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PRELIMINARY RESULTS ON ASYMMETRIC BAXTER-KING FILTER

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Abstract. The paper proposes an extension of the symmetric Baxter-King band pass filter to an asymmetric Baxter-King filter. It turns out the optimal correction scheme of the ideal filter weights is the same as in the symmetric version, i.e, cut the ideal filter at the appropriate length and add a constant to all filter weights to ensure zero weight on zero frequency. Since the symmetric Baxter-King filter is unable to extract the band of frequencies at the very ends of the series, the extension to an asymmetric filter is useful whenever the real time estimation is needed. The paper assesses the filter's properties in extracting business cycle frequencies, in comparison to the symmetric Baxter-King filter and symmetric and asymmetric Christiano-Fitzgerald filter, by using Monte Carlo simulation. The results show that the asymmetric Baxter-King filter is superior to the asymmetric Christiano-Fitzgerald filter for the whole sample space, including the very ends of a sample, thus indicating that the asymmetric Baxter-King filter should be preferred over the asymmetric Christiano-Fitzgerald filter in real time signal extraction exercises.

Key words and phrases. real time estimation, Christiano-Fitzgerald filter, Monte Carlo simulation, band pass filter, asymmetric filter.

Mathematics Subject Classification. Primary 60G35; Secondary 62M20.

1 Introduction

This paper considers a simple extension of the symmetric Baxter-King band pass filter (Baxter and King, 1999) to an asymmetric version of it. Such modification, to the best of my knowledge, has not yet been discussed in the literature. Symmetric filters are not applicable at the very ends of an input signal without the extension of the ends with forecasts. Thus, asymmetric band pass filters are necessary to extract the desired band of frequencies at the ends of an input signal, if forecasting is not used for extending the ends of the input signal.

The closest band pass filter to the Baxter-King filter is Christiano-Fitzgerald band pass filter (Christiano and Fitzgerald, 2003) which, in general, is asymmetric, and whose default specification is optimized for an input signal following a random walk (RW) process, but it allows the input signal to follow other data generating processes (DGP). However, Christiano and Fitzgerald (2003) argue that their default specification of the filter is a good approximation to many DGPs observed in macroeconomic time series and, thus, macroeconomists may opt for it. Although Christiano and Fitzgerald (2003) compares their filter to the symmetric Baxter-King filter, they do not elaborate on an asymmetric version of the Baxter-King filter.

Thus, this paper tries to fill the gap in the literature by formally developing an asymmetric version of the Baxter-King filter and assessing its properties in extracting business cycle frequencies, in comparison to the symmetric Baxter-King filter and symmetric and asymmetric default specification of Christiano-Fitzgerald filter, by using Monte Carlo simulation. The results show that the asymmetric Baxter-King filter is superior to the asymmetric Christiano-Fitzgerald filter at the very ends of a sample, thus indicating that the asymmetric Baxter-King filter should be preferred over the asymmetric Christiano-Fitzgerald filter in real time signal extraction exercises.

The paper is organized as follows. Section 2 develops the filter and Section 3 assesses the performance of the filter by means of Monte Carlo simulation.

2 The Asymmetric Baxter-King Filter

Consider the following orthogonal decomposition of the zero-mean covariance stationary stochastic process, x_t:

$$x_t = y_t + \tilde{x}_t. \tag{1}$$

The process, y_t, has power only in frequencies belonging to the interval $\{[a_1, a_2] \cup [-a_2, -a_1]\} \subset (-\pi, \pi)$, where $0 < a_1 < a_2 < \pi$. The process, $\tilde{x}_t$, has power only in the complement of this interval in $(-\pi, \pi)$. By the spectral representation theorem,

$$y_t = b(L)x_t, \tag{2}$$

where the ideal band pass filter, $b(L)$, is

$$b(L) = \sum_{h=-\infty}^{\infty} b_h L^h, \quad L^h x_t = x_{t-h}, \tag{3}$$

where

$$\begin{aligned} b_h &= \frac{\sin(ha_2) - \sin(ha_1)}{\pi h}, \quad h = \pm 1, \pm 2, \ldots \\ b_0 &= \frac{a_2 - a_1}{\pi}, \quad a_1 = \frac{2\pi}{p_u}, \quad a_2 = \frac{2\pi}{p_l}, \end{aligned} \tag{4}$$

and $p_u, p_l \in (2, \infty)$ define the upper and lower bounds of the wave length of interest. With b_h's specified as in (4), the frequency response function of the ideal filter at frequency ω is

$$\begin{aligned} \beta(\omega) &= 1 \quad \text{for } \omega \in [a_1, a_2] \cup [-a_2, -a_1] \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{5}$$

