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COVARIANCE SEMIGROUP FOR DELAYED GEOMETRIC BROWNIAN MOTION

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Abstract. The paper proposes algorithm for time asymptotic analysis of stochastic linear functional differential equations. An approach is based on extension of the defined by deterministic part of this equation resolving semigroup to linear operator semigroup in the space of countable additive symmetric measures. The weak infinitesimal operator of this semigroup helps to find such Lyapunov-Krasovsky type quadratic functional that gives a necessary and sufficient asymptotic stability condition for the equation defined by selected deterministic part of analyzed stochastic equation. Furthermore, we have got a stochastic process, usable for Ito stochastic differential, by substituting the solution of the analyzed stochastic equation as an argument of this quadratic functional. This property permits to derive an analogue of Ito formula for the above mentioned stochastic process and to discuss equilibrium asymptotic stochastic stability conditions for initial stochastic functional differential equation. As an example, we have deduced necessary and sufficient condition for mean square decrease of stochastic exponent given by Ito type scalar equation with delay.

Keywords: Functional differential equations, Second Lyapunov method, Mean square stability, Delayed stochastic exponent

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1 Introduction

Stochastic exponent or Geometric Brownian motion is most frequently used mathematical model for price index evolution (see, for example, [8] and its refernces there). This stochastic process is defined by scalar linear stochastic differential equation

$$dx(t) = bx(t)dt + \sigma x(t)dw(t) \tag{1}$$

where $w(t)$ is standard Wiener process or Brownian motion. The solution of this equation is constant-sign stochastic process, and if $2b \neq \sigma^2$, this process with probability one tends to 0 (for $2b < \sigma^2$) or to infinity (for $2b > \sigma^2$). But, as it has been indicated in [6], modeling the price process by geometric Brownian motion has been criticized because this model does not take into account the past of analyzed process. In reality, price index evolution does depend depends on its past. We should model drift and diffusion as linear continuous functionals in the space of continuous functions $\mathbb{C}([-h, 0])$ and should deal with linear stochastic functional differential equation

$$dx(t) = f(x_t)dt + g(x_t)dw(t) \tag{2}$$

where $x_t = \{x(t+\theta), -h \le \theta \le 0\}$,

$$f(x_t) = \int\limits_{-h}^{0} x(t+\theta)dF(\theta), \;\; g(x_t) = \int\limits_{-h}^{0} x(t+\theta)dG(\theta)$$

$F(\theta)$ and $G(\theta)$ are functions of bounded variation. This equation in [6] is referred to as *delayed geometric Brownian motion*,whereas in [10], [11] and other papers as *delayed stochastic exponent*. The authors in the above mentioned papers discuss the behavior of delayed stochastic exponent as $t \to \infty$. As it has been shown in [6] and in [2] for that, we can successfully apply integral equation for this. However, this method provides only necessary and sufficient mean square stability condidition and in very complicated form involving improper integrals. For example, a condition for exponential of second moment $\mathbf{E}|x(t)|^2$ even for simplest linear stochastic equation with delay

$$dx(t) = bx(t-1)dt + \sigma x(t-1)dw(t) \tag{3}$$

has a form

$$-\frac{\pi}{2} < b < 0, \;\; \sigma^2 < \frac{1}{\pi}\int\limits_{0}^{\infty} \frac{dz}{(z^2 + 2b\sin z + b^2)} \tag{4}$$

Hence, for time asymptotic analysis of stock price $\{s(t)\}$ authors ([10], [11]) apply the second Lyapunov method with specially constructed quadratic Lyapunov-Krasovsky type functional ([4], [2]). The question under discussion in [10], [11] is behavior of stock price index $s(t)$ model-based by equation

$$ds(t) = [as(t) - bs(t-1)]dt + \sigma s(t-1)dw(t) \tag{5}$$

The formula derived in the above papers, which guaranties an exponential decrease of any solution of (5) with probability one, has a form: $a + \frac{\sigma^2}{2} + |b| < 0$. It should be noted that, according to the most often used dynamic models of financial securities, the term $as(t)$ reflects a dependence of price on demand. Consequently, for realistic market the coefficient a has to be positive. This assertion conflicts with the above inequality. Therefore, the results of the papers [10], [11] are inapplicable for real financial market models. By choosing more appropriate Lyapunov-Krasovsky functional, we have proven in [9] that even for positive a with some constrains on b and σ any solution of equation (5) can be exponentially decreasing with probability one. Our proposed approach is based on the second Lyapunov method combined with analysis of special type extension of resolving semigroup [1] defined by drift equation

$$\frac{d}{dt}\hat{x}(t) = f(\hat{x}_t), \tag{6}$$

This helps us to derive a method and an algorithm, which we can use for equilibrium stability analysis of (3). Besides, applying this method, we can construct [2] sufficiently smooth Lyapunov-Krasovsky type quadratic functional and can successfully use this functional for stochastic stability analysis under permanent random perturbation3 (3).

