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PROCEEDINGS

ASYMPTOTIC METHODS FOR STABILITY ANALYSIS OF MARKOV EVOLUTION FAMILIES BASED ON DIFFUSION APPROXIMATION OF SLOW MOTION

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Abstract. Asymptotic method for stability analysis of linear differential equations with matrix-function frequently switched by fast random evolution and ergodic Markov process is presented. The proposed method is based on diffusion approximation of linear slow motion and random evolution family applying averaging procedure with respect to time and the invariant measure of the Markov process. It is proved that exponential p-stability of the resulting linear stochastic equation non-dependent on time and switched Markov process is sufficient for exponential p-stability of the initial random system.

Keywords: Random Dynamical Syastems; Diffusion Approximation; Stochastic Stability; Averaging Procedures

Mathematics Subject Classification: Primary 60H10, 60H30; Secondary 37H10

1 Introduction

The problem of asymptotic analysis of dynamical systems under small random perturbations has been discussed in many mathematical and engineering papers (see, for example, [3], [6], [7]). Apparently, R. Z. Khasminsky was the first mathematician to have proved ([5]) that the probabilistic limit theorems may be successfully used for differential equations with random right parts. This approach makes it possible to apply for asymptotic analysis of real stochastic dynamical systems not only the Krylov-Bogolyubov averaging procedure but also diffusion approximation (see, for example, [6] and review there). To apply these methods one should introduce a small positive parameter ε and divide dependent on time t variables to fast, which time-derivatives are proportional to negative powers of ε, and slow with having a limit derivatives as $\varepsilon \to 0$. As a rule phase coordinates are slow variables but random perturbations are fast variables with ergodic property [7]. Applying probabilistic limit theorems the authors construct more simple dynamical system, which may be successfully used not only for approximate analysis of initial system on finite time interval but also for asymptotic behaviour of phase coordinates with $t \to \infty$ [7]. But some of engineering applications of dynamical system theory require transfer from initial phase coordinates to cylindrical coordinates. After change of variables initial system becomes splitted to two parts: radial motion as slow motion and fast motion (called by "rotation") with proportional to negative powers of small parameter vector field. As a result we have to deal with system of differential equations in a following form

$$\frac{dx^\varepsilon}{dt} = F(x^\varepsilon,\, y^\varepsilon,\, \xi^\varepsilon(t),\, \varepsilon), \tag{1}$$

$$\frac{dy^\varepsilon}{dt} = \frac{1}{\varepsilon} H(x^\varepsilon,\, y^\varepsilon,\, \xi^\varepsilon(t),\, \varepsilon), \tag{2}$$

where ε is small positive parameter, $F(x, y, \xi, \varepsilon)$ and $H(x, y, \xi, \varepsilon)$ are sufficiently smooth functions, $x^\varepsilon \in \mathbb{R}^n$, $y^\varepsilon \in \mathbb{R}^m$, and $\xi^\varepsilon(t)$ is homogeneous right continuous ergodic Markov process [2] on compact phase space $\mathbb{G}$ with weak infinitesimal operator Q^ε and invariant measure μ. It is clearly that if $F(x, y, \xi, \varepsilon)$ and $H(x, y, \xi, \varepsilon)$ are sufficiently smooth functions system (1)–(2) has unique solution of Cauchy problem $x^\varepsilon(s) = x$, $y^\varepsilon(s) = y$ under condition $\xi^\varepsilon(s) = \xi$ for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\xi \in \mathbb{G}$. We will denote this solution $x^\varepsilon(t, s, x, y, \xi)$, $y^\varepsilon(t, s, x, y, \xi)$. Assuming that $0 \in \mathbb{R}^n$ is equilibrium point for slow motion, that is, $F(0,\, y^\varepsilon,\, \xi^\varepsilon(t),\, \varepsilon) \equiv 0$, this paper analyzes asymptotic stability properties of this equilibrium. We will say that equation (1) is

