}) \neq \mathcal{E}$;

$r(\mathbf{A}) := \max\{|\lambda| : \lambda \in \sigma(\mathbf{A})\}$ – spectral radius of operator $\mathbf{A} \in \mathbb{L}(\mathcal{E})$.

The trivial solution $x_n \equiv 0$ is fixed point of mpping $\mathbf{A}x+\mathbf{F}(x)$ and we will discuss a behavior of iterations (1) in some neighbourhood of it. The trivial solution of (1) is referred to as

- *stable if for any positive number ε and number $n_0 \in \mathbf{N} \cup \{0\}$ there exists such a number $\delta = \delta(\varepsilon, n_0) > 0$, that for any solution x_n of this equation the inequality $\sup_{n \geq n_0} \|x_n\| < \varepsilon$ follows inequality $\|x_{n_0}\| < \delta$;*
- *instable if for some $\varepsilon > 0$, $n_0 \in \mathbf{N} \cup \{0\}$, and any $\delta > 0$ there exists such a solution x_n of this equation that $\|x_{n_0}\| < \delta$ and $\sup_{n \geq n_0} \|x_n\| \geq \varepsilon$;*
- *asymptotically stable if this solution is stable and for any $n_0 \in \mathbf{N} \cup \{0\}$ there exists such a number $\gamma = \gamma(n_0) > 0$ that from inequality $\|x_{n_0}\| < \gamma$ it follows the equality $\lim_{n \to \infty} \|x_n\| = 0$;*
- *exponential stable if for any $n_0 \in \mathbf{N} \cup \{0\}$ there exist such numbers $M = M(n_0) \geq 1$ and $q = q(n_0) \in (0,1)$ that*
$$\forall n \geq n_0 : \ \|x_n\| \leq M q^{n-n_0} \|x_{n_0}\| \tag{3}$$
for any solution of this equation;
- *local exponential stable if for any $n_0 \in \mathbf{N} \cup \{0\}$ there exist such numbers $M = M(n_0) \geq 1$, $q = q(n_0) \in (0,1)$, and $r = r(n_0)$ that for any solution of this equation from inequality $\|x_{n_0}\| < r$ follows an inequality (3).*

In the subsequent text of this paper we will need some of our previous results citing below.

Theorem 1.1 *The following assertions are implications:*

(i) the trivial solution of (2) is exponential stable;

(ii) $r(\mathbf{A}) < 1$;

(iii) the series $\sum_{k=0}^{\infty} \|\mathbf{A}^k\|$ converges.

Theorem 1.2 *Let $\mathcal{E}$ be a complex Banach space, and μ is a boundary point of the set $\sigma(\mathbf{A}) \setminus \{0\}$. For any $\delta > 0$ and $m \in \mathbf{N}$ there exists such a vector ξ that*

$$(1 - \delta)|\mu|^n \leq \|\mathbf{A}^n \xi\| \leq (1 + \delta)|\mu|^n |\xi|$$

for all $n = \overline{0, m}$.

Theorem 1.3 *Let $\mathcal{E}$ be a real Banach space, and μ is a boundary point of the set $\sigma(\mathbf{A}) \setminus \{0\}$. For any $\delta > 0$ and $m \in \mathbf{N}$ there exist such an integer number $m_0 \geq m$ and vector $u \in \mathcal{E}$ with norm $|u| = 1$ that*

$$\|\mathbf{A}^n u\| \leq \left(\sqrt{2} + \delta\right) |\mu|^n \quad \textit{for any} \quad n = \overline{0, m_0}$$

and

$$\|\mathbf{A}^n u\| \geq (1 - \delta)|\mu|^{m_0}$$

The proofs of these results one can find in the papers [1] and [7].

2 Stability by the first approximation.

This Section is devoted to stability analysis of equation (1) by the first approximation. It seems naturally that the linear approximation equation (1) has to be subjected to condition $r(\mathbf{A}) < 1$. But it is wrong to believe that even the aasertion $r(\mathbf{A}) \leq 1$ is necessary in a case $\dim \mathcal{E} = \infty$. Corresponding examples we will give in Section 3. But the proposal in this Section resalts only generalize the well known similar theorems for finite dimensional space $\mathcal{E}$ and therefore an assertion $r(\mathbf{A}) < 1$ is present there.

Theorem 2.1 *Assume that*

(i) the trivial solution of linear equation (2) *is exponential stable;*

(ii) the operators $\mathbf{F}_n, n \geq 0$ satisfy condition of uniform sufficiently small sublinear growth at zero, that is, for some positive number R there exists such a positive number ν that

$$\sup_{n \geq 0} \|\mathbf{F}_n x\| \leq \nu \|x\|, \ \textit{for} \ \|x\| \leq R$$

and

$$\nu \sum_{k=0}^{\infty} \left\|\mathbf{A}^k\right\| < 1 \tag{4}$$

Then the trivial solution of (1) *is local exponential stable.*

Proof. On the basis of the Theorem 3.1 and the first condition of the present theorem one may be certain of convergence of series $\sum_{k=0}^{\infty} \left\|\mathbf{A}^k\right\| := M$. This permits to introduce in the space $\mathcal{E}$ new norm $\|x\|_A = \sum_{k=0}^{\infty} \left\|\mathbf{A}^k x\right\|$ which satisfies inequality $\|x\| \leq \|x\|_A \leq M\|x\|$. Assuming $\|x_n\| \leq R$ one can estimate the value of difference $\triangle\|x_n\|_A = \|x_{n+1}\|_A - \|x_n\|_A$ for solution of equation (1) in a following form:

$$\begin{aligned}
\triangle\|x_n\|_A &= \sum_{k=0}^{\infty}\left\|\mathbf{A}^k x_{n+1}\right\| - \sum_{k=0}^{\infty}\left\|\mathbf{A}^k x_n\right\| = \\
&= \sum_{k=0}^{\infty}\left\|\mathbf{A}^{k+1} x_n + \mathbf{A}^k\mathbf{F}_n x_n\right\| - \sum_{k=0}^{\infty}\left\|\mathbf{A}^k x_n\right\| \leq \\
&\leq \sum_{k=0}^{\infty}\left\|\mathbf{A}^{k+1} x_n\right\| + \sum_{k=0}^{\infty}\left\|\mathbf{A}^k\mathbf{F}_n x_n\right\| - \sum_{k=0}^{\infty}\left\|\mathbf{A}^k x_n\right\| = \\
&= -\|x_n\| + \sum_{k=0}^{\infty}\left\|\mathbf{A}^k\mathbf{F}_n x_n\right\| \leq -\|x_n\| + M\|\mathbf{F}_n x_n\| \leq \\
&\leq -\|x_n\| + M\nu\|x_n\| = (-1+M\nu)\|x_n\| \leq \frac{M\nu-1}{M}\|x_n\|_A \leq \frac{M\nu-1}{M}\|x_n\|_A
\end{aligned}$$

Therefore under condition $\|x_n\| \leq R$ one can apply inequality

$$\|x_{n+1}\|_A \leq q\|x_n\|_A$$

where $q = 1 + \frac{M\nu-1}{M} < 1$ because by assuption of theorem $M\nu < 1$ and $M > 1$ by definition. Taking into account the above inequality and ineqality $\|x\| \leq \|x\|_A \leq M\|x\|$ one can may be sure that

$$\|x_n\| \leq Mq^{n-n_0}\|x_{n_0}\|,\ n \geq n_0$$

for any $\|x_{n_0}\| \leq \dfrac{R}{M}$, where n_0 – any integer number.

