Proceedings of the 12th International Conference "Reliability and Statistics in Transportation and Communication" (RelStat'12), 17–20 October 2012, Riga, Latvia, p. 379–383. ISBN 978-9984-818-49-8
Transport and Telecommunication Institute, Lomonosova 1, LV-1019, Riga, Latvia

STUDYING FATIGUE CURVE APPROXIMATION USING DANIELS' SEQUENCE AND MARKOV CHAIN THEORY

Yuri Paramonov, Vjačeslavs Cimanis, Martinsh Kleinhofs

Aeronautical Institute, Riga Technical University
Lomonosova 1, Riga LV-1019, Latvia
Ph.: +371 7255394. Fax: +371 7089990. E-mail: yuri.paramonov@gmail.com

The definition of random Daniels' sequence (RDS) is introduced. The components of this sequence, $\{s_0, s_1, s_2, ...\}$, show the process of increasing of local stress in some critical weak micro volume (WMV) during fatigue loading with stress level S (amplitude or maximum of cycle) of some specimen of unidirectional composite with cumulative distribution function (cdf) $F_X(x)$ of strength of longitudinal items. Items of this sequence $s_{i+1} = S/(1-\hat{F}(s_i))$, $i = 0,1,2,...$, where $s_0 = S$, $\hat{F}_X(s_i)$ is estimate of cdf , are defined on some sample of observations of random variables with cdf $F_X(x)$. The model explains the existence of random fatigue level and although it is too simple to provide numerical coincidence with experimental fatigue test data, nevertheless it can be used as nonlinear regression model of S-N fatigue curve, some parameters of which can be considered as parameters of cdf of local static strength, so we can predict fatigue curve changes as consequence of static strength parameter changes. Numerical example is given.

Keywords: fatigue curve, Markov chains, composite

1. Introduction

The composite material is now in wide use in different types of structure, for example, in airframe and jet engines. This work is motivated by the need to develop and present quantitative fatigue – life information used is design of such structures. Fatigue data are usually presented in the form of a S-N curve, a log-log plot of cyclic stress or strain S versus the mean or median fatigue life N, which is expressed in cycles to failure. A great number of articles is devoted to this problem. Wide discussion on this matter takes place in [1]. So although for two last decades the main attention of fatigue study is paid to the use of fracture mechanics, nevertheless the study of S-N curve is very important for brittle materials and especially for composite materials. Nearly in every textbook, concerning with fatigue of metal or composite we'll see, first of all, the equation of fatigue curve (Wholer curve) in the form $S^m N = C$, where S is stress amplitude, N – cycle number; m, C are parameters. The Weibull's equation, $S - S_{-1} = C(N + B)^{-\alpha}$ where S_{-1}, α, B and C are parameters, is well known also. In [2], the seven models for the fatigue curve are given. The most interesting is the Random Fatigue Limit model offered by F. G. Pascual and W. Q. Meeker :

$$\log N = \beta_0 + \beta_1 \log(S - \gamma) + \varepsilon,$$

where ε is an error term, random variable $V = \log(\gamma)$ has normal or the smallest extreme value cdf, β_0 and β_1 are some regression parameter.

In [3] the model based on definition of Daniels' sequence (DS) was studied. DS is **determined** sequence of stresses growth in some critical weak micro volume (WMV), $\{s_0, s_1, s_2, ...\}$, during fatigue loading with stress level S (amplitude or maximum of cycle) of some specimen of unidirectional composite (UDC) with cumulative distribution function (cdf) of strength of longitudinal items, $F_X(x)$:

$$s_{i+1} = S/(1 - F_X(s_i)), \quad i = 0, 1, 2, ...,$$

where $s_0 = S$ is the initial rated stress in the undamaged specimen. The application of the DS using Markov chains theory to the description of fatigue curve has been studied in [4].

