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ON STABILITY ANALYSIS OF COCYCLES OVER IMPULSE MARKOV DYNAMICAL SYSTEMS

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Abstract. This paper deals with Cauchy matrix family of a linear differential equation with right part dependent on a step Markov process and an impulse type dynamical system switched by the above process. All the above mentioned stochastic dynamical objects are also dependent on small positive parameter ε, and the infinitesimal operator of Markov process is proportional to ε^{-1}. This means that impulse dynamical system is rapidly switched and one may simplify this using merger procedur to Markov process and averaging procedures to impulse dynamical system and matrix evolution family. Applying these procedures one achieves more simple linear differential equation for matrix evolution family, which becomes now dependent on more simple dynamical systems such as an ordinary differential equation with a right part switched by a lumped Markov process. It is proved that under some hypotheses one may successfully apply these resulting evolution families not only to approximation of the initial family on an arbitrary finite time interval but also to describe a time asymptotic of it.

Key words and phrases. Cocycles, Random Evolutions, Impulse Stochastic Equations, Stochastic Stability, Averaging Procedures.

Mathematics Subject Classification. Primary 60H10, 60H30; Secondary 37H10.

1 Introduction

As it has been metioned in abstract our mathematical model consists of: a right continuous *Step Markov Process* (**SMP**) $\{y^\varepsilon(t),\, t \geq 0\}$ with switching times $\mathbf{S} := \{\tau_k^\varepsilon,\, k \in \mathbb{N}\}$ given on discrete metric space $\mathbb{Y}$ by a weak infinitesimal operator [2] $Q^\varepsilon v(y) := \frac{1}{\varepsilon} Q_1 v(y) + Q_2 v(y)$, where operator Q_1 has 0 is a simple spectrum point of multiplicity d, ε is small positive parameter, $Q_j v(y) = a(y) \sum\limits_{z\in\mathbb{Y}} [v(z) - v(y)] p_j(y,z),\ j = 1,2$ and $v(y)$ is an arbitrary bounded measurable mapping $\mathbb{Y} \to \mathbb{R}$;

Impulse Markov Dynamical System (**IMDS**)[8] switched by the above **SMP**, and given as right continuous m-dimensional vector-function $\{x^\varepsilon(t),\, t \geq 0\}$ satisfying

- a differential equation for $t \notin \mathbb{S}$

$$\frac{dx^\varepsilon}{dt} = f(x^\varepsilon(t),\, y^\varepsilon(t),\, \varepsilon), \tag{1}$$

- a jump condition for $t \in \mathbb{S}$

$$x^\varepsilon(t) = x^\varepsilon(t-0) + \varepsilon g(x^\varepsilon(t-0), y^\varepsilon(t-0), \varepsilon); \tag{2}$$

*Markov Evolution Family (***MEF***) or Markov Multiplicative Cocycle* [5, 6] given as a two parametric Cauchy matrix family $\{X^\varepsilon(t,s),\, t \geq s \geq 0\}$ satisfying a linear differential equation in $\mathbb{R}^n$:

$$\frac{d}{dt}\, X^\varepsilon(t,s) = A(x^\varepsilon(t),\, y^\varepsilon(t),\, \varepsilon)\, X^\varepsilon(t,s). \tag{3}$$

The problem of asymptotic analysis of dynamical systems with random switching has been discussed in many mathematical and engineering papers. Apparently, A. V. Skorokhod was the first mathematician to have proved that the probabilistic limit theorems may be successfully used for differential equations with right parts dependent on step Markov process (English edition [7]). The approach proposed by A.V. Skorokhod and developed by many authors (see, for example, [1] and references there) makes it possible under assumption $d = 1$ to apply for asymptotic analysis of **IMDS** (1)–(2)–(3) not only the averaging procedure by time and invariant measure of **SMP** but also diffusion approximation technique. Some of the above mentioned results have been published also in our previous paper. In this paper we will deal with $d > 1$ and will discuss an ability of proposed by V.S.Krolyuk [4] so called *merger procedure* involving averaging procedure by all invariant measures of operator Q_1^*.

