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EXPONENTIAL STABILITY OF FAST OSCILLATING LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The paper deals with linear finite dimensional functional differential equations dependent on fast oscillating parameters. It is proved that for exponential stability analysis one may apply an averaging procedure to the time dependent generator of two-parameter shift operator family defined by this equation in the space of continuous functions. This method permits to reduce stability analysis to spectrum analysis of a closed operator with compact resolvent.

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Let us consider the functional differential equation in $\mathbf{R}^n$

$$\frac{dx(t)}{dt} = A(t)x_t \tag{1}$$

with initial condition

$$\theta \in [-h, 0] : \quad x(s+\theta) = \varphi(\theta) \tag{2}$$

where x_t is part of n-dimensional continuous vector-function $x(t)$ defined by equality $x_t := \{x(t+\theta), -h \leq \theta \leq 0\}$ for some $h \geq 0$, $\{A(t), t \in \mathbf{R}\}$ is family of linear continuous mappings of the space $\mathbf{C}_n([-h, 0])$ of n-dimensional vector-functions to $\mathbf{R}^n$, defined by formulae

$$\varphi \in \mathbf{C}_n([-h, 0]) : \qquad A(t)\varphi := \int_{-h}^{0} \{d_\theta K(t, \theta)\}\varphi(\theta)$$

where $\{K(t,\theta), t \in \mathbf{R}, \theta \in [-h,0]\}$ is continuous on t and bounded variation on θ matrix-function family. It is well known [1] that under condition $\sup_{t\in\mathbf{R}} ||A(t)|| = C < \infty$ there exists unique solution $\{x(t,s,\varphi), t \geq s\}$ of the initial problem (1)-(2) for any $s \in \mathbf{R}$ and $\varphi \in \mathbf{C}_n([-h,0])$. This solution as a function on t is n-dimensional continuous vector-function and therefore using formula $(T(t,s)\varphi)(\theta) := x(t+\theta,s,\varphi), \theta \in [-h,0]$ one can define on the space $\mathbf{C}_n([-h,0])$ two parameter family of shift operators $\{T(t,s), t \geq s\}$ having a semigroup properties $T(t,\tau)T(\tau,s)$ for any $t \geq \tau \geq s$ [1] and satisfying an operator equation

$$\frac{\partial}{\partial t} T(t,s) = \mathbb{A}(t)T(t,s)$$

where operators $\mathbb{A}(t)$ are defined by equalities

$$(\mathbb{A}(t)\varphi)(\theta) = \begin{cases} \frac{d}{d\theta}, & \text{if } -h \leq \theta < 0, \\ A(t)\varphi, & \text{if } \qquad \theta = 0 \end{cases} \tag{3}$$

for any $\varphi \in \mathcal{D}(\mathbb{A}(t)) := \{\mathbf{C}_n^1[-h,0] : \quad \varphi'(0) = A(t)\varphi\}$. We will refer to this operators as *generator family* of the above two parameter semigroup. If the semigroup $\{T(t,s), t \geq s\}$ exponential decreases for any $s \in \mathbf{R}$, i.e.

$$\forall s \in \mathbb{R}, \forall t \geq s, \exists M > 0, \exists \rho > 0 : \quad ||T(t,s)|| \leq M \exp\{-\rho(t-s)\}$$

we will say that equation (1) is exponential stable.

Our paper deals with subjected to parameter ε functional differential equation

$$\frac{dx^{\varepsilon}(t)}{dt} = A\left(\frac{t}{\varepsilon}\right) x_t^{\varepsilon} \tag{4}$$

which we will analyze for sufficiently small positive ε. The corresponding to (4) two parameter semigroup $\{T^{\varepsilon}(t,s)\}$ has so called fast oscillating generator family $\{\mathbb{A}\left(\frac{t}{\varepsilon}\right)\}$. This generator family defines a behavior of $\{T^{\varepsilon}(t,s)\}$ as $t \to \infty$. If there exists such an operator $\bar{\mathbb{A}} : \mathbf{C}_n^1([-h,0]) \to \mathbf{C}_n([-h,0])$ that $\mathcal{D}([\mathbb{A}(t) - \bar{\mathbb{A}}]) = \mathbf{C}_n([-h,0])$ and