- *locally stable almost sure, if for any $s \in \mathbb{R}$, $\eta > 0$ and $\beta > 0$ there exists such $\delta > 0$ that the inequality*

$$\sup_{\substack{y\in\mathbb{R}^m\\ \xi\in\mathbb{G}}} \mathbb{P}(\sup_{t\geq s} |x^\varepsilon(t, s, x, y, \xi)| > \eta) \;<\; \beta,$$

  follows the condition $x \in B_\delta(0)$, where $B_\delta(0) := \{x \in \mathbb{R}^n \,:\, |x| < \delta\}$;

- *locally asymptotically stochastically stable, if it is locally almost sure stable and there exists such $\gamma > 0$ that the trajectories which do not leave the ball B_γ tend to 0 as $t \to \infty$;*

- *asymptotically stochastically stable, if it is locally almost sure stable and for any $x \in \mathbb{R}^n$, $s \in \mathbb{R}$, and $c > 0$ the equality*

$$\lim_{T\to\infty} \sup_{\substack{y\in\mathbb{R}^m\\ \xi\in\mathbb{G}}} \mathbb{P}(\sup_{t>T} |x^\varepsilon(t, s, x, y, \xi)| > c) = 0$$

  is fulfilled;

- *exponentially p-stable, if there exist such positive numbers M and γ that for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\xi \in \mathbb{G}$, $s \in \mathbb{R}$ and $t > s$ the inequality*

$$\mathbb{E}\,|x^\varepsilon(t, s, x, y, \xi)|^p \leq M|x|^p e^{-\gamma(t-s)}$$

  is fulfilled.

In above definitions ε is a fixed parameter and we will consider the stability of (1) uniformly on $\varepsilon \in (0,\, \varepsilon_0)$ for some positive ε_0. The paper [8] proves that local asymptotical stochastic stability of equilibrium is equivalent to exponential p-stability of linear approximation written in a form

$$\frac{dx^\varepsilon}{ds} = A_0(z^\varepsilon,\, \xi^\varepsilon(s))x^\varepsilon + \varepsilon A_1(z^\varepsilon,\, \xi^\varepsilon(s))x^\varepsilon \tag{3}$$

$$\frac{dz^\varepsilon}{ds} = \frac{1}{\varepsilon} h(x^\varepsilon, z^\varepsilon,\, \xi^\varepsilon(s)) + g_0(x^\varepsilon, z^\varepsilon,\, \xi^\varepsilon(s)) + \varepsilon g_1(x^\varepsilon, z^\varepsilon,\, \xi^\varepsilon(s)) \tag{4}$$

with Markov process given by weak infinitesimal operators $Q^\varepsilon = \frac{1}{\varepsilon}Q$. The operator Q from the latest definitions is supposed to be closed with spectrum $\sigma(Q)$ splitted to two parts $\sigma(Q) = \sigma_{-\rho}(Q) \cup \{0\}$ where $\sigma_{-\rho}(Q) \subset \{Re\lambda \le -\rho < 0$ and zero eigenvalue has multiplicity one. This assumption guarantees [2] an ergodicity of Markov processes defined by infinitesimal operators $\frac{1}{\varepsilon}Q$ with the same invariant measure μ. To avoid of cumbersome formulae an averaging according to Markov phase coordinate $\xi \in \mathbb{G}$ and time t of any function $v(\xi, t)$ we will use following notations:

$$\bar{\bar{v}} := \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{\mathbb{G}} v(\xi, t)\mu(d\xi)dt, \quad \bar{v}(s) := \int_{\mathbb{G}} v(\xi, s)\mu(d\xi)$$