Corollary 2.2 *If the trivial solution of linear equation* (2) *is exponential stable and*

$$\lim_{\|x\|\to 0}\frac{\sup\limits_{n\geq 0}\|F_n x\|}{\|x\|} = 0$$

then the trivial solution of (1) *is local exponential stable.*

Remark 2.3 *If* (4) *is not to hold then the trivial solution f equation* (1) *may not be local exponential stable.*

Example 2.4 *Let us consider scalar difference equation*

$$x_{n+1} = ax_n + \nu|x_n|,\ n \geq 0, \tag{5}$$

where $a, \nu \in (0,1)$*. The formula* (4) *for this equation has a form* $\nu\sum_{k=0}^{\infty} a^k < 1$ *which equivalent to inequality* $a + \nu < 1$*. If* $a + \nu \geq 1$ *then* $x_n = (a+\nu)^n x_0$ *for each* $n \geq 0$ *and trivial solution of* (5) *is not local exponential stable.*

The proof technique of the above theorem may be used for more interesting assertion. Let $\mathbf{F}_n^{[m]} : \mathcal{E} \longrightarrow \mathcal{E}$, $n \geq m \geq 0$, be mappings defined by equalities

$$\mathbf{F}_n^{[m]} = \mathbf{F}_n^{[m-1]}(\mathbf{A} + \mathbf{F}_{n-m}),\ n \geq m \geq 0, \quad \mathbf{F}_n^{[0]} = \mathbf{F}_n$$

where $\mathbf{A}$ and $\mathbf{F}_n,\ n \geq 0$ from equation (1).

Theorem 2.5 *Assume that:*

(i) the trivial solution of equation (2) *is exponential stable;*

(ii) operators $\mathbf{F}_n$*,* $n \geq 0$ *satisfy inequalities*

$$\sup_{n\geq 0} \|\mathbf{F}_n x\| \leq \varphi(\|x\|), \text{ for } \|x\| \leq R,$$

where $R > 0$ *and* $\varphi : [0, R] \longrightarrow [0, +\infty)$ *– is positive continuous definitely increasing function, and* $\varphi(0) = 0$*;*

(iii) $\sup_{n\geq m} \left\|\mathbf{F}_n^{[m]} x\right\| \leq \nu\|x\|$*, for* $\|x\| \leq R$*, and* $\sqrt{\nu} \sum_{k=0}^{\infty} \left\|\mathbf{A}^k\right\| < 1$ *for some integer* m *and positive number* ν*.*

Then the trivial solution of equation (1) *is local exponential stable.*

Proof. Further we will apply the same notations as in the proof of 2.1. It is easily seen that the solutions of (1) satisfies inequalities

$$x_{n+1} = \mathbf{A}x_n + \mathbf{F}_n^{[k]} x_{n-k}, \; n \geq k,$$

for any $k = \overline{0, m}$. Under assumption $\|x_{n-m}\|_A \leq R$ one can write the inequalities

$$\begin{aligned}
\triangle\|x_n\|_A &= \sum_{k=0}^{\infty} \left\|\mathbf{A}^k x_{n+1}\right\| - \sum_{k=0}^{\infty} \left\|\mathbf{A}^k x_n\right\| = \\
&= \sum_{k=0}^{\infty} \left\|\mathbf{A}^{k+1} x_n + \mathbf{A}^k \mathbf{F}_n x_n\right\| - \sum_{k=0}^{\infty} \left\|A^k x_n\right\| \leq \\
&\leq \sum_{k=0}^{\infty} \left\|A^{k+1} x_n\right\| + \sum_{k=0}^{\infty} \left\|A^k F_n x_n\right\| - \sum_{k=0}^{\infty} \left\|A^k x_n\right\| = \\
&= -\|x_n\| + \sum_{k=0}^{\infty} \left\|A^k F_n x_n\right\| \leq -\|x_n\| + M\|F_n x_n\| \leq \\
&\leq -\frac{1}{M}\|x_n\|_A + M\|F_n x_n\| = -\frac{1}{M}\|x_n\|_A + M\left\|F_n^{[m]} x_{n-m}\right\| \leq \\
&\leq -\frac{1}{M}\|x_n\|_A + M\nu\|x_{n-m}\| \leq -\frac{1}{M}\|x_n\|_A + M\nu\|x_{n-m}\|_A
\end{aligned}$$

and therefore

$$\|x_{n+1}\|_A \leq \left(1 - \frac{1}{M}\right)\|x_n\|_A + M\nu\|x_{n-m}\|_A \tag{6}$$

Let n_0 be an arbitrary integer and $\mu \in (0, R)$ is such a number, that for any $\|x_{n_0}\| \leq \mu$ the solution of (1) with this initial condition satisfies inequality

$$\max_{0\leq k\leq m} \|x_{n_0+k}\|_A \leq R.$$

It may be done because continuous function $\varphi : [0, R] \longrightarrow [0, +\infty)$ definitely increases and $\varphi(0) = 0$. Then based on (6) and inequalities

$$\left(1 - \frac{1}{M}\right) + M\nu < 1 \tag{7}$$

one can write $\|x_n\|_A \leq R$ for each $n \geq n_0$. Now from (6) it is easily conclude that

$$\|x_{n+1}\|_A \leq \left(1 - \frac{1}{M} + M\nu\right) \max\{\|x_n\|_A \ldots, \|x_{n-m}\|_A\},$$

for $\|x_{n-m}\|_A \leq R$. Then under condition $\max\limits_{0 \leq k \leq m} \|x_{n_0+k}\|_A \leq R$, where $\left[\frac{n-n_0}{m}\right]$ is ineger part of number $\frac{n-n_0}{m}$ one can write inequalities

$$\|x_n\|_A \leq \left(1 - \frac{1 - M^2\nu}{M}\right)^{\left[\frac{n-n_0}{m}\right]} \max_{0 \leq k \leq m} \|x_{n_0+k}\|_A$$

for any $n \geq n_0$. From this and (7) follows that the trivial solution of equation (1) is local exponential stable.