Baxter and King (1999) have proposed to obtain a symmetric, fixed length approximation to the ideal filter, (3) and (4), by minimizing

$$
\begin{aligned}
Q &= \int_{-\pi}^{\pi} \delta(\omega)\delta(-\omega)d\omega \\
&\text{s.t.} \\
\hat{\beta}(0) &= \sum_{k=-K}^{K} \hat{b}_k = 0 \\
\hat{b}_k &= \hat{b}_{-k},
\end{aligned}
\tag{6}
$$

where $\delta(\omega) = \beta(\omega) - \hat{\beta}(\omega)$ is the discrepancy between the exact and the approximate filters at frequency ω, and the constraint $\hat{\beta}(0) = 0$ is to ensure zero weight on the trend frequency, in line with the assumption $a_1 > 0$. The solution to (6) is a truncation of the ideal filter symmetrically at length K, and addition of a constant $(-\sum_{k=-K}^{K} b_k)/(2K+1)$ to all filter weights to ensure $\hat{\beta}(0) = 0$. Baxter and King (1999) suggest the value of K to be about 3 years, i.e, K=12 for quarterly data, and K=36 for monthly data. The symmetry of the filter together with the condition $\hat{b}_k = \hat{b}_{-k}$ implies that the filter renders stationary time series that is integrated of oder 2 (I(2)) or less. Thus, the symmetric BK filter has trend-reduction property and, therefore, it can be applied to nonstationary, up to I(2) series.

Since the symmetric BK filter can not be used to extract the desired frequencies at the very end of the input series (for the first and the last K observations), a natural extension of the Baxter and King (1999) filter is to allow the approximate filter to be asymmetric, to be able to use the filter in real time. In order to optimally approximate an ideal symmetric linear filter in a Baxter-King sense, the problem is to minimize

$$
\begin{aligned}
Q &= \int_{-\pi}^{\pi} \delta(\omega)\delta(-\omega)d\omega \\
&\text{s.t.} \\
\hat{\beta}(0) &= \sum_{h=-p}^{f} \hat{b}_h = 0.
\end{aligned}
\tag{7}
$$

The condition $\hat{\beta}(0)$ ensures zero weight on zero frequency, thus this asymmetric filter also has a trend-reduction property, however, it alone, without symmetry, is not sufficient to render I(2) process stationary. Thus, the ability of the asymmetric BK filter of real time signal extraction comes at a cost of losing the power to eliminate two unit roots from the input series.

To solve (7), form the Lagrangian

$$\mathcal{L} = Q - \lambda\hat{\beta}(0) \tag{8}$$

with first order conditions (FOCs):

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\hat{b}_h} &= \frac{\partial Q}{\partial\hat{b}_h} - \lambda = 0 \\
\frac{\partial\mathcal{L}}{\partial\lambda} &= -\hat{\beta}(0) = 0.
\end{aligned}
\tag{9}
$$

Since

$$\frac{\partial}{\partial \hat{b}_h}[\delta(\omega)\delta(-\omega)] = \frac{\partial \delta(\omega)}{\partial \hat{b}_h}\delta(-\omega) + \delta(\omega)\frac{\partial \delta(-\omega)}{\partial \hat{b}_h}, \tag{10}$$

and since the frequency response function of the approximating filter is $\hat{\beta}(\omega) = \sum_{h=-p}^{f} \hat{b}_h e^{-i\omega h}$, it follows that

$$\frac{\partial \delta(\omega)}{\partial \hat{b}_h} = -e^{-i\omega h}. \tag{11}$$

(11) implies

$$\frac{\partial Q}{\partial \hat{b}_h} = -\int_{-\pi}^{\pi}[e^{-i\omega h}\delta(-\omega) + \delta(\omega)e^{i\omega h}]d\omega. \tag{12}$$

