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EQUILIBRIUM STOCHASTIC STABILITYS OF MARKOV DYNAMICAL SYSTEMS

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Abstract. In first section of paper we will prove that for linear Markov dynamical systems an equilibrium asymptotic stochastic stability is equivalent to exponential p-stability for sufficiently small positive values p. Then we will prove that exponential p-stability of linearized in vicinity of equilibrium Markov dynamical system guarantees equilibrium asymptotic (local) stochastic stability.This result permits to construct such Lyapunov quadratic functional, which one may use for local equilibrium stochastic stability of sufficiently smooth nonlinear Markov dynamical systems.

Key words and phrases. Markov dynamical systems; stochastic stability; Lyapunov stability.

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1 Stochastic stability of linear differential equations with Markov coefficients

Let $y(t)$ be Feller type Markov process on phase space $\mathbb{Y}$ and with weak infinitesimal operator [Doob] $\mathcal{L}$,$f(x,y)$ be a continuous mapping $\mathbb{R}^n \times \mathbb{Y} \to \mathbb{R}^n$, and $f(0,y) \equiv 0$. The solution of equation

$$\frac{dx(t)}{dt} = f(x(t), y(t)) \tag{1}$$

with initial condition $x(s) = x, y(s) = y$ we will denote $x(t,s,x,y)$. We will say [3] that trivial solution of differential equation (1)

- *locally stable almost sure, if for any $s \in \mathbb{R}$, $\eta > 0$ and $\beta > 0$ there exists such $\delta > 0$ that the inequality*

$$\sup_{\substack{y\in\mathbb{R}^m \\ \xi\in\mathbb{G}}} \mathbb{P}(\sup_{t\geq s} |x(t,s,x,y)| > \eta) < \beta, \tag{2}$$

  follows the condition $x \in B_\delta(0)$, where $B_\delta(0) := \{x \in \mathbb{R}^n : |x| < \delta\}$;

- *locally asymptotically stochastically stable, if it is locally almost sure stable and there exists such* $\gamma > 0$ *that the trajectories which do not leave the ball* B_γ *tend to 0 as* $t \to \infty$*;*
- *asymptotically stochastically stable, if it is locally almost sure stable and for any* $x \in \mathbb{R}^n$*,* $s \in \mathbb{R}$*, and* $c > 0$ *the equality*

$$\lim_{T\to\infty} \sup_{\substack{y\in\mathbb{R}^m \\ \xi\in\mathbb{G}}} \mathbb{P}(\sup_{t>T} |x(t,s,x,y)| > c) = 0 \tag{3}$$

is fulfilled;

- *exponentially p-stable, if there exist such positive numbers* M *and* γ *that for any* $x \in \mathbb{R}^n$*,* $y \in \mathbb{R}^m$*,* $\xi \in \mathbb{G}$*,* $s \in \mathbb{R}$ *and* $t > s$ *the inequality*

$$\mathbb{E}\,|x(t,s,x,y)|^p \le M|x|^p e^{-\gamma(t-s)} \tag{4}$$

is fulfilled.

In this section we will deal with linear differential equations in $\mathbb{R}^n$

$$\frac{dx}{dt} = A(y(t))x, \tag{5}$$

where $A(y)$ is continuous bounded matrix-valued function and $y(t)$ is stochastically continuous Feller Markov process with weak infinitesimal operator Q. The pair $\{x(t),\, y(t)\}$ forms [Skorokhod] homogeneous stochastically continuous Markov process with the weak infinitesimal operator L_0 defined by equality

$$L_0\, v(x,\, y) = (A(y)\, x,\, \nabla_x)\, v(x,\, y) + Q\, v(x,\, y). \tag{6}$$