The special case of the above theorem is following assertion.

Corollary 2.6 *Assume that:*

(i) the first and the second assumptions of Theorem 2.4 are fulfilled;

(ii) there exists such a number $\nu > 0$ that

$$\sup_{n \geq 1} \|\mathbf{F}_n(\mathbf{A}x + \mathbf{F}_{n-1}x)\| \leq \nu\|x\|, \text{ for } \|x\| \leq R,$$

and

$$\sqrt{\nu} \sum_{k=0}^{\infty} \|\mathbf{A}^k\| < 1.$$

Then the trivial solution of equation (1) is local exponential stable.

Remark 2.7 *In Theorem 2.4 and Corollary 2.5 function $\varphi(t)$ may be also of this a type as*

$$\lim_{t \to +0} \frac{\varphi(t)}{t} = +\infty. \tag{8}$$

The next example illustrates possibility of application of Theorem 2.4 or Corollary 2.5 when the Theorem 2.1 is unusable.

Example 2.8 *Let* $\mathbf{H}$ *be a nilpotent operator satisfying equalities* $\mathbf{H}^2 \neq 0$ *and* $\mathbf{H}^3 = 0$*, and* $\mathbf{F} : \mathcal{E} \longrightarrow \mathcal{E}$ *is operator defined by equality*

$$\mathbf{F}(x) = \begin{cases} 0, & \text{if } \ x = 0, \\ \|x\|^{-1/2}\mathbf{H}^2 x + \|x\|\mathbf{H}x, & \text{if } \ x \neq 0. \end{cases}$$

Let us consider equation

$$x_{n+1} = \mathbf{H}x_n + \mathbf{F}(x_n), \ n \geq 0, \tag{9}$$

It is easily seen that

$$\|\mathbf{F}(x)\| \leq \|\mathbf{H}^2\|\sqrt{\|x\|} + \|\mathbf{H}\|\|x\|^2$$

for each $x \in \mathcal{E}$*, and*

$$\left|\|x\|^{-1/2}\left\|\mathbf{H}^2 x\right\| - \|x\|\|\mathbf{H}x\|\right| \leq \|\mathbf{F}(x)\| \tag{10}$$

for $x \in \mathcal{E} \setminus \{0\}$*, (that is the condition of Corollary 2.5 for function* $\varphi(t) = \|\mathbf{H}^2\|\sqrt{t} + \|\mathbf{H}\|t^2$*). This function satisfies equality* (8)*. This and* (10) *make it clear that the Theorem 4 may not be in use for stability analysis of* (1)*. It is obviously that the first assumption of Theorem 5 is also fulfilled. Besides*

$$\|\mathbf{F}(\mathbf{H}x + \mathbf{F}(x))\| = o(\|x\|) \text{ if } \|x\| \to 0 \tag{11}$$

because $\mathbf{H}x + \mathbf{F}(x) \neq 0$ *and therefore*

$$\begin{aligned} \mathbf{F}(\mathbf{H}x + \mathbf{F}(x)) &= \|\mathbf{H}x + \mathbf{F}(x)\|^{-1/2}\mathbf{H}^2(\mathbf{H}x + \mathbf{F}(x)) + \|\mathbf{H}x + \mathbf{F}(x)\|\mathbf{H}(\mathbf{H}x + \mathbf{F}(x)) = \\ &= \|\mathbf{H}x + \mathbf{F}(x)\|^{-1/2}\mathbf{H}^2\left(\mathbf{H}x + \|x\|^{-1/2}\mathbf{H}^2 x + \|x\|\mathbf{H}x\right) + \\ &+ \|\mathbf{H}x + \|x\|^{-1/2}\mathbf{H}^2 x + \|x\|\mathbf{H}x\|\mathbf{H}\left(\mathbf{H}x + \|x\|^{-1/2}\mathbf{H}^2 x + \|x\|\mathbf{H}x\right) = \\ &= \|\mathbf{H}x + \|x\|^{-1/2}\mathbf{H}^2 x + \|x\|\mathbf{H}x\|\left(\mathbf{H}^2 x + \|x\|\mathbf{H}^2 x\right) \end{aligned}$$

Then

$$\|\mathbf{F}(\mathbf{H}x + \mathbf{F}(x))\| \leq \left(\|\mathbf{H}\|\|x\| + \left\|\mathbf{H}^2\right\|\sqrt{\|x\|} + \|\mathbf{H}\|\|x\|^2\right)\left\|\mathbf{H}^2\right\|(1 + \|x\|)\|x\|$$

for any $x \in \mathcal{E}$ *and local exponential stability of the trivial solution of equation* (9) *follows formula* (11)*. It should be mentioned that this assertion is trivial corollary of operator* $\mathbf{H}$ *nilpotency because* $x_n = 0$ *for any* $n \geq 3$*.*

3 Conditions of instability by the first approximation.

In this section we will analyze equation (2) under assumption that $r(\mathbf{A}) > 1$. As in previous Section the proof in many respects uses Banach space renormalization technique. This permits not only sufficiently easily to prove results, which are similar to corresponding results for finite dimentional space $\mathcal{E}$, but also to derive theorems which are specific in a case $\dim \mathcal{E} = \infty$.

Theorem 3.1 *Assume that:*

(i) $r(\mathbf{A}) > 1$*;*

(ii) $\exists r \in [1, r(\mathbf{A})) : \sigma(\mathbf{A}) \cap \{z \in \mathbf{C} : |z| = r\} = \emptyset$*;*

(iii) there exist such positive numbers q and ρ that

$$\sup_{n\geq 0} \|\mathbf{F}_n x\| \leq q\|x\| \text{ for } x \in \{y \in \mathcal{E} : \|y\| \leq \rho\}.$$

Then for sufficiently small q the trivial solution of equation (1) *instable.*

Proof. From the beginning let us assume that $r = 1$ and $\sigma(\mathbf{A}) \cap \{z \in \mathbf{C} : |z| < 1\} \neq \emptyset$. Let P_+ and P_- be spectral projectors corresponding to spectral sets

$$\sigma_+(\mathbf{A}) = \sigma(\mathbf{A}) \cap \{z \in \mathbf{C} : |z| > 1\}$$

and

$$\sigma_-(\mathbf{A}) = \sigma(\mathbf{A}) \cap \{z \in \mathbf{C} : |z| < 1\}$$