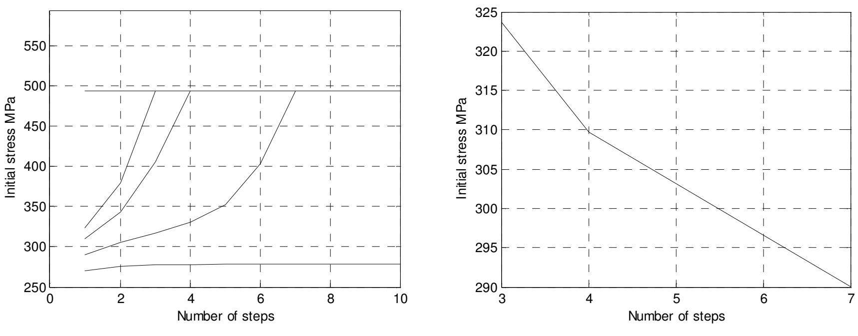
But in this paper the definition of **random** Daniels' sequence (RDS) is introduced and example of using this definition is considered.

2. Random Daniels' Sequence

The components of RDS, $\{s_0, s_1, s_2, ...\}$, correspond to the **random** process: $s_{i+1} = S/(1-\hat{F}_X(s_i))$, $i = 0,1,2,...$, where $s_0 = S$, $\hat{F}_X(s_i)$ is estimate of cdf, which is defined on some sample $(x_1,...,x_n)$ of observations of n random variables with the same cdf $F_X(x)$. These variables are random strengths of n longitudional items of some WMV. If initial stress, S, is upper than some value (RDS-fatigue-limit (RDSFLm)), then stress-sequence grows up to infinity. RDSFLm is defined as the maximum value of S for which there is solution of equation $s = S/(1-\hat{F}_X(s))$. This means that for some i we have: $s_{i+1} = s_i$ and growth of stress is stopped. The value of RDSFLm is equal to $S_{RD} = \max x_i(1-\hat{F}_X(x_i))$, $i = 1,2,...,n$. Growth of local stress corresponds to decreasing of local cross section. Let us define that the failure of specimen takes place if local cross section become less than some value p_C (initial cross section area is equal to one). Then critical local stress corresponding to this event, S_{UT}^*, is defined from equation $F_X(S_{UT}^*) = 1 - p_C$. The number $N = k_m \max\{i : s_i < S_{UT}^*\}$, where k_m is some scale coefficient, can be called as RDS fatigue life (RDSFLf) at stress S.

We have problem in statistical interpretation of RDSFLm. Usually we consider the cdf of some random variable, $F(x)$, as limit ($n \to \infty$) of some empirical cdf $\hat{F}_n(x) = n^{-1}\#\{i : 1 \le i \le n, X_i \le x\} = k(x)/n$, where $X_1,...,X_n$ are independent random variables with continuous cdf F, "#" denotes cardinality, $k(x)$ is number of observation equal or lower than x. But really we can not even to dream to make a test in processing of which we obtain the sample $\{x_1,...,x_n\}$ of observations of realizations of FL. Much easy to understand the meaning of the probability that specimen is still intact (fatigue failure of the specimen does not take place) at fatigue loading at stress level S at fixed number of cycle N. We use abbreviation IPSN to denote this probability. Statistical interpretation of such definition is obvious. It seems that just the knowledge of the value of IPSN should be enough at specific service condition at the required continuance of service. Using definition of RDS we can get the estimate of this probability for the case of simple fatigue test needed to build the S-N curve.

Here we consider the data of fatigue test of carbon-fibre composite [5]. In accordance with [5] it was supposed that the tensile strength of carbon fibre strands has cdf of lognormal distribution, $F_S(x) = \Phi((\log(x) - \theta_0)/\theta_1)$, where $\Phi(.)$ is cdf of standard normal distribution, with parameters $\theta_0 = 6.44$ and $\theta_1 = 0.1816$. These carbon fibre strands are longitudinal items of specimens for fatigue test the result of which was used in [4] for construction of corresponding fatigue curve in framework of DS-model. It has been shown that if we try to calculate DS for corresponding maximum cycle stresses: (S_1, S_2, S_3)=(323.7 309.7 290.1) we see that these stresses are under DSFLm, which is equal in this case to 446.85 MPa. Corresponding DSFLfs are equal to infinity!