It is clearly that there exists family of the matrix-valued functions $\{X(t,s,y),\, t \ge s \ge 0\}$, defined by equality $X(t,s,y)x = x(t,s,x,y)$, where $x(t,s,x,y)$ is the solution of Cauchy problem $x(s,s,x,y) = x$ under condition $y(s) = y$. The matrices $X(t,s,y)$ also satisfy the equation (5) for all $t > s$ and initial condition $X(s,s,y) = I$, where I is matrix unit. This matrix family has the evolution property:

$$X(t,s,y) = X(t,\tau,y(\tau))X(\tau,s,y) \tag{7}$$

for any $y \in \mathbb{Y}, t \ge \tau \ge s \ge 0$. Let us define the Lyapunov p-index of (5) as

$$\lambda^{(p)} = \sup_{x,y} \overline{\lim_{t\to\infty}} \tfrac{1}{pt} \ln \mathbb{E}|X(t,s,y)x|^p. \tag{8}$$

Not so difficult to prove that exponential p-stability of trivial solution of the equation (5) is equivalent to inequality $\lambda^{(p)} < 0$. Because

$$(\mathbb{E}|X(t,s,y)x|^{p_1})^{1/p_1} \le (\mathbb{E}|X(t,s,y)x|^{p_2})^{1/p_2} \tag{9}$$

for any positive $p_1 < p_2$, the inequality

$$\lambda^{(p_1)} \le \lambda^{(p_2)} \tag{10}$$

follows the inequality $p_1 < p_2$ and $\lambda^{(p)}$ is monotone decreasing function as p decreases to 0. It is intuitively clearly, that asymptotic stochastic stability of (5) is equivalent to the condition

$$\exists p_0 > 0,\ \forall p \in (0, p_0) : \lambda^{(p)} < 0.$$

We will assentially use further this assertion and hence it should be proven.

Lemma 1. *If the equation* (5) *is asymptotically stochastically stable then it is exponentially p-stable for all sufficiently small positive p.*

Proof. Let us put in definition of almost sure stability $\eta = 1, \beta = \frac{1}{2}$ and choose so small positive α that the inequality

$$\sup_{\substack{|x|\le 2^{-\alpha}\\ y\in\mathbb{Y}}} \mathbb{P}(\sup_{t\ge 0} |X(t,0,y)x| > 1) < \tfrac{1}{2}.$$

is fulfilled. Due to a linearity of the equation (5) from the above inequality one may write the inequality

$$\sup_{\substack{|x|\le 2^{-\alpha(l-1)}\\ y\in\mathbb{Y}}} \mathbb{P}(\sup_{t\ge 0} |X(t,0,y)x| > 2^{l\alpha}) < \tfrac{1}{2}$$

for any $l \in \mathbb{N}$. Let us denote

$$g_l := \sup_{\substack{|x|\le 1\\ y\in\mathbb{Y}}} \mathbb{P}(\sup_{t\ge 0} |X(t,0,y)x| \ge 2^{l\alpha}).$$

The pair $\{x(t),\ y(t)\}$ is stochastically continuous Markov process and it has the Markov property in the moment $\tau_1(x)$ of exit of the trajectory $x(t, 0, x, y)$ from the ball $B_1(0)$ if $x \in B_1(0)$. Hence

$$g_{l+1} = \sup_{\substack{|x|\le 1\\ y\in\mathbb{Y}}} \mathbb{P}(\sup_{t\ge 0} |X(t,0,y)x| \ge 2^{(l+1)\alpha})$$

$$= \sup_{\substack{|x|\le 1\\ y\in\mathbb{Y}}} \int_{s=0}^{\infty} \int_{\substack{|u|=2^{l\alpha}\\ v\in\mathbb{Y}}} \mathbb{P}_{x,y}(\tau_1(x) \in ds, x(s) \in du, y(s) \in dv)\times$$

$$\times \mathbb{P}(\sup_{t\ge 0} |X(t,0,v)u| > 2^{(l+1)\alpha})$$

$$\le \sup_{\substack{|x|\le 2^{l\alpha}\\ y\in\mathbb{Y}}} \mathbb{P}(\sup_{t\ge 0} |X(t,0,y)x| > 2^{(l+1)\alpha}) \sup_{\substack{|x|\le 1\\ y\in\mathbb{Y}}} \times$$

$$\times \int_{s=0}^{\infty} \int_{\substack{|u|=2^{l\alpha}\\ v\in\mathbb{Y}}} \mathbb{P}_{x,y}(\tau_1(x) \in ds, x(s) \in du, y(s) \in dv)$$

$$\le \frac{1}{2} \sup_{\substack{|x|\le 1\\ y\in\mathbb{Y}}} \mathbb{P}(\sup_{t\ge 0} |X(t,0,y)x| \ge 2^{l\alpha}) = \tfrac{1}{2} g_l.$$