By the property $\int_{-\pi}^{\pi}[f(\omega) + f(-\omega)]d\omega = 2\int_{-\pi}^{\pi} f(\omega)d\omega$ (since $\int_{-\pi}^{\pi} f(\omega)d\omega = \int_{0}^{\pi} f(\omega)d\omega + \int_{-\pi}^{0} f(\omega)d\omega = \int_{0}^{\pi}[f(\omega) + f(-\omega)]d\omega$ is real, then $\int_{-\pi}^{\pi} f(\omega)d\omega = \int_{-\pi}^{\pi} f(-\omega)d\omega$, and the property follows), (12) becomes

$$\frac{\partial Q}{\partial \hat{b}_h} = -2\int_{-\pi}^{\pi} \delta(\omega)e^{i\omega h}d\omega. \tag{13}$$

By the property

$$\begin{aligned}\int_{-\pi}^{\pi} e^{i\omega n}e^{-i\omega m}d\omega = \int_{-\pi}^{\pi} e^{-i\omega(m-n)}d\omega &= 0 \quad \text{for } n \neq m \\ &= 2\pi \quad \text{for } n = m,\end{aligned} \tag{14}$$

obtain

$$\int_{-\pi}^{\pi} \delta(\omega)e^{i\omega h}d\omega = \int_{-\pi}^{\pi}\left[\sum_{k=-\infty}^{\infty} b_k e^{-i\omega k} - \sum_{j=-p}^{f} \hat{b}_j e^{-i\omega j}\right]e^{i\omega h}d\omega = 2\pi[b_h - \hat{b}_h]. \tag{15}$$

Given (15), the FOCs are

$$-4\pi[b_h - \hat{b}_h] - \lambda = 0. \tag{16}$$

If there is no constraint on $\hat{\beta}(0)$, the optimal approximate (in Baxter-King sense) filter is simply derived by truncation of the ideal filter's weights. If there is a constraint on $\hat{\beta}(0)$, then λ must be chosen so that the constraint is satisfied. For this purpose, rewrite (16) as

$$\hat{b}_h = b_h + \theta,$$

where $\theta = \lambda/(4\pi)$. In order to have $\hat{\beta}(0) = \sum_{h=-p}^{f} \hat{b}_h = 0$, the required adjustment is

$$\theta = \frac{-\sum_{h=-p}^{f} b_h}{p + f + 1}, \tag{17}$$

which yields the same optimal weight adjustment scheme as in the symmetric Baxter-King filter case.

The next section describes the results from Monte Carlo simulation to assess the performance of the proposed filter.

3 Comparing Filters By Means Of Monte Carlo Simulation

This section assesses the performance of the proposed filter to extract business cycle frequencies (corresponding to wave length between 1.5 and 8 years) in comparison to i) the original symmetric fixed-length BK filter with K=12 (see (6)), as well as ii) fixed-length symmetric CF filter with K=12 for RW processes, and iii) default asymmetric specification of CF filter for RW processes (Christiano and Fitzgerald, 2003). Thus, the asymmetric CF filter assumes that the first difference of the input signal, i.e, $x_t - x_{t-1}$, is zero-mean covariance stationary process. The symmetric CF filter allows for the input signal to follow RW with drift.

Consider the following data generating process (DGP):

$$y_t = \mu_t + c_t, \tag{18}$$

where

$$\mu_t = \mu_{t-1} + \epsilon_t \tag{19}$$

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + \eta_t \tag{20}$$

$$\epsilon_t \sim nid(0, \sigma_\epsilon^2),\ \eta_t \sim nid(0, \sigma_\eta^2). \tag{21}$$

Equation (18) defines the input signal, y_t, as the sum of a permanent component (stochastic trend), μ_t, and a cyclical component, c_t. The trend, μ_t, in this case is specified as a random walk process. The dynamics of the cyclical component, c_t, is specified as a second order autoregressive (AR(2)) process so that the peak of the spectrum of c_t could be at zero frequency or at business cycle frequencies. Disturbances, ϵ_t and η_t, are assumed to be uncorrelated.

The spectrum of an AR(2) process is

$$f_c(\omega) = \frac{\sigma_\eta^2}{1 + \phi_1^2 + \phi_2^2 - 2\phi_1(1-\phi_2)\cos\omega - 2\phi_2\cos(2\omega)} \tag{22}$$

with a peak at frequency other than zero for

$$\phi_2 < 0 \text{ and } \left|\frac{\phi_1(1-\phi_2)}{4\phi_2}\right| < 1, \tag{23}$$

with the corresponding frequency $\omega = \cos^{-1}[-\phi_1(1-\phi_2)/(4\phi_2)]$ (Box, Jenkins and Reinsel, 1994; Priestley, 1981).