2 Extension of resolving semigroup for linear FDE

Let us remember [1] that the equation (6) assigns on the space $\mathbb{C}([-h,0])$ continuous semigroup $\{\mathbf{X}(t),\, t\geq 0\}$ of class $\mathbf{C}_0$ [3] defined by the equality $(\mathbf{X}(t)\varphi)(\theta)=x(t+\theta,\varphi)$ for all $t\geq 0,\, \varphi\in\mathbb{C}([-h,0])$, and $\theta\in[-h,0]$ where $\{x(t,\varphi)\}$ is the solution of (6) with initial condition $x(s,\varphi)=\varphi(s),\, s\in[-h,0]$. Using this result we will introduce the linear continuous semigroup $\{\mathbf{S}(t):=\mathbf{X}(t)\otimes\mathbf{X}(t),\, t\geq 0\}$ on the space $\mathcal{C}:=\mathbb{C}([-h,0]\times[-h,0])$. It is not difficult to prove that the operator family $\{\mathbf{S}(t),\, t\geq 0\}$ is strongly continuous semigroup on $\mathcal{C}$ with infinitesimal operator

$$(\mathbf{A}q)(\theta_1,\theta_2)=\begin{cases}\left(\frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2}\right)q(\theta_1,\theta_2), \text{if } -h\leq\theta_1<0,-h\leq\theta_2<0,\\ \frac{\partial}{\partial\theta_1}q(\theta_1,0)+\int\limits_{-h}^{0}q(\theta_1,\theta)dF(\theta), \text{if } -h\leq\theta_1<0,\theta_2=0,\\ \frac{\partial}{\partial\theta_1}q(0,\theta_2)+\int\limits_{-h}^{0}q(\theta,\theta_2)dF(\theta), \text{if } \theta_1=0,-h\leq\theta_2<0,\\ \int\limits_{-h}^{0}(q(0,\theta)dF(\theta)+q(\theta,0)dF(\theta)), \text{if } \theta_1=0,\theta_2=0\end{cases}$$

As it has been proven in [2], the spectrum $\sigma(\mathbf{A})$ of operator $\mathbf{A}$ is a set of eigenvalues and the number $\sup\Re\{\sigma(\mathbf{A})\}:=\lambda_0$ is an eigenvalue of the operator $\mathbf{A}$ and defines the type of the semigroup $\{\mathbf{S}(t)\}$, i.e., $\lambda_0=\lim\limits_{t\to\infty}\frac{1}{t}\ln||\mathbf{S}(t)||$. The operator family $\{\mathbf{S}^*(t),\, t\geq 0\}$ also forms a weakly continuous semigroup with weak infinitesimal operator $\mathbf{A}^*$. For any $\mu\in\mathcal{D}(\mathbf{A}^*)$ quadratic functional $v(\varphi):=<\mu\varphi,\,\varphi>$ has Lyapunov derivative $\mathcal{L}_0v$ with respect to (6)

$$(\mathcal{L}_0v)(\varphi):=\lim_{t\to+0}\frac{v(x_t)-v(\varphi)}{t} \tag{7}$$

and this is also quadratic functional given by equality $(\mathcal{L}_0v)(\varphi)=<\mathbf{A}^*\mu\varphi,\varphi>$.

3 Lyapunov equation for linear functional differential equation

It is well known that for stability analysis of linear functional differential equations by the second Lyapunov method we can successfully use a set of quadratic functionals [1]. The trivial solution of (6) is exponentially stable if and only if there exists such positive defined quadratic functional $v(\varphi)$ and positive constant c that

$$\forall\varphi\in\mathbb{C}([-h,0]):(\mathcal{L}_0v)(\varphi)<-c|\varphi(0)|^2 \tag{8}$$

It is not difficult to prove that we may reformulate this assertion as follows there exists such a positive measure $\nu\in\overset{\circ}{\mathbb{K}}{}^*$ that $\mathbf{A}^*\mu=-\nu$ with unique $\mu\in\mathbb{K}^*$. We will refer to this equation as *Lyapunov equation for quadratic functionals.* The above result permits us to derive a following algorithm of Lyapunov-Krasovsky functional $v(\varphi)$ construction:

- solve the equation

$$\mathbf{A}q=-\varphi\otimes\varphi \tag{9}$$

- find

$$v(\varphi):=<\mu\varphi,\varphi>:=[\nu,q] \tag{10}$$

The inequality $\delta(\mu)>0$ is equivalent to exponential decrease of any solution of (6).