The paper [8] discusses the possible results for "nonoscillatory" fast motion defined by identity $\bar{\bar{h}}(x, z, \xi) \equiv 0$. In this case under some assumptions one can apply for stability analysis an averaging procedure for (1) with diffusion approximation of equation (2). Our paper deals with $h(x, z, \xi) \equiv \omega$. Substituting $z^\varepsilon = y^\varepsilon + \omega\tau^\varepsilon$, the above equations we may rewrite in a form

$$\frac{dx^\varepsilon}{ds} = A_0(y^\varepsilon + \omega\tau^\varepsilon,\, \xi^\varepsilon(s))x^\varepsilon + \varepsilon A_1(y^\varepsilon + \omega\tau^\varepsilon,\, \xi^\varepsilon(s))x^\varepsilon \quad (5)$$

$$\frac{dy^\varepsilon}{ds} = g_0(y^\varepsilon + \omega\tau^\varepsilon,\, \xi^\varepsilon(s)) + \varepsilon g_1(y^\varepsilon + \omega\tau^\varepsilon,\, \xi^\varepsilon(s)) \quad (6)$$

$$\frac{d\tau^\varepsilon}{ds} = \frac{1}{\varepsilon}$$

where $y^\varepsilon = z^\varepsilon - \omega\tau^\varepsilon$. Under assumption that functions $A_0(y, \xi), A_1(y, \xi), g_0(y, \xi), g_1(y, \xi)$ are bounded and have two bounded y-derivatives and assumption

$$\forall y \in \mathbb{G} : \quad \overline{\overline{(A_0(y + \omega t, \xi))}} = 0, \quad \overline{\overline{(g_0(y + \omega t, \xi))}} = 0 \quad (7)$$

we will proof that diffusion approximation of (5)-(6) may be successfully applied to exponential stability analysis of the trivial solution $x^\varepsilon(t) \equiv 0$.

2 Diffusion Approximation

We assume that the weak infinitesimal operator Q satisfying ergodic properties in a form

$$\sigma Q = \sigma_{-\rho} \cup \{0\}, \quad \sigma_{-\rho} \subset \{\mathbf{R}e\lambda < -\rho < 0\}, \quad \mathrm{d}im\{0\} = 1 \quad ((\sigma(Q)))$$

This spectral decomposition guarantees [7],[4] inequality

$$g \in \mathbb{C}(\mathbb{G}),\, s \ge t : \sup_{\xi\in\mathbb{G}} |E_{t,\xi}\{g(\xi(s))\} - \bar{g}| \le Me^{-\rho(s-t)}||g|| \quad (e^{tQ})$$

and permits to write solution of equation

$$\Big\{\frac{d}{dt} + Q\Big\}(\mathcal{R}v)(\xi, t) = -u(\xi, t) + \bar{\bar{v}} \quad (8)$$

in a convenient for evaluation integral form

$$(\mathcal{R}v)(\xi,t) := \int_t^{\infty} [[(E_{t,\xi}v)(\xi(s),s) - \bar{v}(s)]ds + \int_0^t [\bar{v}(s) - \bar{\bar{v}}]ds \tag{9}$$

The same like in the papers [9] and [8] our approach is based on construction of function $v_0(x,y) + \varepsilon v_1(x,y,\xi,t) + \varepsilon^2 v_2(x,y,\xi,t)$ as a solution of the equation

$$\begin{aligned} &\Big(\big(A_0(y+\omega t,\xi)x,\nabla_x\big) + \varepsilon\big(A_1(y+\omega t,\xi)x,\nabla_x\big) + \\ +\ &\big(g_0(y+\omega t,\xi),\nabla_y\big) + \varepsilon\big(g_1(y+\omega t,\xi),\nabla_y\big) + \\ +\ &\frac{1}{\varepsilon}\Big\{\frac{d}{dt} + Q\Big\}\Big)\big(v_0(x,y) + \varepsilon v_1 + \varepsilon^2 v_2\big) = \\ =\ &-\varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y,\xi,t) + \varepsilon^3 u_3(x,y,\xi,t) \end{aligned} \tag{10}$$

with sufficiently smooth functions $u_j, j = 1,2,3$. Function v_0 will be defined later under assumption of stability of approximating equation (diffusion approximation). In the above equality we should to equate the terms near the same powers of ε. The first equation has a form

$$\begin{aligned} &\Big(\big(A_0(y+\omega t,\xi)x,\nabla_x\big) + \big(g_0(y+\omega t,\xi),\nabla_y\big)\Big)v_0(x,y) + \\ &+\Big\{\tfrac{d}{dt} + Q\Big\}\, v_1(x,y,\xi,t) = 0 \end{aligned} \tag{11}$$