2 Assumptions and notations.

To achieve the limiting **MEF** for (1)-(2)-(3) this paper assumes that:

(i) $\forall\, y \in \mathbb{Y} \,: 0 < \hat{a}_1 \leq a(y) \leq \hat{a}_2 < \infty$;

(ii) $\forall\, y, z \in \mathbb{Y}\, |p_2(y,z)| \leq c < \infty,\ p_1(y,z) \geq 0,\ \sum_{z \in \mathbb{Y}} p_1(y,z) = 1$;

(iii) 0 is a simple spectrum point of operator Q_1 of multiplicity d and

$$\exists \rho > 0 :\ \sigma(Q_1) \setminus \{0\} \subset \{z \in \mathbf{C} : \Re z < -\rho\};$$

(iv) $f(x,y,\varepsilon) = f_1(x,y) + \varepsilon f_2(x,y)$, $g(x,y,\varepsilon) = g_1(x,y) + \varepsilon g_2(x,y)$, and $f_j(x,y),\ g_j(x,y)\ j = 1,2$ are boundedly (on x and y) continuously differentiable on x functions;

(v) $A(x,y,\varepsilon) = A_1(x,y) + \varepsilon\, A_2(x,y)$ and matrices $A_j(x,y),\ j = 1,2$ are bounded and continuous on x.

In our presentation we will use the following notations and definitions:

- $F_j(x,y) = f_j(x,y) + a(y)g_j(x,y),\ j = 1,2;$
- μ_j – the probabilistic measures with nonintersecting supports $\mathbb{Y}_j$ defined as the solutions of the equation $Q_1^*\mu_j = 0$, where $(Q_1^*\mu_j)(y) := \sum\limits_{z\in\mathbb{Y}_j} a(z)p_1(z,y)\mu_j(z) - a(y)\mu_j(y),\ 1 \le j \le d;$
- $\mathbf{P}_0$ – projective operator in a kernel of Q_1: $1 \le j \le d,\ y \in \mathbb{Y}_j : (\mathbf{P}_0 v)(y) := \sum\limits_{y\in\mathbb{Y}_j} \mu_j(y)\, v(y);$

  Π - an extension of potential: $(\Pi v)(y) := \int\limits_0^\infty \sum\limits_{z\in\mathbb{Y}} P(t,y,z)[v(z) - (\mathbf{P}_0 v)(z)]dt$, where $P(t,y,z)$ is transition probability of a Markov process corresponding to the infinitesimal operator Q_1;
- $\lambda_p(\varepsilon) := \lim\limits_{t\to\infty} \sup\limits_{x,y} \frac{1}{pt} \ln \mathbb{E}^s_{x,y}\{||X^\varepsilon(t,s)||^p\}$– Lyapunov p-index.

Our paper has for an object to analyse asymptotic behaviuor of **MEF** with $t \to \infty$. **MEF** is said to be asymptotic decreasing with probability one if

$$\lim_{T\to\infty} \mathbb{P}\left\{\sup_{t\ge T} ||X^\varepsilon(t,s)|| \ge \delta / x(s) = x, y(s) = y\right\} = 0$$

for any $\delta > 0$, $x \in \mathbb{R}^n$, $y \in \mathbb{Y}$. Here and elsewhere further probability or expectation with indices denote conditional ones, that is, in the above formula one should read $\mathbb{P}\{\bullet / x(s) = x, y(s) = y\}$ instead of $\mathbb{P}^s_{x,y}\{\bullet\}$.

3 Exponential decreasing of MEF.

In this section we will prove that for analysis of **MEF** behaviour as $t \to \infty$ one may use more simple **MEF**

$$\frac{d\hat{x}(t)}{dt} = \hat{F}_1(\hat{x}(t), \hat{y}(t)) \tag{4}$$

where $\{\hat{y}(t)\}$ is homogeneous Markov process (enlardged MP) with state space $\hat{Y} := \{Y_1, Y_2, ..., Y_d\}$, infinitesimal matrix $\Gamma := \{\gamma_k^j\}$, which elements are given by equalities

$$\gamma_k^j = \begin{cases} \sum\limits_{y\in Y_k} \sum\limits_{z\in Y_j} a(y)p_2(y,z)\mu_k(y), \text{if} & j \ne k; \\ -\sum\limits_{\substack{l=1 \\ l\ne j}}^{d} \gamma_l^j, & \text{if} \quad j = k \end{cases}$$

for $k = 1, 2, ..., d$, and defined by this infinitesimal matrix transition probability function $P_0(t,y,z)$. Due to assumption on spectrum structure of the operator Q_1 one can define [2] the projective operator $\mathcal{P}$ by the equalities

$$\forall\ y \in\ Y_k,\ v \in \mathbb{B}(Y) :\ (\mathcal{P}v)(y) \equiv \sum_{z\in Y_k} v(z)\mu_k(z)$$

for each $k = \overline{1, d}$ and the linear continuous operator $\hat{\Pi}: \mathbb{B}(\hat{Y}) \to \mathbb{B}(\hat{Y})$ by equality

$$(\hat{\Pi}v)(y) := \int_0^\infty \sum_{z \in \hat{Y}} P_0(t, y, z)(v - \mathcal{P}v)(z)dt \tag{5}$$

We will refere to operator (5) as *a potential of enlarged Markov process.*

Theorem 3.1 *(***Merger principle** [8]*). Under the above assumptions the family of processes* $\{x^\varepsilon(s), 0 \le s \le T\}$ *for any* $T > 0$ *weak converges as* $\varepsilon \to 0$ *to the solution of* (4) *with corresponding initial condition.*

Theorem 3.2 *Under the above assumptions if defined by* (4) *evolution family exponentially decrease in the mean with power p then there exists such* $\varepsilon_p > 0$ *that initial* **MEF** *exponentially decrease in the mean with power p for any* $\varepsilon \in (0, \varepsilon_p)$.