These operators define spectral decomposition of space $\mathcal{E}$ $\mathcal{E}_+ = P_+\mathcal{E}$, $\mathcal{E}_- = P_-\mathcal{E}$ and restrictions $\mathbf{A}|_{E_+}$ and $\mathbf{A}|_{E_-}$ of operator $\mathbf{A}$ on these subspaces. By definition the spectrum of the above restrictions coincide with sets $\sigma_+(\mathbf{A})$ and $\sigma_-(\mathbf{A})$, besides $0 \notin \sigma_+(\mathbf{A})$. Therefore operator $\mathbf{A}|_{E_+} : \mathcal{E}_+ \longrightarrow \mathcal{E}_+$ is reversible and spectral radiuses of operators $\mathbf{A}|_{E_-}$ and $\big(\mathbf{A}|_{E_+}\big)^{-1}$ less than 1. By the Theorem 1.1 the serieses

$$\sum_{k=1}^{\infty} \left\|(\mathbf{A}|_{E_+})^{-k}\right\| \text{ and } \sum_{k=0}^{\infty} \left\|(\mathbf{A}|_{E_-})^{k}\right\|.$$

are convergent and this permits to define in the space $\mathcal{E}$ a new norm

$$\|x\|_A = \sum_{k=1}^{\infty} \left\|(\mathbf{A}|_{E_+})^{-k} P_+ x\right\| + \sum_{k=0}^{\infty} \left\|(\mathbf{A}|_{E_-})^{k} P_- x\right\|$$

Owing inequality $m\|x\| \leq \|x\|_A \leq M\|x\|$ for all $x \in \mathcal{E}$, where $m = \min\left\{1, \frac{1}{\|\mathbf{A}|_{E_+}\|-1}\right\} > 0$ and $M = \sum\limits_{k=1}^{\infty} \left\|(\mathbf{A}|_{E_+})^{-k}\right\| + \sum\limits_{k=0}^{\infty} \left\|(\mathbf{A}|_{E_-})^{k}\right\| < \infty$ one makes sure that the norms $\|\cdot\|$ and $\|\cdot\|_A$ are equivalent. It follows the inequalities

$$
\begin{aligned}
\min\left\{1, \frac{1}{\|\mathbf{A}|_{E_+}\|-1}\right\}\|x\| \leq & \\
\leq & \min\left\{1, \frac{1}{\|\mathbf{A}|_{E_+}\|-1}\right\}\|P_+x\|+\|P_-x\| \leq \\
\leq & \frac{1}{\|\mathbf{A}|_{E_+}\|-1}\|P_+x\|+\|P_-x\| = \\
= & \sum_{k=1}^{\infty}\left\|\mathbf{A}|_{E_+}\right\|^{-k}\|P_+x\|+\|P_-x\| = \\
= & \sum_{k=1}^{\infty}\left\|\mathbf{A}|_{E_+}\right\|^{-k}\left\|\left(\mathbf{A}|_{E_+}\right)^{k}\left(\mathbf{A}|_{E_+}\right)^{-k}P_+x\right\|+\|P_-x\| \leq \\
\leq & \sum_{k=1}^{\infty}\left\|\mathbf{A}|_{E_+}\right\|^{-k}\|\mathbf{A}|_{E_+}\|^{k}\left\|\left(\mathbf{A}|_{E_+}\right)^{-k}P_+x\right\|+\|P_-x\| = \\
= & \sum_{k=1}^{\infty}\left\|\left(\mathbf{A}|_{E_+}\right)^{-k}P_+x\right\|+\|P_-x\| \leq \\
\leq & \sum_{k=1}^{\infty}\left\|\left(\mathbf{A}|_{E_+}\right)^{-k}P_+x\right\|+\sum_{k=0}^{\infty}\left\|\left(\mathbf{A}|_{E_-}\right)^{k}P_-x\right\| = \\
= & \|x\|_A \leq \left(\sum_{k=1}^{\infty}\left\|(\mathbf{A}|_{E_+})^{-k}\right\|+\sum_{k=0}^{\infty}\left\|(\mathbf{A}|_{E_-})^{k}\right\|\right)\|x\|
\end{aligned}
$$

Now one can apply the above constructed projective operators to solution x_n of equation (1) $\|P_+x_n\|_A = \sum\limits_{k=1}^{\infty}\left\|(\mathbf{A}|_{E_+})^{-k}P_+x_n\right\|$, $\|P_-x_n\|_A = \sum\limits_{k=0}^{\infty}\left\|(A|_{E_-})^{k}P_-x_n\right\|$ and write a decomposition $\|x_n\|_A = \|P_+x_n\|_A + \|P_-x_n\|_A$ Let us estimate each item taken separately:

$$
\begin{aligned}
\triangle\|P_+x_n\|_A &= \|P_+x_{n+1}\|_A - \|P_+x_n\|_A = \\
&= \|P_+Ax_n + P_+\mathbf{F}_nx_n\|_A - \|P_+x_n\|_A \geq \\
&\geq \|P_+Ax_n\|_A - \|P_+x_n\|_A - \|P_+\mathbf{F}_nx_n\|_A = \\
&= \|P_+x_n\| - \|P_+\mathbf{F}_nx_n\|_A \geq \frac{1}{M}\|P_+x_n\|_A - \|P_+\mathbf{F}_nx_n\|_A
\end{aligned}
$$

and $\triangle\|P_-x_n\|_A \leq -\frac{1}{M}\|P_-x_n\|_A + \|P_-\mathbf{F}_nx_n\|_A$. Therefore

$$
\|P_+x_{n+1}\|_A - \|P_+x_0\|_A \geq \sum_{k=0}^{n}\left(\frac{1}{M}\|P_+x_k\|_A - \|P_+\mathbf{F}_kx_k\|_A\right),
$$

$$
\|P_-x_{n+1}\|_A - \|P_-x_0\|_A \leq \sum_{k=0}^{n}\left(-\frac{1}{M}\|P_-x_k\|_A + \|P_-\mathbf{F}_kx_k\|_A\right)
$$

and one may write an inequality

$$
\begin{aligned}
\|x_{n+1}\|_A &\geq \|P_+x_{n+1}\|_A - \|P_-x_{n+1}\|_A \geq \\
&\geq \|P_+x_0\|_A - \|P_-x_0\|_A + \sum_{k=0}^{n}\left(\frac{1}{M}\|x_k\|_A - \|\mathbf{F}_kx_k\|_A\right)
\end{aligned}
$$