Hence $g_l \le \frac{1}{2^l}$ for any $l \in \mathbb{N}$. Let us denote

$$\zeta := \sup_{t\ge 0} |x(t,0,x,y)|^p.$$

It is clearly to see that for all $p > 0$, $x \in \mathbb{R}^n$, $y \in \mathbb{Y}$ it may be written

$$\mathbb{E}\,\zeta \le |x|^p \sup_{|x|\le 1} \mathbb{E}\,\zeta \le \sum_{l=1}^{\infty} 2^{l\alpha p}\mathbb{P}(\sup_{t\ge 0} |x(t,0,x,y)| \ge 2^{(l-1)\alpha})$$

$$\le \sum_{l=1}^{\infty} 2^{l\alpha p} 2^{-l}|x|^p := K_1|x|^p.$$

Therefore random variable ζ has expectation for all $x \in \mathbb{R}^n$, $y \in \mathbb{Y}, p \in (0, \alpha^{-1})$. According to Lemma's conditions the solution of (5) $x(t,0,x,y)$ tends to 0 almost sure as t tends to ∞ uniformly on $y \in \mathbb{Y}$ and by the Lebesque Theorem one can write

$$\lim_{t\to\infty} \sup_{y\in\mathbb{Y}} \mathbb{E}|x(t+s,s,x,y)|^p = 0$$

for all $x \in \mathbb{R}^n$, $p \in (0, \alpha^{-1})$. Besides, not complicatedly to verify that this convergence is uniform on x in the ball $B_1(0)$ and $s \ge 0$, i.e.

$$\lim_{t\to\infty} \sup_{\substack{x\in B_1(0)\\ y\in\mathbb{Y}}} \mathbb{E}|x(t+s,s,x,y)|^p = 0.$$

Now we can choose a number T so large then the inequality

$$\sup_{y\in\mathbb{Y}} \mathbb{E}|x(t+s,s,x,y)|^p \le |x|^p e^{-1}$$

is fulfilled and further, by using the inequality

$$\mathbb{E}|x(lT,0,x,y)|^p = \int_{\mathbb{R}^n}\int_{\mathbb{Y}} \mathbb{P}(x,y,(l-1)T,du,,dv)\mathbb{E}|x(T,0,u,v)|^p$$

$$\le e^{-1}\mathbb{E}|x((l-1)T,0,x,y)|^p,$$

where $\mathbb{P}(x,y,t,du,,dv)$ is transition probability of homogeneous Markov process $\{x(t),y(t)\}$, one can write

$$\mathbb{E}|x(t,0,x,y)|^p \le K_1 e^{-[\frac{t}{T}]T}|x|^p,$$

where $[a]$ is integer of number a. This inequality completes the proof.

To analyze the behaviour of solutions of (5) one may use well known the Dynkin formula [2]

$$\mathbb{E}^{(u)}_{x,y} v(x(\tau_r(t)),y(\tau_r(t))) = v(x,y) + \mathbb{E}^{(u)}_{x,y}\left\{\int_u^{\tau_r(t)} (L_0 v)(x(s),y(s))\,ds\right\}, \tag{11}$$

where the indexes of expectation denote the condition $x(u) = x$, $y(u) = y$ and $\tau_r(t) = \min\{\tau_r,\, t\}$, $\tau_r = \inf\{t > u : x(t,u,x,y) \notin B_r(0)\}$. If $u = 0$, then upper index will be absent.