4 Linear Stochastic Functional Differential Equations

Let us consider stochastic functional differential equation (2) . This equation defines [2] Markov process in the space $\mathbb{C}([-h,0])$ with weak infinitesimal operator

$$\mathcal{L}v(\varphi) = \lim_{t\to+0}\frac{1}{t}[\mathbf{E}\{v(x_t)/x_0 = \varphi\} - v(\varphi)] \tag{11}$$

According to the perturbation theory algorithm equation (2) is a result of permanent stochastic perturbations of linear FDE (6), which was analyzed in previous section. We have proven in [2] that for any solution $x(t)$ of (2) and any $\mu \in \mathcal{D}(\mathbf{A}^*)$ stochastic process $< \mu x_t, x_t >$ has stochastic Ito differential

$$d < \mu x_t, x_t >= \mathcal{L} < \mu x_t, x_t > dt + 2g(x_t) < \mu\mathbf{1}, x_t > dw(t) \tag{12}$$

where

$$\mathcal{L} < \mu\varphi, \varphi >= \mathcal{L}_0 < \mu\varphi, \varphi > +\delta(\mu)|g(\varphi)|^2$$

If we select μ from $\mathbf{A}^*\mu = -\nu$ then $\mathcal{L}_0 < \mu\varphi, \varphi >=< \mathbf{A}^*\mu\varphi, \varphi >= - < \nu\varphi, \varphi >$ and we can write the above for stability analysis of (2) in simpler form

$$\mathcal{L} < \mu\varphi, \varphi >= - < \nu\varphi, \varphi > +\delta(\mu)|g(\varphi)|^2 \tag{13}$$

with $\mu \in \mathbb{K}^*, \nu \in \overset{\circ}{\mathbb{K}}{}^*$. If the right-hand part of (13) is strongly positive quadratic functional the stochastic process $< \mu x_t, x_t >$ is a positive supermartingale that satisfies inequality $\frac{d}{dt}\mathbf{E}_\varphi\{< \mu x_t, x_t >\} < -c\mathbf{E}_\varphi\{|x(t)|^2\}$ with some positive constant c for any initial function $x_0 = \varphi \in \mathbb{C}([-h,0])$. The latter inequality guaranties[2] exponential decrease of any solution of equation (2). The success of proposed method depends on the choice of a quadratic functional $\mu \in \mathbb{K}^*$ for analysis of deterministic linear functional differential equation corresponding to (2). We will illustrate this by an analysis of the equation (3). A necessary and sufficient condition of exponential decreasing of any solution for this equation has a form: $-\frac{\pi}{2} < b < 0$. As consistent with our proposal algorithm at first we have to solve equation

$$\left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right) q(\theta_1, \theta_2) = -\varphi(\theta_1)\varphi(\theta_2) \tag{14}$$

for $-h \le \theta_1 < 0, -h \le \theta_2 < 0$. As we are looking for a symmetric function, we can find solution of (14) for $-1 \le \theta_2 \le \theta_1 < 0$ and extend this function to the set $-1 \le \theta_1 \le \theta_2 < 0$ applying symmetry $q(\theta_1,\theta_2) = q(\theta_2,\theta_1)$.It is not difficult to find a general solution of the above equation:

$$q(\theta_1, \theta_2) = r(\theta_2 - \theta_1) + \int_{\theta_2}^{\theta_2-\theta_1} \varphi(u - \theta_2 + \theta_1)\varphi(u)du \tag{15}$$

with an arbitrary smooth function$\{r(t),\ t \in [-1, 0]\}$.If $\theta_1 = 0, -1 \le \theta_2 \le 0$ the above function $q(\theta_1, \theta_2)$ satisfies differential equation

$$\frac{\partial}{\partial\theta_2} q(0, \theta_2) + bq(\theta_2, -1) = -\varphi(0)\varphi(\theta_2)$$

with boundary condition $2bq(0,-1) = -|\varphi(0)|^2$. Substitution (15) results in a differential equation for $\{r(t),\ -1 \le t \le 0\}$