Suppose that $\|x_k\| < \rho$ for $k = \overline{0,n}$. Then $\|F_n x_n\|_A \le M\|F_n x_n\| \le Mq\|x_n\| \le \frac{M}{m} q\|x_n\|_A$ and for initial vector $x_0 = P_+ x_0$ we can use an inequality

$$\|x_{n+1}\|_A \ge \sum_{k=0}^{n} \left(\frac{1}{M} - \frac{M}{m} q \right) \|x_k\|_A + \|x_0\|_A \tag{12}$$

Note that for $0 < q < \dfrac{m}{M^2}$ value $\dfrac{1}{M} - \dfrac{M}{m} q$ is positive and therefore

$$\|x_n\|_A \ge \left(1 + \frac{1}{M} - \frac{M}{m} q \right)^n \|x_0\|_A \tag{13}$$

An instability of trivial solution of (1) follows the above inequality because for $\varepsilon < \rho$ and for any $\|x_0\| = \|P_+ x_0\|$ there exists such anumber $n \in \mathbb{N}$ that $\|x_n\| \ge \varepsilon$. Note that if $\sigma(\mathbf{A}) \cap \{z \in \mathbf{C} : |z| < 1\} = \emptyset$ the theorem can be proved with the help of norm $\|x\|_A = \sum_{k=1}^{\infty} \left\|A^{-k} x\right\|$ in the same way as for $\sigma(\mathbf{A}) \cap \{z \in \mathbf{C} : |z| < 1\} \ne \emptyset$. First we shall show that for solution of (1) under assumption $\max_{0 \le k \le n} \|x_k\| < \rho$ the formula (12) is true. Secondly as it has been done before one can establish the inequality (13) which convinces of instability of trivial solution of (1). Thus we have proved Theorem 3.1 for a case $r = 1$.

Let us assume now that $r \in (1, r(\mathbf{A}))$. Side by side with (1) we consider equation

$$x_{n+1} = \mathbf{A} x_n + \breve{\mathbf{F}}_n x_n, \; n \ge 0 \tag{14}$$

where

$$\breve{\mathbf{F}}_n x = \begin{cases} \mathbf{F}_n x, & \text{for } \|x\| \le \rho, \\ \dfrac{\|x\|}{\rho} \mathbf{F}_n \dfrac{\rho}{\|x\|} x, & \text{for } \|x\| > \rho. \end{cases}$$

Owing inequality $\|\breve{\mathbf{F}}_n x\| \le q\|x\|$ for each $x \in \mathcal{E}$ one can easy be certain that the trivial solutions of equtions (1) and (2) are stable or instable concurrently. Substituting in (14)

$$x_n = r^n y_n \tag{15}$$

we will have for y_n equation

$$y_{n+1} = r^{-1} \mathbf{A} y_n + r^{-n-1} \breve{\mathbf{F}}_n r^n y_n, \; n \ge 0 \tag{16}$$

where $r(r(A))^{-1} > 1, \sigma(r^{-1}\mathbf{A}) \cap \{z \in \mathbf{C} : |z| = 1\} = \emptyset$, and $\|r^{-n-1} \breve{\mathbf{F}}_n r^n x\| \le r^{-1} q\|x\| \le q\|x\|$ for any $x \in \mathcal{E}$. As it follows from our previous results, this inequality quarantees instabiliuty of the trivial solution of equation (16). With regard to equation (15) and inequality $r > 1$ one may assert that the trivial solution of equation (14) is instable.

Let us remark that for a case $\dim \mathcal{E} < \infty$ it follows that spectrum set $\sigma(\mathbf{A})$ consists of finite number of points and therefore one may resign the third assertion of Theorem 3.1. This convins of the following assertion. But if $\dim \mathcal{E} = \infty$ the below example makes it clear that un the above this assertion may not be rejected.

Example 3.2 *([2]). Let* $\mathbf{B} \in \mathbb{L}(\mathcal{E}), \sigma(\mathbf{B}) = \{z \in \mathbf{C} : |z| \leq 1\}$, *and* $(\mathbf{B}_m)_{m\geq 0}$ *be a sequnce of nilpotent operators acting in Banach space* $\mathcal{E}$ *satisfying following assumptions:*

$$\lim_{n\to\infty} \|\mathbf{B}_n - \mathbf{B}\| = 0 \tag{17}$$

Applying the results of [10] one can construct the above mentiontioned operators for example in the spaces l_2 *or* $L_2([0,1])$. *Assume that in* (1) $\mathbf{A} = e^{-\varepsilon\mathbf{I}+\mathbf{B}}$ *with* $\varepsilon \in (0,1)$, *and* $\mathbf{F}^{[m]}x = \left(e^{-\varepsilon I+B_m} - e^{-\varepsilon\mathbf{I}+\mathbf{B}}\right)x,\ x \in \mathcal{E}$ *By definition* $r(\mathbf{A}) > 1$, *and*

$$\|\mathbf{F}^{[m]}x\| \leq \left\|e^{-\varepsilon\mathbf{I}+\mathbf{B}_m} - e^{-\varepsilon\mathbf{I}+\mathbf{B}}\right\| \|x\|,\ x \in \mathcal{E}$$

Therefore it follows from (17) *that* $\lim\limits_{m\to\infty} \|\mathbf{F}^{[m]}x\| = 0$ *and for any* $q > 0$ *due to assumption* (17) *one can choose such an integer* m *that* $\|F^{[m]}x\| \leq q\|x\|$ *for any* $x \in \mathcal{E}$. *Besides equation* (1) *of our example may be rewritten in a following form*

$$x_{n+1} = e^{-\varepsilon\mathbf{I}+\mathbf{B}_m}x_n,\ n \geq 0,$$

and $r\left(e^{-\varepsilon\mathbf{I}+\mathbf{B}_m}\right) = e^{-\varepsilon} < 1,\ m \geq 1$. *Therefore the trivial solution of defined in our example difference equation* (1) *is asymptotically stable for whatever positive number* q.

It should be mentioned that if $\dim\mathcal{E} = \infty$ even under assumptions $\lim\limits_{\|x\|\to 0} \frac{\sup_{n\geq 0}\|\mathbf{F}_n x\|}{\|x\|} = 0$ and $r(\mathbf{A}) > 1$ the trivial solution of (1) may be asymptotically stable. Corresponding example one can find in [5]. To resign the second assertion permits more rigid condition on behaviour of function $\mathbf{F}_n(x)$ as $||x|| \to 0$. In our previous paper [1] we have prove a following result.

Theorem 3.3 *Assume that:*

(i) $r(\mathbf{A}) > 1$;

(ii) there exist such positive number a, p, *and* ρ *that* $\sup\limits_{n\geq 0} \|\mathbf{F}_n x\| \leq a\|x\|^{1+p}$ *for any* $x \in \{y \in \mathcal{E} : \|y\| \leq \rho\}$.