If for all $t \geq u \geq 0$ there exist the expectations $\mathbb{E}_{x,y} v(x(t), y(t))$ and $\mathbb{E}_{x,y}(L_0 v)(x(t), y(t))$ one can use the Dynkin formula (11) in the more simple form

$$\mathbb{E}^{(u)}_{x,y} v(x(t), y(t)) \;=\; v(x, y) \;+\; \int_u^t \mathbb{E}^{(u)}_{x,y}(L_0 v)(x(s), y(s))\, ds. \tag{12}$$

Sometimes it is necessary to use the Lyapunov functions depending also on argument t. If the function $v(t, x, y)$ belongs (as the function of arguments x and y) to the region of definition of infinitesimal operator L_0 and has continuous t-derivative, one may use the Dynkin formula (11) in the form

$$\begin{aligned}\mathbb{E}^{(u)}_{x,y} v(\tau_r(t), x(\tau_r(t)), y(\tau_r(t))) \;&=\\ = v(u, x, y) \;&+\; \mathbb{E}^{(u)}_{x,y}\left\{ \int_u^{\tau_r(t)} \Big(\tfrac{\partial}{\partial s} + L_0\Big) v(s, x(s), y(s))\, ds \right\},\end{aligned}$$

or formula (12) in the form

$$\begin{aligned}&\mathbb{E}^{(u)}_{x,y} v(t, x(t), y(t)) =\\ &= v(u, x, y) + \int_u^t \mathbb{E}^{(u)}_{x,y} \left\{ \Big(\frac{\partial}{\partial s} + L_0\Big) v(s, x(s), y(s)) \right\} ds.\end{aligned} \tag{13}$$

Besides Dynkin formula and the Second Lyapunov method one can use also well known the supermartingale inequality [1] for positive supermartingale $\{\xi(t), \mathfrak{F}^t\}$ with filtration $\mathfrak{F}^t$ in the form

$$\mathbb{P}(\sup_{t \geq u} \xi(t) \;\geq\; c) \;\leq\; \tfrac{1}{c}\mathbb{E}\xi(u). \tag{14}$$

Lemma 2. *The trivial solution of equation* (5) *is exponentially p-stable if and only if there exists the Lyapunov function $v(x, y)$, which satisfies the conditions*

$$c_1|x|^p \leq v(x, y) \leq c_2|x|^p, \quad c_1 > 0 \tag{15}$$

$$L_0 v(x, y) \;\leq\; -\,c_3|x|^p, \quad c_3 > 0 \tag{16}$$

for all $x \in \mathbb{R}^n$, $y \in \mathbb{Y}$ with some positive p.

Proof. Let there exists above mentioned the Lyapunov function. It is clearly to verify that

$$\Big(\tfrac{\partial}{\partial s} + L_0\Big)\Big(v(x, y)e^{\frac{c_3}{c_2}t}\Big) \;\leq\; 0,$$

and then one can write

$$\mathbb{E}_{x,y} v(x(t), y(t)) e^{\frac{c_3}{c_2}t} \;\leq\; v(x, y) \;\leq\; c_2|x|^p$$

for all $t > 0$, $x \in \mathbb{R}^n$ and $y \in \mathbb{Y}$. Hence

$$\mathbb{E}_{x,y}|x(t)|^p \le \tfrac{1}{c_1} e^{-\frac{c_3}{c_2}t}\mathbb{E}_{x,y} v(x(t),y(t)) e^{\frac{c_3}{c_2}t} \le \tfrac{c_2}{c_1} e^{-\frac{c_3}{c_2}t}|x|^p$$

and the equation (5) is exponentially p-stable. By using the solutions $x(t+s,s,x,y)$ of the equation (5) one can construct for any $T > 0$ function

$$v(x,y) := \int_0^T \mathbb{E}|x(s+t,s,x,y)|^p\, dt, \tag{17}$$

which do not dependent on s owing to homogeneity of Markov process $y(t)$. It is easily to verify that under conditions that the matrix $A(y)$ is uniformly bounded, that is, $\sup_{y\in\mathbb{Y}} ||A(y)|| := a < \infty$ this function satisfies the conditions (15). Let L_0 be the weak infinitesimal operator of the pair $\{x(t),\, y(t)\}$. If the trivial solution of equation (5) is exponentially p-stable, one can write the inequality

$$\begin{aligned} L_0\, v(x,y) &= \lim_{\delta\to 0} \tfrac{1}{\delta}[\int_0^T \mathbb{E}_{x,y}\{\mathbb{E}_{x(\delta),y(\delta)}|x(t)|^p\}\, dt - \int_0^T \mathbb{E}_{x,y}|x(t)|^p\, dt] \\ &= \lim_{\delta\to 0} \tfrac{1}{\delta}[\int_0^T \mathbb{E}_{x,y}|x(t+\delta)|^p\, dt - \int_0^T \mathbb{E}_{x,y}|x(t)|^p\, dt] \\ &= \mathbb{E}_{x,y}|x(T)|^p - |x|^p \le (Me^{-\gamma T} - 1)|x|^p, \end{aligned}$$

where M and γ are constants from from definition of exponential p-stability stability. Now we can put $T = (ln2 + lnM)/\gamma$ and proof is complete.