Proof. Due to exponential decrease of the p-moments of the solutions of (4) and a boundedness of the x-derivative of $\tilde{F}_1(x, y)$ one can define function

$$y \in Y_k : \ v^{(p)}(x, y) \equiv \hat{v}^{(p)}(x, k) := \int_0^T \mathbb{E}_{x,k}|\hat{x}(t)|^p \, dt, \ \ k = \overline{1, h},$$

with so large constant T that the above function satisfies the inequalities $m_1 \, |x|^p \le v^{(p)}(x, y) \le m_2 \, |x|^p$ with some positive constants m_1, m_2, and the inequality

$$(\hat{F}_1(x, k), \bigtriangledown) \, \hat{v}^{(p)}(x, k) + \Gamma\hat{v}^{(p)}(x, k) \le -m_3 \, \hat{v}^{(p)}(x, k)$$

is hold with some positive constant m_3 for any $k = \overline{1, h}$ and $x \in \mathbb{R}^n$. To prove the theorem we will use the Lyapunov function

$$v_\varepsilon^{(p)}(x, \, y) := v^{(p)}(x, y) + \varepsilon\tilde{\Pi}\{F_1(x, \, y), \, \bigtriangledown) \, v^{(p)}(x, y) + Q_1 \, v^{(p)}(x, y)\},$$

which satisfies the inequalities $\hat{m}_1 \, |x|^p \le v_\varepsilon^{(p)}(x, \, y) \le \hat{m}_2 \, |x|^p$ with some positive constants $\hat{m}_1$, $\hat{m}_2$ for any $\varepsilon \in (0, 1)$. By definition of the operator $\tilde{\Pi}$ one can write the equality

$$\begin{aligned} (F_1(x, y), \, \bigtriangledown) \, v^{(p)}(x, y) &+ Q_1 \, v^{(p)}(x, y)\} + Q_0 v_1^{(p)}(x, y) \\ = (\tilde{F}_1(x, y), \, \bigtriangledown) \, v^{(p)}(x, y) &+ \mathcal{P}Q_1 \, v^{(p)}(x, y)\} + \varepsilon r(x, y, \varepsilon) \\ = (\hat{F}_1(x, k), \, \bigtriangledown) \, \hat{v}^{(p)}(x, k) &+ \Gamma \, \hat{v}^{(p)}(x, k)\} + \varepsilon r(x, y, \varepsilon) \\ \le -m_3\hat{v}^{(p)}(x, k) &+ \varepsilon\alpha(\varepsilon)|x|^p \end{aligned}$$

and therefore

$$(F_1(x, y), \, \bigtriangledown) \, v^{(p)}(x, y) + Q_1 \, v^{(p)}(x, y)\} + Q_0 v_1^{(p)}(x, y) \le -m_3\hat{v}^{(p)}(x, k) + \varepsilon\alpha(\varepsilon)|x|^p$$

for any $y \in Y_k$ and $k = \overline{1, d}$, where $\alpha(\varepsilon)$ is infinitesimal as $\varepsilon \to 0$. Due to the above inequalities there exists such positive constants ε_p that

$$\mathbb{L}(\varepsilon)\, v_\varepsilon^{(p)}(x,\, y) \leq -\frac{m_3}{2}\, v_\varepsilon^{(p)}(x,\, y)$$

for any $\varepsilon \in (0,\, \varepsilon_p)$. Using Dynkin formula [2] for the stochastic process

$$\xi(s) := v_\varepsilon^{(p)}(x^\varepsilon(s),\, y(s/\varepsilon))\, e^{\frac{m_3}{4} s}$$

one can get the inequalities

$$e^{\frac{m_3}{4} s}\, \mathbb{E}_{x,y}\, |x^\varepsilon(s)|^p \leq v_\varepsilon^{(p)}(x,\, y) \leq \hat{m}_3\, |x|^p,$$

for any $s \geq 0$ and proof is completed.

Note. By using the supermartingale property of the above defined stochastic process $\xi(s)$ one can make sure that *under the conditions of the Theorem 7 the trivial solution of the* (3) *is asymptotically stochastically stable for all sufficiently small positive* ε.

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