Then the trivial solution of (1) *is instable.*

In this paper we prove more stronger result, weakening the second assertion of the above therem.

Theorem 3.4 . *Assume that:*

(i) $r(\mathbf{A}) > 1$;

(ii) there exists such a continuous monotone function $\{q(y), 0 \leq y \leq \rho\}$ *that* $q(0) = 0$ *and* ρ *that* $\sup\limits_{n\geq 0} \|\mathbf{F}_n x\| \leq q(\|x\|)\|x\|$;

(iii) there exist such a number $\nu \in (0, \rho]$ *and a sequence* $\{f(n), n \in \mathbb{N}\}$ *that* $\|\mathbf{A}^n\| \leq f(n)(r(\mathbf{A}))^n$ *for any* $n \in \mathbb{N}$ *and series* $\sum\limits_{k=1}^{\infty} f(k)q\left(\nu(r(A))^{-k}\right)$ *converges.*

Then the trivial solution of (1) *is instable.*

Proof.. At first we assume that $\mathcal{E}$ is a complex Banach space. Let δ, P and γ are such positive numbers that $1+\delta < P < 2$, and

$$\sum_{k=0}^{\infty} f(k)q\left(\gamma r^{-k}\right) < \frac{(P-1-\delta)r}{P} < \frac{r}{2} \tag{18}$$

Taking into account (18), a monotony of function $q(y)$, and choosing such a number $\varepsilon \in (0, \gamma P^{-1}r^{-1})$, and an integer $n(\varepsilon) > 0$ that

$$\frac{\gamma}{r} \le \varepsilon P r^{n(\varepsilon)} \le \gamma \tag{19}$$

one can be certain of the inequality

$$\sum_{k=0}^{n-1} f(n-1-k)q\left(\varepsilon P r^{k}\right) < \frac{(P-1-\delta)r}{P}, \tag{20}$$

for any $n = \overline{1, n(\varepsilon)}$. Based on Theorem 1.2 one can find such a vector $\xi \in \{x \in \mathcal{E} : \|x\| = 1\}$ that

$$(1-\delta)r^n \le \|\mathbf{A}^n \xi\| \le (1+\delta)r^n, \quad n = \overline{1, n(\varepsilon)} \tag{21}$$

Let us split the solution x_n of equation (1) with initial condition $x_0 = \xi \in \{x \in \mathcal{E} : \|x\| = 1\}$ in a following form

$$x_n = x_{1,n} + x_{2,n}, \tag{22}$$

where $x_{1,n} = \mathbf{A}^n x_0$, and $x_{2,n} = \sum\limits_{k=0}^{n-1} \mathbf{A}^{n-1-k} F_k x_k,\ n \ge 1$

In compliance with (21) and the second asseretion of theorem there exists such an integer $m \in [0, n(\varepsilon)]$, that

$$\|x_n\| \le \varepsilon P r^n \tag{23}$$

for any $n = \overline{0, m}$. Therefore

$$\begin{aligned} \|x_{2,n}\| &= \left\| \sum_{k=0}^{n-1} \mathbf{A}^{n-1-k} F_k x_k \right\| \le \sum_{k=0}^{n-1} \|A^{n-1-k}\| \, \|F_k x_k\| \le \\ &\le \sum_{k=0}^{n-1} f(n-1-k) r^{n-1-k} q\left(\varepsilon P r^k\right) \varepsilon P r^k = \\ &= \varepsilon r^n \frac{P}{r} \sum_{k=0}^{n-1} f(n-1-k) q\left(\varepsilon P r^k\right) \end{aligned}$$

and $\|x_{1,n}\| \le (1+\delta)\varepsilon r^n$ for any $n \in [1, m]$. Then

$$\forall n = \overline{1, m} : \|x_n\| \le \left(1 + \delta + \frac{P}{r} \sum_{k=0}^{n-1} f(n-1-k) q\left(\varepsilon P r^k\right)\right) \varepsilon r^n,$$

and, because from (20) follows inequality

$$1 + \delta + \frac{P}{r} \sum_{k=0}^{n-1} f(n-1-k) q\left(\varepsilon P r^k\right) < P,$$

for $n=\overline{1,n(\varepsilon)}$, we may apply (23) for any $n\in[1,n(\varepsilon)]$. Applying (19)–(21), we can find lower bound for $\|x_{n(\varepsilon)}\|$:

$$\begin{aligned}
\|x_{n(\varepsilon)}\| &\geq \|x_{1,n(\varepsilon)}\|-\|x_{2,n(\varepsilon)}\|\geq\\
&\geq (1-\delta)\varepsilon r^{n(\varepsilon)}-\varepsilon r^{n(\varepsilon)}\frac{P}{r}\sum_{k=0}^{n(\varepsilon)-1}f(n(\varepsilon)-1-k)q(\varepsilon Pr^k)=\\
&= \varepsilon r^{n(\varepsilon)}\left(1-\delta-\frac{P}{r}\sum_{k=0}^{n(\varepsilon)-1}f(n(\varepsilon)-1-k)q\left(\varepsilon Pr^k\right)\right)\geq\\
&\geq \varepsilon r^{n(\varepsilon)}\left(1-\delta-\frac{P}{r}\cdot\frac{(P-1-\delta)r}{P}\right)\geq\frac{\gamma(2-P)}{Pr}=a>0
\end{aligned}$$

Therefore $\|x_{n(\varepsilon)}\|\geq a$ for any arbitrarily small $\varepsilon=\|x_0\|$ and the proof of theorem for a complex Banach space is completed. Now let $\mathcal{E}$ be a real Banach space. Like before we can find such positive numbers $\delta,\ P$ and γ that $\sqrt{2}+\delta<P<2$ and

$$\sum_{k=0}^{\infty}f(k)q\left(\gamma r^{-k}\right)<\frac{\left(P-\sqrt{2}-\delta\right)r}{P}<\frac{r}{2} \tag{24}$$