Corollary 1. *In the conditions of Lemma 2 the trivial solution of equation* (5) *is asymptotically stochastically stable.*

Proof. Due to formula (16) for $\overline{v}(t,x,y) = v(x,y)e^{\frac{c_3}{c_2}t}$ one may conclude that random process

$$\xi(t) := v(x(t),y(t))e^{\frac{c_3}{c_2}t}$$

is positive supermartingale. Hence

$$\begin{aligned} \sup_{y\in\mathbb{Y}} \mathbb{P}(\sup_{t\ge 0}|x(t,0,x,y| > \varepsilon) &= \sup_{y\in\mathbb{Y}} \mathbb{P}(\sup_{t\ge 0}|x(t,0,x,y|^p > \varepsilon^p) \\ &\le \sup_{y\in\mathbb{Y}} \mathbb{P}_{x,y}(\sup_{t\ge 0}\{\tfrac{1}{c_1}v(x(t),y(t))\} > \varepsilon^p) \\ &= \sup_{y\in\mathbb{Y}} \mathbb{P}_{x,y}(\sup_{t\ge 0}\{\tfrac{1}{c_1}\xi(t)e^{-\frac{c_3}{c_2}t}\} > \varepsilon^p) \\ &\le \sup_{y\in\mathbb{Y}} \mathbb{P}_{x,y}(\sup_{t\ge 0}\xi(t) > \varepsilon^p c_1) \le \tfrac{1}{\varepsilon^p c_1}\mathbb{E}_{x,y}\xi(0) \le \tfrac{c_2}{\varepsilon^p c_1}|x|^p \end{aligned}$$

and trivial solution of (5) is stochastically stable almost sure. Now to prove asymptotic stochastic stability one can apply the supermartingale inequality (14) and write the inequalities

$$\sup_{y\in\mathbb{Y}}\mathbb{P}(\sup_{t\geq u}|x(t,u,x,y| \ > \ c) = \sup_{y\in\mathbb{Y}}\mathbb{P}(\sup_{t\geq u}|x(t,u,x,y|^p \ > \ c^p)$$
$$\leq \sup_{y\in\mathbb{Y}}\mathbb{P}^{(u)}_{x,y}(\sup_{t\geq u}\{\tfrac{1}{c_1}v(x(t),y(t))\} \ > \ c^p)$$
$$\leq \sup_{y\in\mathbb{Y}}\mathbb{P}^{(u)}_{x,y}(\sup_{t\geq u}\{\tfrac{1}{c_1})\xi(t)e^{-\frac{c_3}{c_2}t}\} \ > \ c^p)$$
$$\leq \sup_{y\in\mathbb{Y}}\mathbb{P}^{(u)}_{x,y}(\sup_{t\geq u}\{\tfrac{1}{c_1})\xi(t)e^{-\frac{c_3}{c_2}u}\} \ > \ c^p) \leq \tfrac{1}{c^p c_1}\mathbb{E}\xi(u) \leq \tfrac{c_2}{c^p c_1}|x|^p e^{-\frac{c_3}{c_2}u}.$$

2 Stochastic stability by linear approximation

In this section we will consider the quasilinear equation

$$\frac{d\tilde{x}}{dt} \ = \ A(y(t))\tilde{x} \ + \ g(\tilde{x}, \ y(t)), \tag{18}$$

under conditions that the matrix $A(y)$ and Markov process $y(t)$ satisfy the conditions of the Section 1, the function $g(x,y)$ has bounded continuous x-derivative with conditions $g(0,y)\equiv 0$, and for any $r>0$ its x-derivative is uniformly bounded at any ball $B_r(0)$, i.e.