Data are generated from (18) with $\phi_1 = 1.2$ and different values for ϕ_2 to control the location of the peak in the spectrum of the cyclical component. I also vary the ratio of standard deviations of the disturbances, $\sigma_\epsilon/\sigma_\eta$, to change the relative importance of components of y_t. Such DGP can create series with spectral characteristics typical to macroeconomic variables, such as gross domestic product and inflation (Watson, 1986; Guay and St-Amant, 2005). The idea of such simulation is taken from Ahamada and Jolivaldt (2010) who, in turn, take it from Guay and St-Amant (2005).

Particularly, 10000 samples of length 401 are created, with the first 200 observations of each sample dropped off as burn-in. The vector $[\phi_1, \phi_2]$ is set to five different values, as shown in Table 1. The value of $\sigma_\epsilon/\sigma_\eta$ is set to change from 0 to 9.9 with step size 0.15 (Watson (1986)

ϕ_1	ϕ_2	Fundamental period of the cycle (yrs)
1.2	-0.25	$\approx \infty$
1.2	-0.35	$\gg 8$
1.2	-0.44	8.2
1.2	-0.5	3.5
1.2	-0.8	1.9

Table 1: Five different values of $[\phi_1, \phi_2]$ for the DGP.

estimated this ratio for the U.S. GNP to be 0.75).

The performance of filters is assessed by comparing the correlation of the true cyclical component with the estimated cyclical component, and by comparing the true AR(2) regression coefficients for the cycle with the fitted AR(2) regression coefficients.

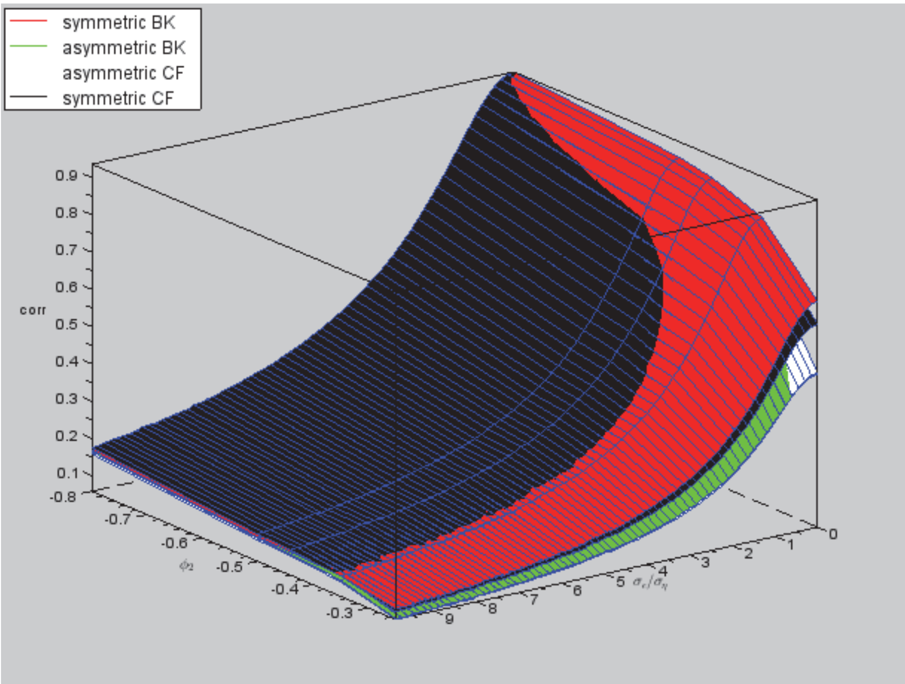

Figure 1: Average correlation between the true and estimated cyclical components for given $[\phi_1, \phi_2]$ and $\sigma_\epsilon/\sigma_\eta$ values. The correlation is estimated for the whole sample interval except for the first and the last K=12 observations.