For any $\varepsilon\in(0,\gamma P^{-1}r^{-1})$ there exists such an integer $n(\varepsilon)>0$, that $\frac{\gamma}{r}\leq\varepsilon Pr^{n(\varepsilon)}\leq\gamma$. Applying Theorem 1.3 one can choose a number $m_0\geq n(\varepsilon)$ and a vector $u\in E$, $\|u\|=1$, which permits write inequalities

$$\forall n=\overline{0,m_0}:\|\mathbf{A}^n u\|\leq\left(\sqrt{2}+\delta\right)|\mu|^n \tag{25}$$

and

$$\|A^n u\|\geq(1-\delta)|\mu|^{m_0} \tag{26}$$

Besides owing monotony of sequaence $f(n)$ from (24) follows inequality

$$\sum_{k=0}^{n-1}f(n-1-k)q\left(\varepsilon Pr^k\right)<\frac{\left(P-\sqrt{2}-\delta\right)r}{P} \tag{27}$$

for all $n=\overline{1,m_0}$. Let us choose such a number $\varepsilon_1\in(0,\varepsilon)$ that

$$\frac{\gamma}{r}\leq\varepsilon_1 Pr^{m_0}\leq\gamma \tag{28}$$

and estimate the solution x_n of (1) with initial condition $x_0=\varepsilon_1 u$, splitting this in a form (22). Fomula (25) and the second assertion of theorem guarantee existence such an integer $m\in[0,m_0]$ that

$$\|x_n\|\leq\varepsilon_1 Pr^n \tag{29}$$

for any $n = \overline{0,m}$. Then

$$\begin{aligned}
\|x_{2,n}\| &= \left\|\sum_{k=0}^{n-1} \mathbf{A}^{n-1-k} F_k x_k\right\| \le \sum_{k=0}^{n-1} \|\mathbf{A}^{n-1-k}\| \, \|\mathbf{F}_k x_k\| \le \\
&\le \sum_{k=0}^{n-1} f(n-1-k) r^{n-1-k} q\left(\varepsilon_1 P r^k\right) \varepsilon_1 P r^k = \\
&= \varepsilon_1 r^n \frac{P}{r} \sum_{k=0}^{n-1} f(n-1-k) q\left(\varepsilon_1 P r^k\right) \\
\|x_{1,n}\| &\le \left(\sqrt{2}+\delta\right) \varepsilon_1 r^n
\end{aligned}$$

and one can apply inequality

$$\|x_n\| \le \left(\sqrt{2}+\delta+\frac{P}{r}\sum_{k=0}^{n-1} f(n-1-k) q\left(\varepsilon_1 P r^k\right)\right) \varepsilon_1 r^n$$

for each $n = \overline{1,m}$. Because from (27) follows formula

$$\sqrt{2}+\delta+\frac{P}{r}\sum_{k=0}^{n-1} f(n-1-k) q\left(\varepsilon_1 P r^k\right) < P$$

for any $n = \overline{1,m_0}$, we have proved inedquality (29) for any integer $n \in [1, m_0]$.

To find lower bound of $\|x_{m_0}\|$ one can use the formulae (25)–(27) and derive inequalities

$$\begin{aligned}
\|x_{m_0}\| &\ge \|x_{1,m_0}\| - \|x_{2,m_0}\| \ge \\
&\ge (1-\delta)\varepsilon_1 r^{m_0} - \varepsilon_1 r^{m_0} \frac{P}{r} \sum_{k=0}^{m_0-1} f(m_0-1-k) q\left(\varepsilon_1 P r^k\right) = \\
&= \varepsilon_1 r^{m_0} \left(1-\delta-\frac{P}{r}\sum_{k=0}^{m_0-1} f(m_0-1-k) q\left(\varepsilon_1 P r^k\right)\right) \ge \\
&\ge \varepsilon_1 r^{m_0} \left(1-\delta-\frac{P}{r}\cdot\frac{\left(P-\sqrt{2}-\delta\right) r}{P}\right) \ge \frac{\gamma\left(1+\sqrt{2}-P\right)}{Pr} = a > 0
\end{aligned}$$

Therefore the value of chosen solution $\|x_{m_0}\|$ with satisfying equality $\varepsilon_1 = \|x_0\|$ initial condition remains not less then $a > 0$ for any arbitrary small number ε_1. The proof is completed.

Example 3.5 *Let us consider difference equation*

$$x_{n+1} = \mathbf{A}x_n + \begin{cases} (1-\ln\|x_n\|)^{-2-p}\mathbf{B}x_n, & \text{if } x_n \ne 0, \\ 0, & \text{if } x_n = 0, \end{cases} \tag{30}$$

where $p > 0$, operator $\mathbf{A} \in \mathbb{L}(\mathcal{E})$ satisfies inequality

$$\forall n \in \mathbb{N} : \|\mathbf{A}^n\| \le M(1+n)2^n,$$

$\sigma(\mathbf{A}) = \{t : 0 \leq t \leq 2\}$, $\mathbf{B} \in \mathbb{L}(\mathcal{E})$ – nontrivial operator, and $\mathcal{E}$ – a complex Banach space. Now we choose sequence $f(n) = M(1+n)$ and function

$$q(y) = \begin{cases} \|\mathbf{B}\| |1 - \ln y|^{-2-p}, & \text{for } y > 0, \\ 0, & \text{if} \quad y = 0, \end{cases}$$

and substitute these in series $\sum_{k=0}^{\infty} f(k) q\left(\nu(r(A))^{-k}\right)$ from the third assertion of Theorem 3.4:

$$\sum_{k=0}^{\infty} \frac{M\|\mathbf{B}\|(1+k)}{(1 - \ln \nu + k \ln 2)^{2+p}}$$

Not so difficult to proof that this series converges for any $\nu \in (0,1)$. From the above we can be sure that for equation (30) all assertions of theorem 3.4 are satisfied and therefore the trivial solution of (30) is instable.

Remark 3.6 *Theorem 3.3 is a sequence of Theorem 3.4.*

Proof. Let us define sequence $f(n) = \max_{s \in [0,n] \cap (\mathbb{N} \cup \{0\})} \|\mathbf{A}^s\| (r(\mathbf{A}))^{-s}$ and function

$$q(y) = \begin{cases} \left(\hat{f} \left(\frac{1}{\ln r(\mathbf{A})} \ln \frac{1}{y} \right) \right)^{-1} (1 - \ln y)^{-1-p}, & \text{for } y \in (0,1], \\ 0, & \text{if } y = 0, \end{cases}$$

where $p > 0$ and $\hat{f}(t)$ is such a continuous monotony function that restriction $\hat{f}|_{\mathbb{N} \cup \{0\}}$ onto $\mathbb{N} \cup \{0\}$ coinside with above defined $f(n)$. By definition

$$\hat{f} \left(\frac{1}{\ln r(\mathbf{A})} \ln \frac{1}{y} \right) \geq \hat{f} \left(\frac{1}{\ln r(\mathbf{A})} \ln \frac{v}{y} \right)$$

for each $y \in (0, v]$ and $v \in (0,1]$. Therefore

$$\begin{aligned} \sum_{k=0}^{\infty} f(k) q\left(v(r(A))^{-k}\right) &\leq \sum_{k=0}^{\infty} f(k) q\left((r(A))^{-k}\right) = \\ &= \sum_{k=0}^{\infty} f(k) \left(\hat{f}(k)\right)^{-1} (1 + k \ln r(A))^{-1-p} = \\ &= \sum_{k=0}^{\infty} (1 + k \ln r(A))^{-1-p} < \infty \end{aligned}$$