$$\lim_{T\to\infty} \left|\left| \frac{1}{T} \int_s^{s+T} [\mathbb{A}(t) - \bar{\mathbb{A}}] dt \right|\right| = 0 \tag{5}$$

we call it *an averaged generator*. It is easily understood that the above equality is equivalent to the equality

$$\lim_{T\to\infty} \sup_{\substack{s\in\mathbf{R} \\ ||\varphi||=1}} \left|\left| \frac{1}{T} \int_s^{s+T} [\mathbb{A}(t) - \bar{\mathbb{A}}]\varphi dt \right|\right| = \lim_{T\to\infty} \sup_{\substack{s\in\mathbf{R} \\ ||\varphi||=1}} \left| \frac{1}{T} \int_s^{s+T} [A(t) - \bar{A}]\varphi dt \right| = 0 \tag{6}$$

and therefore an averaged generator for (4) may be defined in a following form

$$(\bar{\mathbb{A}}\varphi)(\theta) = \begin{cases} \frac{d}{d\theta}, & \text{if } -h \leq \theta < 0, \\ \bar{A}\varphi, & \text{if } \qquad \theta = 0 \end{cases} \tag{7}$$

where $\bar{A}\varphi := \lim\limits_{T\to\infty} \frac{1}{T}\int\limits_0^T A(t)\varphi dt$. As it has been proved in [1] the operator given by equality (7) is closed operator with spectrum $\sigma(\bar{\mathbb{A}}) = \{z \in \mathbf{C} : det\left[Iz - \bar{\mathbb{A}}\mathbf{e}\right] = 0\}$ where $\mathbf{e}$ is n-dimensional vector-function with components $\{\exp\{z\theta\},\ \theta \in [-h, 0]\}$.

Side by side with initial problem (2)-(4) we will analyze the initial problem (2) for averaged equation

$$\frac{du(t)}{dt} = \bar{A}u_t \tag{8}$$

denoting this solution $u(t, s, \varphi) \equiv u(t - s, 0, \varphi) := S(t - s)\varphi$, where $\{S(t), t \geq 0\}$ is resolving semigroup [1] for (8) with generator (7).

Lemma. *Assume that* $\sup\limits_{t\in\mathbf{R}} ||A(t)|| = C < \infty$, *there exists an average generator* $\bar{\mathbb{A}}$ *satisfying the condition* (5) *and* $\sigma\{\bar{\mathbb{A}}\} \subset \{\mathbf{R}ez < 0\}$. *Then for any* $Q > 0$ *and* $\delta > 0$ *one can choose such a positive number* $\hat{\varepsilon}(Q, \delta)$ *that for any* $\varphi \in \mathbf{K}_1, t_0 \in \mathbf{R}$, *and* $\varepsilon \in \hat{\varepsilon}(Q, \delta)$ *the following inequality is valid*

$$g(Q, \varepsilon) := \max_{0\leq t\leq Q, s\in\mathbf{R}} \left|\left| \int\limits_{t_0}^{t_0+t} [\mathbb{A}\left(\frac{s}{\varepsilon}\right) - \bar{\mathbb{A}}]S(s - t_0)ds \right|\right| \leq \delta \tag{9}$$

Proof. By definition of the norm in $\mathbf{C}_n([-h, 0])$ the quantity (9) may be presented formula

$$g(Q, \varepsilon) = \max_{\substack{0\leq t\leq Q, s\in\mathbf{R} \\ ||\varphi||=1}} \left| \int\limits_{t_0}^{t_0+t} [A\left(\frac{s}{\varepsilon}\right) - \bar{A}]u_s(t_0, \varphi)ds \right|$$