Figure 1 shows an average correlation between the true and estimated cyclical components for given $[\phi_1, \phi_2]$ and $\sigma_\epsilon/\sigma_\eta$ values. The correlation is estimated for the whole sample interval except for the first and the last K=12 observations, since fixed-length symmetric filters do not produce the estimated cycle for those observations. These K=12 observations are deleted from the output of the asymmetric filters for a fair comparison between symmetric and asymmetric filters. Figure 1 shows a similar behavior between the filters - their performance decreases with $\sigma_\epsilon/\sigma_\eta$, which is an expected result. When $\sigma_\epsilon/\sigma_\eta = 0$, the input signal is the true cycle, so the output signal (estimated cycle) correlates highly with the input. As $\sigma_\epsilon/\sigma_\eta$ increases, the influence of the permanent component in the input increases, thus making harder for filters to extract the cycle, thus the estimated correlation between the true and estimated cycles, $\hat{\rho}(c, \hat{c})$, decreases.

Figure 1 also shows that the performance of all filters decreases with an increasing ϕ_2. The value of $\phi_2 = -0.8$ together with $\phi_1 = 1.2$ corresponds to the length of the cycle 1.9 years, which is close to the usually defined minimum length of a business cycle, 1.5 years. The value of $\phi_2 = -0.44$ together with $\phi_1 = 1.2$ produces the cycle of length approximately 8.2 years, which is close to the usually defined maximum length of a business cycle, 8 years. With higher then $\phi_2 = -0.44$ values, the length of the true cycle rapidly increases. Although with $\phi_2 = -0.25$ the cycle still is considered stationary ($\phi_1 + \phi_2 < 1$), it is a close approximation to a nonstationary process in a finite sample (Campbell and Perron, 1991). Thus, Figure 1 shows expected deterioration in performance of BK filters as ϕ_2 increases. The similar deterioration in performance of the CF filters with an increasing length of the cycle was less expected. Another unexpected result is the inferior performance of asymmetric filters to their shorter symmetric counterparts. The results also show that the symmetric BK filter is superior to the symmetric CF filter for $0 \leq \sigma_\epsilon/\sigma_\eta < 0.5$ if cycle length is longer than 2 years. For most of the rest of the region, particularly - cycle length less than 8 years, given $\sigma_\epsilon/\sigma_\eta \geq 1$ - the symmetric CF filter is slightly superior to the symmetric BK filter. For the remainder, i.e., $0.5 \leq \sigma_\epsilon/\sigma_\eta < 1$, the symmetric CF filter shows superiority when cycle is relatively short (up to about 3.5 years), and the symmetric BK filter shows superiority when the cycle is longer than approximately 3.5 years.

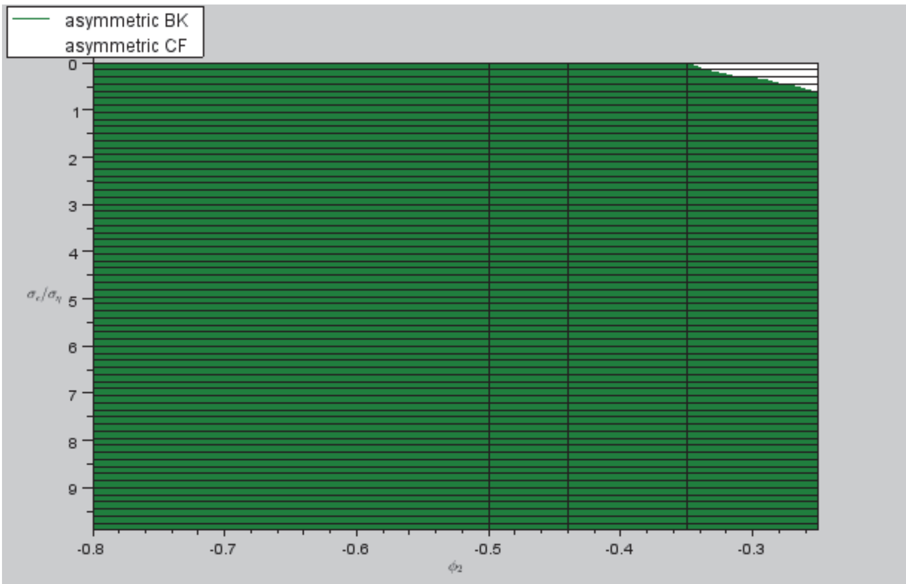

Figure 2: Relative superiority of the asymmetric BK and CF filters in the sample, except for the first and the last K=12 observations. The horizontal axis represent cycle length, while the vertical axis represent the importance of permanent component in the series. The results suggest that, on average in the sample, the asymmetric BK filter is superior to the asymmetric CF filter.