Assumption (!0) allows to find v_1:

$$v_1(x,y,\xi,t) = \Big(\big((\mathcal{R}A_0)(y+\omega t,\xi)x,\nabla_x\big) + \big((\mathcal{R}g_0)(y+\omega t,\xi),\nabla_y\big)\Big)v_0(x,y) \tag{12}$$

Equating the terms near ε, as well as adding and subtracting term $\mathcal{L}_0 v_0(x,y)$, one can write equation

$$\begin{aligned} &\Big\{\Big(\big(A_0(y+\omega t,\xi)x,\nabla_x\big) + \big(g_0(y+\omega t,\xi),\nabla_y\big)\Big)v_1(x,y,\xi,t) + \\ +\ &\Big(\big(A_1(y+\omega t,\xi)x,\nabla_x\big) + \big(g_1(y+\omega t,\xi),\nabla_y\big)\Big)v_0(x,y) - \\ -\ &\underline{\mathcal{L}_0 v_0(x,y)} + \Big\{\frac{d}{dt} + Q\Big\}\, v_2(x,y,\xi,t)\Big\} + \underline{\mathcal{L}_0 v_0(x,y)} = -u_1(x,y) \end{aligned} \tag{13}$$

which is basic for our approach. At first we should substitute (12) and, applying Fredholm alternative, define a diffusion approximation of initial system (5)-(6) as Markov process with infinitesimal operator

$$\begin{aligned} \mathcal{L}_0 v(x,y) :\ =\ &\Big[\big(\overline{\overline{\mathcal{R}A_1}},\ \nabla_x\big) + \big(\overline{\overline{\mathcal{R}g_1}},\ \nabla_y\big)\Big]v(x,y) + \\ +\ &\Big[\overline{\overline{(A_0x,\nabla_x)(\mathcal{R}A_0x,\nabla_x)}} + \overline{\overline{(g_0,\nabla_y)(\mathcal{R}A_0x,\nabla_x)}} + \\ +\ &\overline{\overline{(A_0x,\nabla_x)(\mathcal{R}g_0,\nabla_y)}} + \overline{\overline{(g_0,\nabla_y)(\mathcal{R}g_0,\nabla_y)}}\Big]v(x,y) \end{aligned} \tag{14}$$

This process may be given as the system of stochastic Ito equations

$$dx = a_1(y)xdt + \sum_{k=1}^{n+m} b_{1k}(y)xdw_k(t) \quad (15)$$

$$dy = a_2(y)dt + \sum_{k=1}^{n+m} b_{2k}(y)dw_k(t) \quad (16)$$

3 Stability analysis

For exponential p-stability analysis of defined by (15)-(16) slow motion one can apply proposed in [5] second Lyapunov method with Lyapunov function $v_0(x, y)$ satisfying Lyapunov equation

$$(\mathcal{L}_0 v_0)(x, y) = -u_1(x, y) \quad (17)$$

If the defined by equation (15)-(16) slow motion is asymptotically stable, there exist such constants $M > 0, \rho > 0$ that $\mathbf{E}_{x,y}\{|x(t)|^p\} \leq Me^{-\rho t}|x|^p$ for any $x \in \mathbb{R}^n, y \in \mathbb{R}^m$. Therefore we may choose

$$u_1(x, y) := \int_0^\infty \mathbf{E}_{x,y}\{|x(t)|^p\}dt \quad (18)$$

Under smoothness (by y) assumption of functions $A_0(y, \xi), A_1(y, \xi), g_0(y, \xi), g_1(y, \xi)$ one can proof [5] that function (18) and solution of equation (17) staisfy inequalities