The assertion $\sigma\{\bar{\mathbb{A}}\} \subset \{\mathbf{R}ez < 0\}$ guarantees an existence of positive numbers M and γ [1] that $||S(t - s)||| \leq Mexp\{-\gamma(t - s)\}$ for any $t \geq s$. Besides inequality $||\bar{\mathbb{A}}|| \leq C$ follows the definition of this operator. Therefore to prove (9) owing the first assertion of lemma one may use inequality

$$\max_{\substack{0\leq s_1\leq t\leq s_2 \\ s_1\in\mathbf{R}}} \left|\left| \int\limits_{s_1}^{t} [\mathbb{A}\left(\frac{s}{\varepsilon}\right) - \bar{\mathbb{A}}]S(s - t_0)ds \right|\right| = \max_{\substack{0\leq t\leq T, s\in\mathbf{R} \\ ||\varphi||=1}} \left| \int\limits_{s_1}^{t} [A\left(\frac{s}{\varepsilon}\right) - \bar{A}]u_s(t_0, \varphi)ds \right| \leq 2CM(s_2 - s_1)$$

Let $t_0 < t_1 < \cdots < t_{m-1} < t_m = t_0 + Q$ be a partition of the segment $[t_0, t_0 + Q]$ defined by points $t_k = k\frac{Q}{m}, k = 0, 1, \ldots m,\ m \in \mathbf{N}$. Then

$$\begin{aligned} g(T, \varepsilon) &= \max_{0\leq k\leq m-1} \max_{t_k\leq t\leq t_{k-1}} \left| \int\limits_{t_0}^{t} [A\left(\frac{s}{\varepsilon}\right) - \bar{A}]u_s(t_0, \varphi)ds \right| = \\ &\leq \max_{1\leq k\leq m} \left| \int\limits_{t_0}^{t_k} [A\left(\frac{s}{\varepsilon}\right) - \bar{A}]u_s(t_0, \varphi)ds \right| + 2\frac{MTC}{m} \end{aligned}$$

or

$$\begin{aligned} g(Q,\varepsilon) &\leq \max_{1\leq k\leq m}\left|\sum_{l=0}^{k-1}\int_{t_l}^{t_{l+1}}[A\left(\frac{s}{\varepsilon}\right)-\bar{A}]u_s(t_0,\varphi)ds\right|+\frac{\delta}{2}\leq \\ &\leq m\max_{0\leq k\leq m-1}\left|\left|\int_{t_k}^{t_{k+1}}[\mathbb{A}\left(\frac{s}{\varepsilon}\right)-\bar{\mathbb{A}}]ds\right|\right|M+\frac{\delta}{2} \\ &\leq MQ\sup_{s\in\mathbf{R}}\left|\left|\frac{m\varepsilon}{Q}\int_{s}^{s+\frac{Q}{m\varepsilon}}[\mathbb{A}(t)-\bar{\mathbb{A}}]dt\right|\right|+\frac{\delta}{2} \end{aligned}$$

for any $m > \frac{4MQC}{\delta}$. Now one can take an advantage of formula (6) and chose number $\hat{\varepsilon}$ in such a way that for any $\varepsilon \in (0,\hat{\varepsilon})$

$$\sup_{\psi\in\mathbf{K}_1,s\in\mathbf{R}}\left|\left|\frac{m\varepsilon}{Q}\int_{s}^{s+\frac{Q}{m\varepsilon}}[\mathbb{A}(t)-\bar{\mathbb{A}}]dt\right|\right|<\frac{\delta}{2MQ}$$

and proof is completed.

Theorem. *Under assertions of the above lemma there exists such a number $\hat{\varepsilon}$ that for any $\varepsilon \in (0,\hat{\varepsilon})$ equation (1) is exponentially stable.*

Proof. For any $\varphi \in \mathbf{C}_n([-h,0]), \varepsilon > 0, s \in \mathbf{R}$, and $t \geq 0$ one can write an integral inequality

$$\begin{aligned} ||[T^{\varepsilon}(t,s)-S(t-s)]\varphi|| &\leq \left|\left|\int_s^t\left[\mathbb{A}\left(\frac{\tau}{\varepsilon}\right)T^{\varepsilon}(\tau,s)-\mathbb{A}\left(\frac{\tau}{\varepsilon}\right)S(\tau-s)\right]\varphi d\tau\right|\right|+ \\ +\left|\left|\int_s^t[\mathbb{A}\left(\frac{\tau}{\varepsilon}\right)-\bar{\mathbb{A}}]u_{\tau}(s,\varphi)d\tau\right|\right| &\leq g(\varepsilon,t)M||\varphi||+C\int_s^t||[T^{\varepsilon}(\tau,s)-S(\tau-s)]\varphi||d\tau \end{aligned}$$

