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ASYMPTOTICAL DECOMPOSITION METHODS FOR MOMENT STABILITY ANALYSIS OF MARKOV DIFFERENCE EQUATIONS

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Abstract. The paper deals with classical approach to equilibrium asymptotical stability analysis adapting the first and the second Lyapunov methods to Markov difference equations. Having proved stability theorem by a linear approximation the convergence problem of the linear stochastic iterations is discussed. It is shown that for moment stability analysis one may use specially constructed semi-group of linear continuous operator in functional space. The mean square Lyapunov index can be calculated as a real spectral radius of the generator of this semi-group. Under some assumptions this moment semi-group can be reduced to finite dimensional space. Besides, one may apply the second Lyapunov method to analysis of spectrum of the above generator.

Key words. Asymptotical stability, Lyapunov method, Markov difference equations, Lyapunov index, spectrum asymptotical decomposition.

Mathematics Subject Classification: 93E15

1 Analysis of the first moments

Linear difference equations with Markov coefficients in $\mathbb{R}^n$:

$$x_t = A(\xi_t)\, x_{t-1} \tag{1.1}$$

where $\{\xi_t, t \in \mathbb{Z}\}$ is a homogeneous Markov Feller chain with phase space $\mathbb{Y}$ as metric and compact space and transition probabilities $P(y, dz)$ will be analyzed. Henceforward a linear *n*-dimensional column-vector space $\mathbb{R}^n$ will be viewed as Euclid space with a scalar product $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n$: $(u,v) = u^T v$. Let assume that a Markov sequence $\vec{\xi} := \{\xi_t,\, t \in \mathbb{Z}\}$ is given in a filtrated probability space $(\Omega, \mathfrak{F}, \mathfrak{F}^t, \mathsf{P})$, where $\{\mathfrak{F}^t\}$ is a minimal filtration harmonizing it.

The following denotations will be used $s \in \mathbb{R}:\ X_s^s = I;\ t > s:\ X_s^t := \prod_{k=s+1}^{t} A(\xi_k)$. The solution of (1.1) can be written in a form $x_t = X_t^s x_s$ for all $s \in \mathbb{R},\ t \ge s$. Let define an operator in a space of continuous n-dimensional reproductions $\mathbb{C}(\mathbb{Y} \to \mathbb{R}^n) := \mathbb{C}_n(\mathbb{Y})$:

$$y \in \mathbb{Y}, u \in \mathbb{C}_n(\mathbb{Y}):\ (\mathsf{A}u)(y) = \int_{\mathbb{Y}} A^T(z) u(z) P(y, dz) \tag{1.2}$$

The operator A is linear continuous operator in $\mathbb{C}_n(\mathbb{Y})$.

Lemma 1 For any $s \in \mathbb{R},\ t > 0,\ v \in \mathbb{C}_n(\mathbb{Y}),\ x \in \mathbb{R}^n$

$$E\left\{\left(X_s^{s+t} x, v(\xi_{s+t})\right) / \mathfrak{F}^s\right\} = \left(x, \left(\mathsf{A}^t v\right)(\xi_s)\right). \tag{1.3}$$

Theorem 1 Let elements of sequence $\{\xi_t, t \in \mathbb{Z}\}$ are independent and identically distributed. Then

(i) operator A leaves as invariant a subspace $\mathbb{R}^n \subset \mathbb{C}_n(\mathbb{Y})$ and restriction $\bar{\mathsf{A}}$ of operator A in this subspace is defined by equality

$$v \in \mathbb{R}^n:\ \bar{\mathsf{A}} v = \bar{A}^T v \tag{1.4}$$

where $\bar{A} = E\{A(\xi_0)\}$;

(ii) for each $s \in \mathbb{Z}$, each $t > s$ and each $\mathfrak{F}^t$-harmonized solution $\{x_t, t \ge 0\}$ of equation (1.1) the following equality is into force:

$$E\{x_t\} = \bar{A}^{t-s} E\{x_s\}. \tag{1.5}$$

The equation (1.1) is mean reducible, if such a continuous matrix function $\{B(y), y \in \mathbb{Y}\}$ and such a matrix Λ exist [4.], that for all $s \in \mathbb{R}$ and $t > s$ the following equality is into force

$$E\left\{B(\xi_t) x_t / \mathcal{F}^s\right\} = \Lambda^{t-s} B(\xi_s) x_s. \tag{1.6}$$

Further the possibility of (1.1) mean reducibility will be considered in the case when the matrix function $\{A(y), y \in \mathbb{Y}\}$ is near to constant and can be given in a form of uniformly converging sequence:

$$A(y) = A_0 + \varepsilon A(y, \varepsilon) := A_0 + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k A_{k+1}(y) \tag{1.7}$$

where $\varepsilon \in (0,1)$ is a small parameter.