$$\exists c_1 > 0, c_2 > 0, c_3 > 0, c_4 > 0 : \forall\, y \in \mathbb{G}, \forall\, x \in \mathbb{R} :$$

$$c_1|x|^p \leq v_0(x, y) \leq c_2|x|^p, \quad (19)$$

$$c_3|x|^p \leq u_1(x, y)| \leq c_4|x|^p, \quad (20)$$

Now one can find the second term $v_2(x, y, \xi, t)$ of desired Lyapunov function:

$$\begin{aligned} v_2 = &\Big(\big((\mathcal{R}A_1)(y+\omega t, \xi)x, \nabla_x\big) + \big((\mathcal{R}g_1)(y+\omega t, \xi), \nabla_y\big)\Big)v_0(x, y) + \\ + &\Big[\big(A_0(y+\omega t, \xi)x, \nabla_x\big)\big((\mathcal{R}A_0)(y+\omega t, \xi)x, \nabla_x\big) + \\ + &\big(g_0(y+\omega t, \xi), \nabla_y\big)\big((\mathcal{R}A_0)(y+\omega t, \xi)x, \nabla_x\big) + \\ + &\big(A_0(y+\omega t, \xi)x, \nabla_x\big)\big((\mathcal{R}g_0)(y+\omega t, \xi), \nabla_y\big) + \\ + &\big(g_0(y+\omega t, \xi), \nabla_y\big)\big((\mathcal{R}g_0)(y+\omega t, \xi), \nabla_y\big)\Big]v_0(x, y) - \mathcal{L}_0 v_0(x, y)\Big] \end{aligned} \quad (21)$$

The last two terms u_2, u_3 in Lyapunov equation (10) one can define in "a residual" form

$$\begin{aligned} &u_2(x, y, \xi, t) = \\ = &\Big(\big(A_0(y+\omega t, \xi)x, \nabla_x\big) + \big(g_0(y+\omega t, \xi), \nabla_y\big)\Big)v_2(x, y, \xi, t) + \\ + &\Big(\big(A_1(y+\omega t, \xi)x, \nabla_x\big) + \big(g_1(y+\omega t, \xi), \nabla_y\big)\Big)v_1(x, y, \xi, t) \qquad u_3(x, y, \xi, t) = \\ = &\Big(\big(A_1(y+\omega t, \xi)x, \nabla_x\big) + \big(g_1(y+\omega t, \xi), \nabla_y\big)\Big)v_2(x, y, \xi, t) \end{aligned} \quad (22)$$

Owing to boundedness of the operator-resolvent $\mathcal{R}$ and as a result of inequalities (19), (20) we make sure that the above defined functions (22),(22), (12), and (21) satisfy inequalities

$$\exists c > 0 : \forall y \in \mathbb{G}, \forall x \in \mathbb{R} :$$
$$|u_2(x, y, \xi, t)| \leq c|x|^p, |u_3(x, y, \xi, t)| \leq c|x|^p, |v_2(x, y, \xi, t)| \leq c|x|^p, |v_3(x, y, \xi, t)| \leq c|x|^p$$

Therefore one may choose such a sufficiently small positive number ε_0 that

$$\forall y \in \mathbb{G}, \forall x \in \mathbb{R} :$$
$$\frac{c_1}{2}|x|^p \leq v_0(x, y) + v_1(x, y, \xi, t) + v_2(x, y, \xi, t) \leq \frac{c_2}{2}|x|^p,$$
$$\frac{c_3}{2}|x|^p \leq u_1(x, y) + u_2(x, y, \xi, t) + u_3(x, y, \xi, t) \leq \frac{c_4}{2}|x|^p$$

These inequalities for solution of Lyapunov equation (10) guarantee [8] exponential p-stability of defined by (5)-(6) slow motion for all sufficiently small $\varepsilon > 0$.

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