$$\dot r(t) = -br(-1-t) + p(t) \tag{16}$$

where $p(t) = -b \int_{-1}^{-1-t} \varphi(u+1+t)\varphi(u)du - \varphi(t)\varphi(0)$, and boundary condition

$$2br(-1) = -|\varphi(0)|^2 \tag{17}$$

Having differentiated equation (16) with respect to t we can rewrite the above equation as second order ordinary differential equation

$$\ddot r(t) + b^2 r(t) = bp(-1-t) + \dot p(t) \tag{18}$$

and can easily find solution of this equation in a following form:

$$r(t) = c_1 \cos(bt) + c_2 \sin(bt) + \frac{1}{b}\int_{-1}^{t} \sin(b(t-s))(bp(-1-s) + \dot p(s))ds$$

with arbitrary constants c_1 and c_2. As equation (16) has been differentiated, the solution $r(t)$ of equation (18) above found should satisfy both boundary condition (17) and also an additional boundary condition $\dot r(0) = -br(-1) + p(0)$. The boundary conditions permit to write the system of linear equations for constants c_1 and c_2, and find a function $q(\theta_1, \theta_2) = -(\mathbf{A}^{-1}\varphi \otimes \varphi)(\theta_1, \theta_2)$. Now we can find a quadratic functional, which we will use for analysis of stochastic equation (2). For this equation with delayed diffusion it is more convenient to use the quadratic functional as a sum $< \mu\varphi, \varphi > = < \mu_0\varphi, \varphi > + \gamma \int_{-1}^{0} |\varphi(\theta)|^2 d\theta$ where $< \mu_0\varphi, \varphi > = - < (\mathbf{A}^*)^{-1}\delta_0\varphi, \varphi >$. The positive constant γ will be selected later. We can find μ_0 by applying the above computed solution $q(\theta_1, \theta_2)$:

$$\begin{aligned} < \mu_0\varphi, \varphi > = [\delta_0, q] &= q(0,0) = -\frac{1-\sin b}{2b\cos b}|\varphi(0)|^2 \\ &- \frac{\sin b}{b}\left[\int_{-1}^{0} |\varphi(s)|^2 ds + \varphi(-1)\varphi(0)\right] \end{aligned} \tag{19}$$

Substituting $< \mu\varphi, \varphi >$ in (13) results in the equation

$$\mathcal{L} < \mu\varphi, \varphi > = -|\varphi(0)|^2 + \gamma|\varphi(0)|^2 - \gamma|f(-1)|^2 - \sigma^2 \frac{1-\sin b}{2b\cos b}|\varphi(-1)|^2$$

Therefore choosing $\gamma = -\sigma^2(1-\sin b)(2b\cos b)^{-1}|\varphi(-1)|^2$, we can write necessary and sufficient condition for exponential decrease of the second moment in a form of the inequality $\sigma^2 < -2b\cos b/(1-\sin b)$. Applying the above proposed algorithm we can also find the necessary and sufficient condition for exponential decrease of the second moment of the delayed exponent (4). For stability analysis of this equation it is convenient to introduce a new parameter $c = a/b$. If $\sigma = 0$, the equilibrium stability region of the deterministic equation is $\mathbf{G} = \mathbf{G}_1 \cup \mathbf{G}_2$ where

$\mathbf{G}_1 = \{c < -1, b > 0\} \cup \{c > 1, b < 0\}$, and $\mathbf{G}_2 = \left\{-1 \le c < 0, 0 < b < \arccos(c)/\sqrt{1-c^2}\right\}$. The necessary and sufficient conditions for equilibrium mean square exponential stability of (4) we can write in the following form:

$$\sigma^2 < \begin{cases} 2b\sqrt{1-c^2}\frac{\sqrt{1-c^2}-\sin(b\sqrt{1-c^2})}{\cos(b\sqrt{1-c^2})+c}, & \text{if } \{b,c\} \in \mathbf{G}_1, \\ 2b\sqrt{c^2-1}\frac{e^{2b\sqrt{c^2-1}}-2\sqrt{c^2-1}e^{2b\sqrt{c^2-1}}-1}{1+2ce^{2b\sqrt{c^2-1}}+e^{2b\sqrt{c^2-1}}}, & \text{if } \{b,c\} \in \mathbf{G}_2 \end{cases}$$

The functional $v(\varphi)$, constructed by algorithmproposed in this paper, is also well-suited for quasilinear stochastic functional differential equation analysis (see, for example, [2]).

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