The operator (1.2), which corresponds to matrix (1.7), can be expressed in a form $\mathsf{A}(\varepsilon) = \mathsf{A}_0 + \varepsilon \mathsf{A}(\varepsilon)$, hereto the operator A_0 leaves as invariant subspace $\mathbb{R}^n$, and it can be represented as a tensor product of operators $\mathsf{A}_0 = \mathcal{P} \otimes A_0^T$:

$$h \in \mathbb{C}(\mathbb{Y}), g \in \mathbb{R}^n:\ \mathsf{A}_0(h \otimes g) = \mathcal{P} h \otimes A_0^T g,$$

where $\mathcal{P}$ is a Markov operator. The tensor representation of operator allows to simplify finding the spectrum and resolvent using the spectrum and resolvent of operators which define it [3.]. Due to exponential ergodicity the operator A_0 spectrum can be expressed in a form:

$$\sigma\left(\mathsf{A}_0\right)=\left\{\lambda_1\lambda_2 : \lambda_1 \in \sigma\left(\mathcal{P}\right), \lambda_2 \in \sigma\left(A_0\right)\right\}=\sigma\left(A_0\right)\cup\sigma_\rho, \tag{1.8}$$

where $\sigma_\rho\left(A_0\right):=\left\{\lambda_1\lambda_2 : \lambda_1 \in \sigma\left(\mathcal{P}\right), \lambda_2 \in \sigma_\rho\right\}$. As main assumption for mean reducibility of the equation (1.1) is disjunction of sets in spectrum decomposition (1.8), that is, $\sigma\left(A_0\right)\cap\sigma_\rho=\varnothing$.

It makes possible to offer an asymptotical method, which is based on the decomposition of operator $\mathsf{A}\left(\varepsilon\right)$ spectral projection [2.] by powers of a small parameter ε.

Conjugated space of $\mathbb{C}_n\left(\mathbb{Y}\right)$ [1.] is a space of vector-valued measures $\mathbb{C}_n\left(\mathbb{Y}\right)^*$, and scalar product of elements $v\in\mathbb{C}_n\left(\mathbb{Y}\right)$ and $g\in\mathbb{C}_n\left(\mathbb{Y}\right)^*$ is defined by equality

$$\left\langle g,v\right\rangle := \int_{\mathbb{Y}}\left(g\left(dy\right),v\left(y\right)\right). \tag{1.9}$$

Lemma 2 If all above mentioned assumptions are into force, then for sufficiently small $\bar{\varepsilon}>0$ and all $\left|\varepsilon\right|<\bar{\varepsilon}$ a difference equation is mean reducible, hereto the matrix function $\left\{B\left(y,\varepsilon\right), y\in\mathbb{Y}\right\}$ is a basis in operator $\mathsf{A}\left(\varepsilon\right)$ root subspace which corresponds to the spectrum $\sigma_0\left(\varepsilon\right)$ part which is defined by equality $\lim_{\varepsilon\to 0}\sigma_0\left(\varepsilon\right)=\sigma_0$, but matrix $\Lambda\left(\varepsilon\right)$ is operator $\mathsf{A}\left(\varepsilon\right)$ restriction matrix to this root subspace. For each $\left|\varepsilon\right|<\bar{\varepsilon}$ $n\times n$-matrix function of basis $\left\{B\left(y,\varepsilon\right), y\in\mathbb{Y}\right\}$ and constant $n\times n$-matrix $\Lambda\left(\varepsilon\right)$ unambiguously are defined by equality

$$y\in\mathbb{Y},\left|\varepsilon\right|<\bar{\varepsilon}:\ \left(\mathsf{A}\left(\varepsilon\right)B\right)\left(y,\varepsilon\right)=B\left(y,\varepsilon\right)\Lambda^T\left(\varepsilon\right) \tag{1.10}$$

For the description of the construction algorithm for basis matrix $B\left(y,\varepsilon\right)$ and matrix $\Lambda\left(\varepsilon\right)$ the decompositions of these matrices in a form of uniformly converged sequences by powers of a small parameter ε: $\Lambda\left(\varepsilon\right):=\Lambda_0+\varepsilon\sum_{k=0}^{\infty}\Lambda_{k+1}$ and $B\left(y,\varepsilon\right):=B_0+\varepsilon\sum_{k=0}^{\infty}B_{k+1}\left(y\right)$, and also the decomposition of operator $\mathsf{A}\left(\varepsilon\right)$ in a form of uniformly converged sequence by powers of a small parameter ε: $\mathsf{A}\left(\varepsilon\right):=\mathsf{A}_0+\varepsilon\sum_{k=0}^{\infty}\varepsilon^k\mathsf{A}_{k+1}$ are used, where $\left(\mathsf{A}_j v\right)\left(y\right)=\int_{\mathbb{Y}}A_j^T\left(z\right)v\left(z\right)P\left(y,dz\right)$. For each sufficiently small ε these decompositions can be substituted in the expression (1.10). Equating coefficients of equal powers of ε the equations can be obtained for finding the unknown elements of series for $\Lambda\left(\varepsilon\right)$ and $B\left(y,\varepsilon\right)$:

$$\mathsf{A}_0B_0=B_0\Lambda_0^T \tag{1.11}$$

$$\mathsf{A}_0B_1-B_1\Lambda_0^T=B_0\Lambda_1^T-\mathsf{A}_1B_0 \tag{1.12}$$

$$\mathsf{A}_0B_2-B_2\Lambda_0^T=B_0\Lambda_2^T+B_1\Lambda_1^T-\mathsf{A}_0B_2-\mathsf{A}_1B_1 \tag{1.13}$$

Let define an operator

$$\begin{aligned} & y \in \mathbb{Y}, v \in \hat{\mathbb{C}}: \\ (\mathbb{L}v)(y) &:= (\mathsf{A}_0 v)(y) - v(y) A_0^T := \\ &:= \int_{\mathbb{Y}} A_0^T \left(v(z) - v(y) \right) P(y, dz) + A_0^T v(y) - v(y) A_0^T := \\ &:= (\mathbb{H}v)(y) + (\mathbb{G}v)(y) \end{aligned} \tag{1.14}$$

for the elements of continuous matrix functions space $\hat{\mathbb{C}}$. Looking at $\hat{\mathbb{C}}$ as at $\mathbb{R}^{n^2}$, similarly as in $\mathbb{C}_n(\mathbb{Y})$ case, count additive matrix-valued measure $\hat{\mathbb{C}}^*$ can be found, which will be as conjugated space, and a scalar product of elements $g \in \hat{\mathbb{C}}^*$ and $v \in \hat{\mathbb{C}}$ can be defined by formula

$$\langle g, v \rangle := Sp\left\{ \int_{\mathbb{Y}} v^T(y) g(dy) \right\}, \tag{1.15}$$

where $Sp\{\ \}$ is a matrix trace. Taking unit matrix $B_0 := I$ as basis in $\mathbb{R}^n$ and substituting it in the equation (1.11) , $\Lambda_0^T = A_0^T$ can be find, that is, $\Lambda_0 = A_0$. Using Fredholm theorem about normal solvability one can the necessary and sufficiently conditions to ensure that a solution exists. The matrix Λ_2 can be found from the equation (1.12), afterwards also $B_2(y)$. Then the next equations can be written for finding Λ_3, $B_3(y)$ and so on until the needed accuracy of matrix $\Lambda(\varepsilon)$ decomposition is obtained. Since $\mathbb{Y}$ is compact and matrices $\{B_j(y), j = 1, 2, \ldots\}$ are continuous, the elements of obtained basis $B := I + \varepsilon B_1 + \varepsilon^2 B_2 + \ldots$ are linearly independent for sufficiently small ε.

2 Covariance analysis

In this section dynamics of the second moment matrix of difference equation (1.1) solution will be analyzed, that is, behaviour of matrix as matrix function of argument *t*

$$Q_t := E\left\{ x_t x_t^T \right\}. \tag{2.1}$$

At first it should be noted that a real $n \times n$ matrix space $\mathbb{M}_n$ can be viewed as n^2-dimensional Euclid space $\mathbb{R}^{n^2}$ with scalar product $[q, g] := Sp\{qg^T\}$. A set of symmetric $n \times n$ matrices $\hat{\mathbb{M}}_n$ in a form

$$q := \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} \tag{2.2}$$

makes linear closed subspace in $\mathbb{M}_n$. $\hat{\mathbb{M}}_n$ can be identified with Euclid space $\mathbb{R}^{\frac{n(n+1)}{2}}$ with column vectors in form

$$\vec{q} := \left(q_{11}, q_{12}, \dots, q_{1n};\; q_{22}, q_{23}, \dots, q_{2n};\; q_{(n-1)(n-1)}, q_{(n-1)n};\; q_{nn}\right)^T \tag{2.3}$$

and scalar product $(q,g) := q^T g$. For matrices sequence $(xx)_t := x_t x_t^T$ a linear difference equation in space $\mathbb{M}_n$ can be written as

$$(xx)_t = A(\xi_t)(xx)_{t-1} A^T(\xi_t) := \vec{A}(\xi_t)(xx)_{t-1}. \tag{2.4}$$