which guarantees that$||[T^{\varepsilon}(t,s)-S(t-s)]\varphi|| \leq g(\varepsilon,t)M||\varphi||exp\{C(t-s)\}$. Now for a given $T > 0$ one can chose $\hat{\varepsilon}(Q)$ so small that $g(\varepsilon,Q) \leq \frac{1}{4Mexp\{CQ\}}$ and then $||[T^{\varepsilon}(s+Q,s)-S(Q)]\varphi|| \leq \frac{||\varphi||}{4}$ for all $\varepsilon \in (0,\hat{\varepsilon}(Q)), s \in \mathbf{R}$, and $\varphi \in \mathbf{C}_n([-h,0])$. Because under condition $\sigma(\bar{\mathbb{A}}) \subset \{\mathbf{R}ez < 0\}$ the inequality $|u(t,s,\varphi)| \leq M||\varphi||exp\{-\gamma(t-s)\}$ is valid for any $\varphi \in \mathbf{C}_n([-h,0]), s \in \mathbf{R}$, and $t \geq s$ with some positive constants M and γ we can put $\hat{Q} = \frac{1}{\gamma}ln(4M)$ and to write inequality

$$||T^{\varepsilon}(s+\hat{Q},s)\varphi|| \leq ||S(\hat{Q})\varphi||+||T^{\varepsilon}(s+\hat{Q},s)\varphi-S(\hat{Q})\varphi|| \leq \frac{||\varphi||}{2}$$

for any $\varphi \in \mathbf{C}_n([-h,0]), s \in \mathbf{R}$, and $\varepsilon \in (0,\hat{\varepsilon}(\hat{Q}))$. Besides for chosen $\hat{Q}$ under condition $\sup_{t\in\mathbf{R}}||A(t)|| = C < \infty$ the solutions of (4) may be majorized in the following way

$$\max_{0\leq t\leq\hat{Q}}||T^{\varepsilon}(s+\hat{Q},s)\varphi|| = \max_{0\leq t\leq\hat{Q}}||x^{\varepsilon}_{s+t}(s,\varphi)|| \leq ||\varphi||exp\{C\hat{Q}\} = ||\varphi||(4M)^{\frac{1}{\gamma}}$$

for any $\varphi \in \mathbf{C}_n([-h,0]), s \in \mathbf{R}$, and $\varepsilon \in (0, \hat{\varepsilon}(Q))$. Therefore taking the numbers $k = 1, 2, ...$ one can construct an exponentially decreasing scala

$$\begin{aligned} ||T^\varepsilon(s+(k+1)\hat{Q}, s)\varphi|| &\le \frac{1}{2}\max_{0\le t\le \hat{Q}} ||T^\varepsilon(s+k\hat{Q}, s)\varphi|| \le \cdots \le \left(\frac{1}{2}\right)^k ||\varphi|| \\ \max_{0\le t\le \hat{T}} ||T^\varepsilon(s+k\hat{Q}+t, s)\varphi|| &\le ||T^\varepsilon(s+k\hat{Q}, s)\varphi||(4M)^{\frac{1}{\gamma}} \le \left(\frac{1}{2}\right)^k (4M)^{\frac{1}{\gamma}}||\varphi|| \end{aligned}$$

which majorizes solutions of (4) and guarantees an exponential decreasing of two parametric family $T^\varepsilon(t,s)$ with $t \to \infty$ uniformly on $||\varphi|| = 1, s \in \mathbf{R}, \varepsilon \in (0, \hat{\varepsilon}(Q))$ and proof is completed.