The simulation results show (not supplied to to space limitation) that the distance between the average correlation between the true and estimated cyclical components for given $[\phi_1, \phi_2]$ and $\sigma_\epsilon/\sigma_\eta$ values from asymmetric BK and CF filters is practically nil at all points in the sample, when the first and the last K=12 observations are dropped. Figure 2 shows the regions of (small) relative superiority of the asymmetric BK and CF filters. Figure 2 shows that the asymmetric BK filter is superior to asymmetric CF filter for all $\sigma_\epsilon/\sigma_\eta$ values and for any interesting length of the cycle. A slightly surprising finding from Figure 2 is the inability of asymmetric CF filter to perform better than the asymmetric BK filter in the region of high influence of the permanent component (corresponds to lower part of the graph).

Now, let us compare the performance of the asymmetric filters for the K=12 observations of the sample, where the fixed-length symmetric filters can not be applied. Figure **??** shows the estimated correlation of the true and estimated cycles at each of the first three observations of the sample, calculated across the 10000 replications, and averaged over both symmetric ends. The correlation graphs at other observations are skipped due to space limitation. The results show that the filters give close result at points closer to the center of the sample. As the estimation point approaches the end of the sample, filters become more asymmetric, and the difference in their performance becomes more obvious. Thus, the asymmetric filters perform roughly equally well at points that are at least about 3 years (for quarterly data) away from the end of the sample. Otherwise, the asymmetric BK filter becomes increasingly superior to the asymmetric CF filter for any cycle length and for any share of permanent component in the input signal considered in the simulation. Thus, based on the results illustrated in Figure **??**, it is recommended to use the asymmetric BK filter rather than the asymmetric CF filter for the business-cycle frequency extraction in real time, i.e, at the very end of the sample.

As for the comparison of the true AR(2) process of the cycle, and the estimated AR(2) regression coefficients, Table 2 shows that the length of the cycle extracted by the filters, when the influence of the permanent component in the input series is sufficiently high, i.e., about $\sigma_\epsilon/\sigma_\eta > 5$, is about constant, regardless of the true length of the cycle. This result shows the potential limitation of the considered filters.

	ϕ_1	ϕ_2	Fundamental period of the cycle (yrs)
true	1.2	-0.25	$\approx \infty$
	1.2	-0.35	$\gg 8$
	1.2	-0.44	8.2
	1.2	-0.5	3.5
	1.2	-0.8	1.9
symmetric BK	1.699	-0.886	3.56
asymmetric BK	1.689	-0.884	3.48
asymmetric CF	1.696	-0.879	3.60
symmetric CF	1.623	-0.848	3.23

Table 2: The true AR(2) parameters and cycle length, and the estimated AR(2) parameters and cycle length by the four filters, when the influence of the permanent component in the input series is sufficiently high, i.e., about $\sigma_\epsilon/\sigma_\eta > 5$. In such case, the estimated AR(2) parameters and the length of the extracted cycle are about constant, regardless of the true length of the cycle.

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References

[1] AHAMADA, I., JOLIVALDT, P.: *Classical vs wavelet-based filters Comparative study and application to business cycle*. Documents de travail du Centre d'Economie de la Sorbonne 10027, Universite Pantheon-Sorbonne (Paris 1), Centre d'Economie de la Sorbonne, 2010.
[2] BAXTER, M., KING, R. G.: *Measuring business cycles: approximate band-pass filters for economic time series*. In The Review of Economics and Statistics, MIT Press, Vol. 81(4), pp. 575-593, 1999.
[3] BOX, G., JENKINS, G. M., REINSEL, G.: *Time Series Analysis: Forecasting and Control*, 3rd Edition, Prentice Hall, 1994.
[4] CAMPBELL, J. Y., PERRON, P.: *Pitfalls and opportunities: what macroeconomists should know about unit roots*. NBER Technical Working Papers 0100, National Bureau of Economic Research, Inc., 1991.
[5] CHRISTIANO, L. J., FITZGERALD, T. J.: *The band pass filter*. In International Economic Review, Vol. 44(2), pp. 435-465, 2003.
[6] GUAY, A., ST-AMANT, P.: *Do the Hodrick-Prescott and Baxter-King filters provide a good approximation of business cycles?*. In Annales d'Economie et de Statistique, issue 77, 2005.
[7] PRIESTLEY, M. B.: *Spectral Analysis and Time Series*, Academic Press, Inc., 1981.
[8] WATSON, M. W.: *Univariate detrending methods with stochastic trends*. In Journal of Monetary Economy, Elsevier, Vol. 18, pp. 49-75, 1986.