To prove that

$$\lim_{y \to +0} \frac{y^{\varepsilon}}{q(y)} = 0 \quad \text{for any} \quad \varepsilon > 0 \tag{31}$$

one may apply a substitution $y = (r(A))^{-t}$ and rewrite (31) in following form

$$\lim_{t \to +\infty} \frac{(r(\mathbf{A}))^{-\varepsilon t}}{q((r(\mathbf{A}))^{-t})} = 0 \quad \text{for any} \quad \varepsilon > 0 \tag{32}$$

Taking into accountan Gelfand formula $r(\mathbf{A}) = \lim_{n\to+\infty} \sqrt[n]{\|\mathbf{A}^n\|}$ and equalities

$$\begin{aligned}\frac{y^\varepsilon}{q(y)} &= \frac{\hat{f}(t)(1+t\ln r(\mathbf{A}))^{1+p}}{(r(A))^{\varepsilon t}} \le \frac{f([t+1])(1+t\ln r(\mathbf{A}))^{1+p}}{(r(A))^{\varepsilon[t]}} = \\ &= \frac{f([t+1])}{\sqrt{(r(\mathbf{A}))^{\varepsilon[t]}}} \cdot \frac{(1+t\ln r(\mathbf{A}))^{1+p}}{\sqrt{(r(\mathbf{A}))^{\varepsilon[t]}}},\end{aligned}$$

$$\lim_{t\to+\infty} \frac{f([t+1])}{\sqrt{(r(\mathbf{A}))^{\varepsilon[t]}}} = 0,$$

$$\lim_{t\to+\infty} \frac{(1+t\ln r(\mathbf{A}))^{1+p}}{\sqrt{(r(\mathbf{A}))^{\varepsilon[t]}}} = 0,$$

we can get formula (32), which is equivalent to (31). Therefore if one may apply Theorem 3.4 then, based on (31), one also may apply Theorem 3.3.

It is well known [12] that the spectrum $\sigma(\mathbf{A})$ may be presented as a sum of disjoint sets

$$\sigma(\mathbf{A}) = \sigma_p(\mathbf{A}) \cap \sigma_c(\mathbf{A}) \cap \sigma_r(\mathbf{A})$$

where

$$\lambda \in \sigma_p(\mathbf{A}) \Leftrightarrow \{\exists x \neq 0 : (\mathbf{A} - \lambda\mathbf{I})x = 0\};$$

$$\lambda \in \sigma_c(\mathbf{A}) \Leftrightarrow \{\overline{Im(\mathbf{A} - \lambda\mathbf{I})} = \mathcal{E}, \exists x \notin Im(\mathbf{A} - \lambda\mathbf{I})\};$$

$$\lambda \in \sigma_r(\mathbf{A}) \Leftrightarrow \{\overline{Im(\mathbf{A} - \lambda\mathbf{I})} \neq \mathcal{E}\}.$$

Here and further an overline over a metric set denotes a closure of it. Stability analysis of (1) becomes simpler if there exists such a number $\delta < 1$ that the set $\{z \in \mathbf{C} : |z| > \delta$ contains only eigenvalues of operator $\mathbf{A}$ (for example, $\dim \mathcal{E} < \infty$, $\mathbf{A}$ is compact opertor). But sometimes, as it has been shown by our research, one can succesfully use bound points of $\sigma(\mathbf{A})$, eleminating a part of spectrum $\sigma_{ess.a}(\mathbf{A}) \subset \sigma(\mathbf{A})$ called *essentially approximative spectrum.*

Definition 3.7 *([6]) Complex number λ is an essentially approximative spectrum point iff there exists such an essentially divergent sequence $\{x_n, n \in \mathrm{N}\} \subset \mathcal{E}$ that $\lim_{n\to\infty} ||(\mathbf{A} - \lambda\mathbf{I})x_n|| = 0$.*

In [6] has been proved folloﬁng results.

Lemma 3.8 *For any $\mathbf{A} \in L(\mathcal{E})$*

$$\sigma(\mathbf{A}) \cap \{z \in \mathbb{C} : |z| = r\} \neq \emptyset \Leftrightarrow \sigma(\mathbf{A}) \cap \{z \in \mathbb{C} : |z| = r\} \neq \emptyset \tag{33}$$

Theorem 3.9 *Let us assume that:*

(i) $\sigma_{ess.a}(A) \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset$;

(ii) there exist such a continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ and operator sequnce $\mathbf{K}_n \in \mathcal{K}(\mathcal{E}), n \ge 0$ that $\varphi(0) = 0$ and $\|\mathbf{F}_n x\| \le \varphi(\|\mathbf{K}_n x\|)$ for all $(n, x) \in \mathbb{N} \times \mathcal{E}$.

Then the trivial solution of (1) *is instable.*

Applying the above results, we can reasonably simply generelize Theorem 3.1.

Theorem 3.10 *Let us assume that:*

(i) $r(\mathbf{A}) > 1$*;*

(ii) there exists a sequence of compact opoerators $\{\mathbf{K}_n, n \in \mathbb{N}\} \subset \mathcal{K}(\mathcal{E})$*, and* $q_0 := \sup_{n \geq 0} \|\mathbf{K}_n\| < \infty$*;*

(iii) $\|\mathbf{F}_n x\| \leq \|\mathbf{K}_n x\|$ *for all* $(n, x) \in \mathbb{N} \times \mathcal{E}$*.*

Then for sufficiently small q_0 *the trivial solution of* (1) *is instable.*

Proof. If $\sigma(\mathbf{A}) \cap \{z \in \mathbb{C} : |z| = r\} = \emptyset$ for some $r \in [1, r(A))$ the proof of theorem follows from Theorem 3.1. If $\sigma(\mathbf{A}) \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset$ then by (33) $\sigma_{ess.a}(\mathbf{A}) \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset$ and one can apply Theorem 3.4. The proof is completed.

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Current address

Jevgeņijs Carkovs, professor
Probability and Statistics Chair, Riga Technical University,
Kaļķu iela 1, Riga, LV-1658, Latvia, tel. +371 26549111
e-mail: carkovs@latnet.lv

Vasyl Slyusarchuk, professor
National University of Water Management and Nature Resources Use,
Soborna Str., 11, 33000, Rivne.