a b

Figure1. Daniels sequence of local stresses (a) and the corresponding D-fatigue curve for S =323.7, 309.7 and 290.1 MPa (b) for for k_S =1.6, k_m =1, p_C =0.1, S_{UT}^* =494 MPa

So in framework of DS-model using cdf of strength of strands the failure of specimens can be explained only by existence of significant local stress concentration. Results of calculations of DSs for the same set of initial stress (S_1, S_2, S_3) taking into account the stress concentration coefficient, $k_S = 1.6$, are given on Figure 1a. For illustration of possibility to explain the existence of fatigue limit phenomenon using DS-model the results of calculation for S = 270 MPa are given also. In the last case DSFLf is equal to infinity. On Figure 2 we see Monte Carlo modelling of 10 different RDSs for S = 290.1 MPa and different k_S for n = 1000. The estimate of values of IPSN (pFlr in Fig.2) are given also (the approximate IPSN can be estimated using Daniels' normal approximation of strength of bundles of threads [3], but this is subject of another paper).

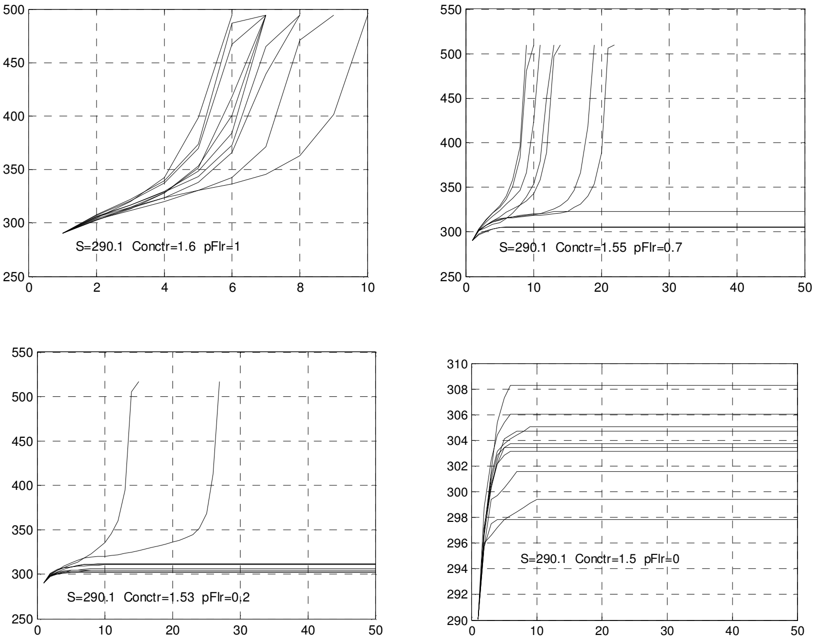


Figure 2. RDS for S = 290.1 MPa and different k_S (On Fig. 2 it is denoted by Conctr)

On Figure 1a we see the DFLf (the order number of DS up to failure of specimen) for $k_m = 1$. Corresponding fatigue curve is shown on Figure 1b. The values of the DFLfs are very small: 3, 4, 7. Thus, although DS allows making quality explanation of fatigue failure of material, can explain phenomenon of fatigue limit, but the quantity prediction is very poor. And it does not explain the scatter of fatigue life.

But the possibility of explanation of phenomenon of fatigue limit is very attractive. The explanation of the scatter of fatigue life and more appropriate description of fatigue curve in framework of model based on definition of DS are given in [5] using theory of Markov chain with space of states based on DS. Here we try to use model based on definition of RDS and some other (more "physical") approach taking into account the accumulation of energy necessary to failure the composite longitudinal items.