$$\sup_{\substack{y\in\mathbb{Y}\\ x\in B_r(0)}} \ ||D_x g(x,y)|| := \ g_r \ < \infty \tag{19}$$

Theorem 1.*If the equation* (5) *is asymptotically stochastically stable and* $\lim_{r\to 0} g_r \ = \ 0$, *then the trivial solution of equation* (18) *is asymptotically stochastically stable.*

Proof. Side by side with the equation (18) we will consider the equation (5) as an equation of its linear approximation. Due to Lemma 1 and Lemma 2 we can construct the Lyapunov function (17) with some small positive p. Because the matrix-valued function $D_x x(t,0,x,y)$ is the Cauchy matrix of the equation (5) it permits the estimation

$$\sup_{y\in\mathbb{Y}}\mathbb{E}||D_x x(t+s,s,x,y)||^p \ \leq \ h_2 e^{-\gamma t}$$

with some positive constants h, γ for all $t \ > \ 0$. Therefore the above Lyapunov function satisfies the conditions (15)-(16) and by construction for all $x \neq 0$ has x-derivative satisfying the inequalities

$$\begin{aligned}
|\nabla_x v(x,y)| \ &= \ |\int_0^T \mathbb{E}\{\nabla_x|x(t+s,s,x,y)|^p\}\,dt| \\
&\leq \ p\int_0^T \mathbb{E}\{|x(t+s,s,x,y)|^{(p-2)}|\{D_x x(t+s,s,x,y)\}\,x(t+s,s,x,y)|\}\,dt \\
&\leq \ p|x|^{(p-1)}\int_0^T \sup_{y\in\mathbb{Y}}\mathbb{E}||D_x x(t+s,s,x,y)||^p\,dt \leq \ c_3|x|^{(p-1)}
\end{aligned}$$

with some positive c_3. Because the above estimations do not dependent on initial time moment s we will put for simplicity $s = 0$. Now one can estimate the function $Lv(x, y)$ where L is weak infinitesimal operator of the pair $\{\tilde{x}(t),\, y(t)\}$:

$$\begin{aligned} Lv(x,y) \;&:= (A(y)x + g(x,y), \nabla_x)v(x,y) + Q\,v(x,y) \\ &= L_0\,v(x,y) \;+\; (g(x,y), \nabla_x)\,v(x,y) \\ &\le -\tfrac{1}{2}|x|^p + c_3|x|^p|g(x,y)| \le (g_r c_3 - \tfrac{1}{2})|x|^p \end{aligned}$$

for all $y \in \mathbb{Y},\, x \in B_r(0), r\, >\, 0$. Hence, due to Dynkin formula, we may use inequality

$$\begin{aligned} \mathbb{E}^{(u)}_{x,y} v(\tilde{x}(\tau_r(t)), y(\tau_r(t))) \;&=\; v(x,y) \;+\; \mathbb{E}^{(u)}_{x,y}\{\int_u^{\tau_r(t)} (Lv)(\tilde{x}(s), y(s))\,ds\} \\ &\le v(x,y) + (g_r c_3 - \frac{1}{2})\mathbb{E}^{(u)}_{x,y}\{\int_u^{\tau_r(t)} |\tilde{x}(s)|^p\,ds\} \end{aligned} \tag{20}$$

for all $y \,\in\, \mathbb{Y},\, x \,\in\, B_r(0), r \,>\, 0, t \,\ge\, u \,\ge\, 0$. If r is sufficiently small number the second summand in the right hand part of inequality (16) is nonpositive. Hence the stochastic process $v(\tilde{x}(\tau_r(t)), y(\tau_r(t)))$ is supermartingale and we can write the inequalities

$$\begin{aligned} &\mathbb{P}_{x,y}(\sup_{t\ge 0} |\tilde{x}(t)| \,>\, \varepsilon) = \mathbb{P}_{x,y}(\sup_{t\ge 0} |\tilde{x}(t)|^p \,>\, \varepsilon^p) \\ &= \mathbb{P}_{x,y}(\sup_{t\ge 0} |\tilde{x}(\tau_r(t))|^p \,>\, \varepsilon^p) \le \mathbb{P}_{x,y}(\sup_{t\ge 0} v(\tilde{x}(\tau_r(t)), y(\tau_r(t))) \\ &>\, c_1\varepsilon^p) \le \frac{v(x,y)}{c_1\varepsilon^p} \le \frac{c_2\delta^p)}{c_1\varepsilon^p} \end{aligned} \tag{21}$$