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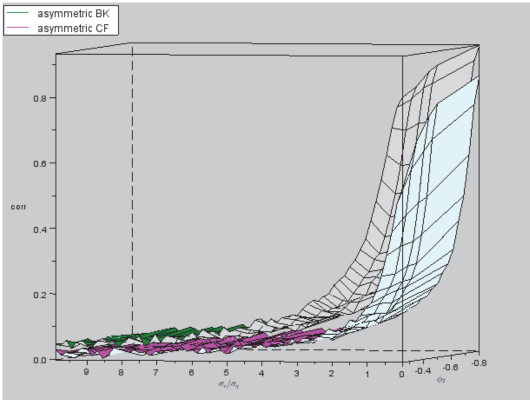

Figure 3: Correlation at obs. 3

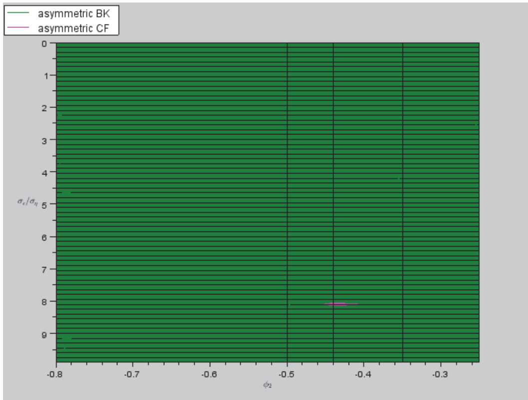

Figure 4: view at 3 from the top

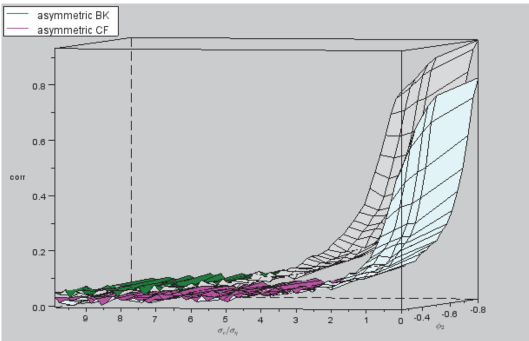

Figure 5: Correlation at obs. 2

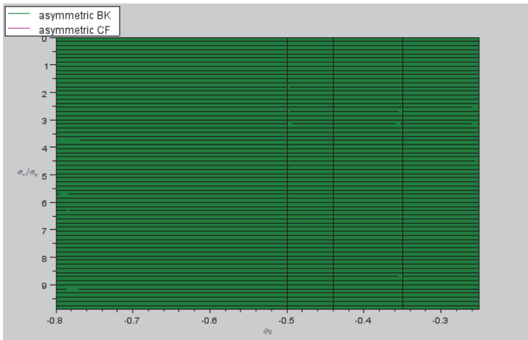

Figure 6: view at 5 from the top

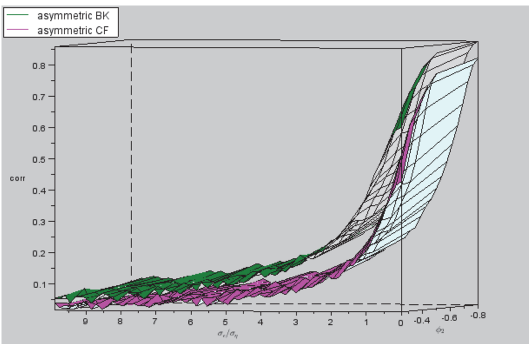

Figure 7: Correlation at obs. 1

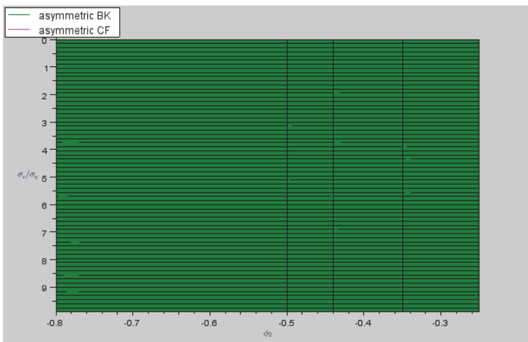

Figure 8: view at 7 from the top