Example. Let us consider scalar delay-differential equation

$$\frac{dx^\varepsilon(t)}{dt} = ax^\varepsilon\left(t - 1 + h\sin\left(\frac{t}{\varepsilon}\right)\right) \tag{10}$$

with constant $h \in (-1,1)$. The operator $A(t)$ for this equation is defined by formula $A(t)\varphi := a\varphi(-1 + h\sin t)$ which to involve identity $A(t) \equiv A(t+2\pi)$. Therefore no so difficult to be certain of the hypothesis (6) validity because

$$\begin{aligned} \int_s^{s+T} A\left(\frac{t}{\varepsilon}\right)\varphi dt &= \varepsilon\left(\int_{\frac{s}{\varepsilon}}^{\frac{s}{\varepsilon}+\frac{T}{\varepsilon}} A(t)\varphi dt\right) = \varepsilon\left[\frac{T}{2\pi\varepsilon}\right]\int_0^{2\pi} A(t)\varphi dt + \\ +\varepsilon\left(\int_0^{\frac{T}{\varepsilon}-2\pi\left[\frac{T}{2\pi\varepsilon}\right]} A(t)\varphi dt\right) &= \varepsilon\left[\frac{T}{2\pi\varepsilon}\right]2\pi\bar{A}\varphi + +\varepsilon\left(\int_0^{\frac{T}{\varepsilon}-2\pi\left[\frac{T}{2\pi\varepsilon}\right]} A(t)\varphi dt\right) \end{aligned}$$

where

$$\bar{A}\varphi := \frac{1}{2\pi}\int_0^{2\pi} \varphi(-1 + h\sin t)dt \tag{11}$$

and $[\alpha]$ – integer port of number α. Now to find a border of exponential stability region as a subset of band $\{a \in \mathbf{R}, -1 < h < 1\}$ for average equation

$$\frac{du(t)}{dt} = \frac{a}{2\pi}\int_0^{2\pi} u(t - 1 + h\sin\theta)d\theta \tag{12}$$

we will apply the D-partition method [1] to characteristic equation. This approach gives us the border of stability in a parametric form

$$\begin{aligned} \frac{a}{2\pi}\int_0^{2\pi} \sin\{\lambda(-1 + h\sin\theta)\}d\theta &= \lambda \\ \frac{a}{2\pi}\int_0^{2\pi} \cos\{\lambda(-1 + h\sin\theta)\}d\theta &= 0 \end{aligned}$$

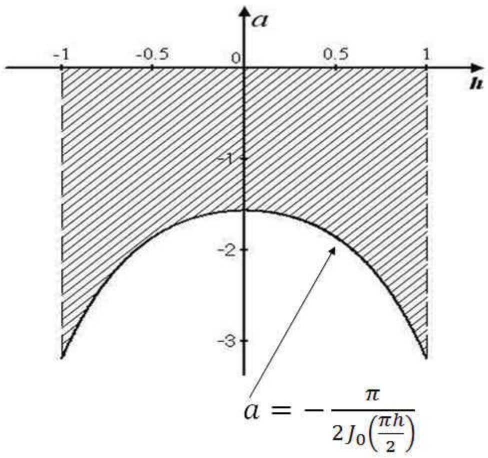

Figure 1: Stability region.

or

$$-aJ_0(\lambda h)\sin\lambda = \lambda, \quad aJ_0(\lambda h)\cos\lambda = 0$$

where $J_0(t)$ – Bessel function of zero order. Because $J_0(0) = 1$ we can substitute the solution of the first equation $\lambda = \frac{\pi}{2} + \pi k, k = 0, \pm 1, \pm 2, ...$ in the second equation and to write equations for borders of D-partitions. Not complicated analysis of derived regions permits to write the region of exponential stability for functional differential equation (12) in a form of inequalities

$$-1 < h < 1, -\frac{\pi}{2J_0\left(\left(\frac{\pi}{2}\right)h\right)} < a < 0$$

Therefore if the parameters a and h in the equation (10) belong to the shaded area of the above figure and $\varepsilon > 0$ is sufficiently small any solution of this equation exponentially tends to zero as $t \to \infty$. Looking at the figure 1 one may draw a conclusion that fast periodical perturbation of delay can enlarge stability region of retarding dynamical systems.

The proposal method may be successfully applied ([3], [2]) also to asymptotic analysis of retarding dynamical system subjected to rapid Markov switching.

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