for all $y \in \mathbb{Y},\, x \in B_\delta(0), \delta \in (0, \varepsilon), \varepsilon \in (0, r)$ and sufficiently small $r \,>\, 0$. The local stability almost sure immediately follows from these inequalities. Let us define function

$$h_R(r) = \begin{cases} 1, & \text{for } x \in [0, R) \\ \frac{2R-r}{R}, & \text{for } x \in [R, 2R) \\ 0, & \text{for } x \ge 2R. \end{cases}$$

The differential equation

$$\frac{dx_R}{dt} = A(y(t))x_R + h_R(|x_R(t)|)g(x_R, y(t)) \tag{22}$$

has unique solution of the Cauchy problem $x_R(0) = x$ because function $h_R(|x|)g(x, y)$ satisfies the Lipschitz condition with constant c_{2R}. Hence the pair $\{x_R(t), y(t)\}$ is Markov process with weak infinitesimal operator L_R defined by equality

$$\begin{aligned} L_R v(x,y) \;&= (A(y)x, \nabla_x)v(x,y) + (h_R(|x|)g(x,y), \nabla_x)v(x,y) + Qv(x,y) \\ &= L_0 v(x,y) + (h_R(|x|)g(x,y), \nabla_x)v(x,y) \end{aligned}$$

and choosing R such small that $(c_{2R}c_3 - \frac{1}{2}) := -c_4 < 0$ one can write the inequality

$$L_R v(x,y) \le -c_4|x|^p.$$

Therefore

$$\mathbb{E}^{(u)}_{x,y} v(x_R(t), y(t)) \le v(x,y) - c_4 \int_u^t \mathbb{E}^{(u)}_{x,y} |x_R(s)|^p \, ds$$
$$\le v(x,y) - \frac{c_4}{c_1} \int_u^t \mathbb{E}^{(u)}_{x,y} v(x_R(s), y(s)) \, ds \qquad (23)$$

for all $t \ge u \ge 0$. Hence the stochastic process $v(x_R(t), y(t))$ is positive supermartingale and one can write

$$\mathbb{P}_{x,y}(\sup_{t \ge s} |x_R(t)| > \varepsilon) = \mathbb{P}_{x,y}(\sup_{t \ge s} |x_R(t)|^p > \varepsilon^p)$$
$$\le \mathbb{P}_{x,y}(\sup_{t \ge s} v(x_R(t), y(t)) > c_1 \varepsilon^p) \le \frac{1}{c_1 \varepsilon^p} \mathbb{E}_{x,y} v(x_R(s), y(s)) \qquad (24)$$

for all $y \in \mathbb{Y}$, $x \in B_R(0), \varepsilon \in (0, R)$ and sufficiently small $R > 0$. It is not complete to get the inequality

$$\mathbb{E}_{x,y} v(x_R(t), y(t)) \le v(x,y) e^{-\frac{c_4}{c_1} t} \le c_2 |x|^p e^{-\frac{c_4}{c_1} t}$$

from the inequality(19), and then f it can be written

$$\mathbb{P}_{x,y}(\sup_{t \ge s} |x_R(t)| > \varepsilon) \le \frac{c_2 |x|^p}{\varepsilon^p c_1} e^{-\frac{c_4}{c_1} s}.$$

Hence all solutions of the equation (22) which have start at $t = 0$ in the ball $B_\varepsilon(0)$ with $\varepsilon \in (0, R)$ and sufficiently small R tend to 0 with probability one. But up to time of the ball $B_\varepsilon(0)$ leaving the solutions of the equations (18) and (22) with the same initial conditions in the ball $B_\varepsilon(0)$ are coincident. So all solutions of (18) which do not leave the ball $B_\varepsilon(0)$ with sufficiently small ε tend to zero with probability one and the proof is complete.

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