3. Approximation of Fatigue Curve Using the RDS Definition and Theory of Markov Chains

First, let us remind the model founded on the theory of Markov chain with space of states based on DS (see [4]). A simple Markov chain is considered. The first r states of it are related with items of Daniels' sequence $\{s_0, s_1, ..., s_{r-1}\}$, s_r is absorbing state (local stress is equal or more than S^*_{UT} (taking into account the stress concentration)) . We assume that the only transitions to the nearest 'senior' states can take place. So the following matrix of transition probabilities takes place:

$$P = \begin{bmatrix} q_1 & p_1 & 0 & & & \cdots & 0 \\ 0 & q_2 & p_2 & 0 & & \cdots & 0 \\ 0 & 0 & q_3 & p_3 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & & & \cdots & 0 & q_r & p_r \\ 0 & & & \cdots & 0 & 0 & 1 \end{bmatrix}, \quad q_i = 1-p_i,\ I = 1,...,r.$$

The main characteristics of this type of Markov chain are well known. Time to failure (time to absorption) $T = X_1 + X_2 + ... + X_r$, where X_i (time the process spends in i-th state), $i = 1,...,r$, are independent random variables. Random variable X_i has geometric distribution with probability mass function $P(X_i = n) = (1 - p_i)^{n-1} p_i$, $i = 1,2,...$ Expectation value and variance are equal to $E(X_i) = 1/p_i$ and $V(X_i) = (1 - p_i)/p_i^2$. Probability generating function for random variable T is equal to $G_T(z) = \sum_{i=1}^{\infty} p_T(i) z^i = \prod_{i=1}^{r} \frac{zp_i}{1 - z(1 - p_i)}$. The cumulative distribution function of the number of steps up to specimen failure (number of steps of Markov chain up to absorption in absorbing state), T_A, is defined by equation $F_{T_A}(t, S, \eta) = \pi P^t b$, $t = 1,2,3,....$, where π is a row vector : $\pi = (1,0,...,0)$ (in general case the distribution of some defects can be taken into account), the vector b is vector column (0, … 0,1)′. All these formulae are well known. New steps, which has been offered in [4], are as follows: 1) the connection of probabilities p_i, $i = 1,...,r$, with parameter of composite material component tensile strength distribution and parameters of cycles of fatigue loading and 2) the connection of Markov chain state space with DS: order number of Markov chain state co-insider with the order number of DS.

In what follows, for definiteness, loading by a pulsing cycle is assumed; S is the maximum (nominal) stress of the cycle, and η is the vector-parameter (its components are parameters of the distribution function of strength,...). It is assumed that one step of Markov chain in general case corresponds to k_M cycles (the k_M is also a component of the vector η). Then fatigue life (the fatigue cycle number up to specimen failure), T, is equal to $k_m T_A$. The p-quantile fatigue curve which defines the fatigue life $t_p(S)$ (the number of cycles) corresponding to the probability of failure p under an initial normal stress S and the corresponding mean fatigue curve are defined by equations

$$t_p(S) = k_m F_{T_A}^{-1}(p; S, \eta), \quad E(T(S)) = \int_0^{\infty} t dF_{T_A}(t; S, \eta).$$

By fitting experimental data we can get the estimate of the parameter η (first of all, the values k_m and k_S), by using either the nonlinear method of least squares or the method of maximum likelihood.

On Figure 3 we see example of fitting of the data of [5] using Markov chain model and the same cdf of tensile strength of strands as, for example, it is on Figure 1, assuming that $k_S = 1.6$ and $k_m = 12.2847$. The items of matrix P are defined in the following way:

$p_1 = \Phi((\log(S) - \theta_0)/\theta_1)$; $s_2 = S/(1 - p_1)$; $p_{ic} = \Phi((\log(s_i) - \theta_0)/\theta_1)$, $p_i = (p_{ic} - p_{(i-1)c})/(1 - p_{(i-1)c})$, $s_{i+1} = S/(1 - p_{ic})$, $i = 1,2,...,r$.