The above defined linear operator $\vec{A}(\xi_t)$ family leaves as invariant symmetric matrices space $\hat{\mathbb{M}}_n$ for each fixed value of argument ξ_t, therefore, if it is more convenient for use, instead of (2.4) the corresponding linear difference equation in space $\mathbb{R}^{\frac{n(n+1)}{2}}$ can be analyzed.
Let denote *V* Banach space, which consists of symmetric $n \times n$ matrix functions $\{q(y), y \in \mathbb{Y}\}$ with norm $\|q\| := \sup_{y\in\mathbb{Y}, \|x\|=1} \left|(q(y)x, x)\right|$.
Let define a linear continuous operator in space *V* using matrix function $\{A(y), y \in \mathbb{Y}\}$ and transition probabilities of Markov chain

$$(\mathsf{A}q)(y) := \int_{\mathbb{Y}} A^T(z) q(z) A(z) P(y, dz) \tag{2.5}$$

All results from the section 0 can be adapted to the analysis of this operator. In this section it will be shown that the exponentially mean square stability of (1.1) is rather easy to determine analyzing a positive real spectrum of operator (2.5).
If the matrix function $\{A(y), y \in \mathbb{Y}\}$ is almost constant and can be represented in a form of uniformly convergent series (1.7), where $\varepsilon \in (0,1)$ is a small parameter, then considerations can be made similarly as in the section 0 for mean square reducibility of equation, changing a dimension from *n* to $\frac{n(n+1)}{2}$ of matrix $\Lambda(\varepsilon)$ and matrix-basis $B(\varepsilon)$.

3 Equations with independent coefficients

If a sequence $\{\xi_t, t \in \mathbb{N}\}$ consists of independent random variables having identical distribution $p(dy)$, then analysis of covariance of (1.1) gets simpler. In this case, similarly as in analysis of the first moments, the operator's A defined by formula (2.5) restriction $\hat{\mathsf{A}}$ on space of constant real symmetric $n \times n$-matrices $\mathbb{M}_n$:

$$\hat{\mathsf{A}} q := \mathbb{E}\left\{A^T(y_t) q A(y_t)\right\} = \int_{\mathbb{Y}} A^T(y) q A(y) p(dy)$$

and the cone of positive defined matrices $\overset{\circ}{\mathsf{K}}_n := \hat{\mathbb{M}}_n \cap \overset{\circ}{\mathbb{K}}$ can be used.
If a sequence $\{\xi_t, t \in \mathbb{N}\}$ consists of independent random variables and the equation (1.1) is exponentially mean square stable, then maximal by module spectrum point $r\{\mathsf{A}\}$ of operator A is less than unit. It allows rather easy to analyse the solutions of *m*-dimensional scalar difference equations behaviour

$$x_{n+m} = \sum_{k=0}^{m-1} a_{k+1} x_{n+k} + c \sum_{k=0}^{m-1} h_{k+1} \xi_{n+k+1} x_{n+k} , \tag{3.1}$$

where $\{\xi_k\}$ is a sequence of identically distributed independent random variables with mean value zero and unit variance. This equation can be rewritten in vector form in space $\mathbb{R}^m$:

$$\vec{X}_{n+1} = A\vec{X}_n + c\sum_{k=0}^{m-1} \xi_{n+k+1} H_{k+1} \vec{X}_n , \tag{3.2}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_m & a_{m-1} & a_{m-2} & \dots & a_1 \end{pmatrix}, \quad H_k = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & h_{m-k} & 0 & \dots & 0 \end{pmatrix}.$$

It can be shown that the second moment of any solution of equation (3.2) is exponentially decreasing if and only if for any $0 \le \rho < 1$ a positive defined matrix solution of equation

$$A^T q A + c^2 \sum_{k=1}^{m} H_k^T q H_k = \rho q$$

exists. Therefore, if eigenvalues of matrix *A* are located inside circle $\{|z| < 1\}$, then such a positive number $c^2 < r^2$ exists, that the second moment of each solution $\mathbb{E}|x_n|^2$ of equation (3.1) tends to zero if $n \to \infty$, but in case if $c^2 > r^2$ then unlimited increasing solution exists.
A system of *m* linear equations for numbers q_{jm}, $j = 1, 2, \dots, m$ can be easy found:

$$q_{im} - \sum_{l=1}^{i} a_l q_{(m-i-1+l)m} - \sum_{l=1}^{m-i} a_{m-i-l+1} q_{(m-l+1)m} = 0, \ i = 1, 2, \dots, m-1 \tag{3.3}$$

$$q_{mm}\left(1 - \sum_{i=1}^{m}\left(a_{m-i+1}^2 + r^2 h_{m-i+1}^2\right)\right) - 2\sum_{l=1}^{m-1}\sum_{s=1}^{l} a_{m-l+1} a_{l-s+1} q_{lm} = 0 \tag{3.4}$$

Because of number r^2 existence, this equation should have nontrivial solution and therefore a determinant of equations system (3.3)-(3.4) should be equal to zero. Taking into account the form of analysed equations system the following conclusion can be made, that its determinant is a linear function of parameter r^2 and this number can be found as ratio of two parameters.

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