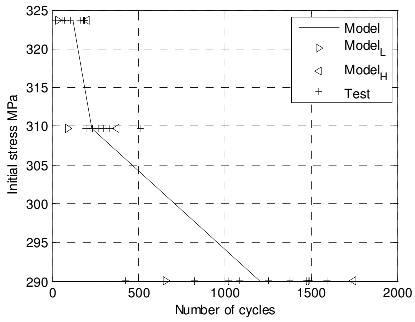


Figure 3. Fatigue test data (+) and Markov model mean fatigue curve for $k_S = 1.6$ and $k_m = 12.2847$; simbols (▶ ◀) show two standard deviation intervals

4. Approximation of Fatigue Curve Using the RDS Definition and Taking into Account the Energy Needed to Failure

Let S_{UT} be the mean ultimate strength of one longitudinal item. The energy needed to failure of one longitudinal item is supposed to be proportional to S_{UT}^2. Similarly, the energy absorbed for failure in one cycle of fatigue loading of the specific longitudinal item with specific random tensile strength, S_k, is supposed to be proportional to S_k^2; $k(s_i)$ defines the number of the longitudinal items with the strength in interval (s_{i-1}, s_i). So, it can be supposed that the time to failure of critical WMV is defined by equation

$$T = k_E N_{WMV} S_{UT}^2 / \sum_{i=1}^{r} k(s_i) s_i^2 \ ,$$

where

k_E – is coefficient of proportionality;

N_{WMV} – is number of longitudinal item in one WMV;

r – is order number of first item of RDS, which is more than S_{UT}^* :

$r = \min\{i : s_i > S_{UT}^*\}$.

The corresponding fatigue curve (diagram) is shown on Figure 4.

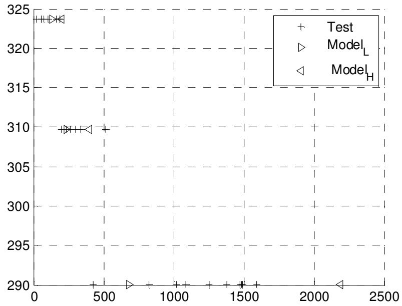


Figure 4. Approximation of fatigue curve using the RDS definition and taking into account the energy needed to failure. Simbols (►◄) correspond to 10th and 990th order statistics in calculation of fatigue lifes using 1000 Monte Carlo trials

Conclusions

In [4] has been shown that using the Daniels' sequence and cdf of strength of longitudinal items (strands) we can explain the existence of fatigue limit of unidirectional composite. But its value is too large and fatigue failure under loading at stress level lower than its value can be explained only by the local stress concentration (or local decreasing of strength). But "predicted DS-fatigue life" is too small. Reasonable good fitting of fatigue test data of carbon-fibre composite specimen was obtained using Markov chain model with states of space based on Daniels' sequence taking into account the local stress concentration and some scale factor. In this paper the similar result was obtained using random Daniels' sequence taking into account the accumulation of energy necessary to produce the failure of the longitudinal items of composite. The examples of calculation of the probability that specimen is still intact (fatigue failure of the specimen does not take place) at fatigue loading at stress level S at fixed number of cycle N are given also. This new model does not provide much more precise approximation of S-N curve but it provides much more "physical" interpretation of it. It seems that the model can be improved by development more appropriate model of energy needed for fatigue failure of longitudinal items. Although the model is too simple and does not provide precise numerical coincidence with experimental fatigue test data but it can explain existence of random fatigue limit and it can be used as nonlinear regression model of S-N fatigue curve. By the use of this model we can try to predict fatigue curve changes as consequence of tensile strength parameter changes.

References

1. Harris, B. (2003). *Fatigue in Composites*. Cambridge, England: Woodhead publishing limited.
2. Pascual, F. G., Meeker, W. Q. (1999). Estimating Fatigue Curves with the Random Fatigue-limit Model. *Technometrics*, 41, 277–302.
3. Paramonov, Yu., Kuznetsov, A., Kleinhofs, M. (2011). *Reliability of fatigue-prone airframes and composite materials.* Riga: RTU.
4. Paramonov, Yu, Cimanis, V. (2012). Fatigue curve approximation using Daniels' sequence and Markov chain. In Proceedings of the XXIV International Research and Practice Conference "Theory and Practice in Physical, Mathematical and Technical Sciences", May 03 to 13, 2012. Great Britain: International Academy of Science and Higher Education (IASHE).
5. Kleinhofs, M. (1983). *Investigation of static strength and fatigue of composite material used in aircraft structure.* Candidate of science degree thesis, Riga Polytechnic Institute. Riga, Latvia.