Eddy current problem for a moving medium with varying properties

Valentina Koliskina, and Inta Volodko

Abstract**—**Analytical solution of eddy current problem for a moving medium is obtained in the present paper. A single-turn circular coil is located above a conducting two-layer medium. The upper layer is moving in a horizontal direction with constant velocity. The lower layer of the medium is fixed. The electrical conductivity and magnetic permeability of the upper layer are exponential functions of the vertical coordinate. The solution is found by the method of Fourier integral transform. The change in impedance of the coil is obtained in terms of double integral containing Bessel functions. A particular case of a moving half-space with varying properties is considered in detail. The solution can be generalized for the case of a moving multilayer medium.

Keywords**—**eddy current testing, multilayer medium, electrical conductivity, magnetic permeability, Fourier transform

I. Introduction

EDDY current method is widely used in practice in order to test properties of electrically conducting materials. The theory of the method is well-developed in the literature for the case where a coil with alternating current is located above a multilayer conducting medium with constant electrical conductivity and magnetic permeability [1]-[3].

There are examples in engineering where the properties of a conducting layer (the electrical conductivity or magnetic permeability) can vary with respect to one spatial coordinate. Examples include the following applications: (a) analysis of depletion of aluminium in blades of gas turbines [4] where the electrical conductivity of the metal alloy changes with respect to the vertical coordinate and (b) the change of the magnetic permeability of ferromagnetic metals as a result of surface hardening [5], [6]. In particular, experimental data in [5] indicate that the magnetic permeability of ferromagnetic metals in such cases can be well approximated by an exponential function of the vertical coordinate.

Two approaches are usually used in the literature in order to model eddy current problem for a multilayer medium with varying properties. One method is based on the solution of eddy current problem for a multilayer medium with constant properties. In this case variability of electrical conductivity or magnetic permeability is represented by a piecewise-constant function. In other words, the region where the parameters of the medium change with respect to the vertical coordinate is divided into a large number of relatively thin layers where the properties of each layer are assumed to be constant. Such an approach is used, for example, in [7] where up to 50 layers of constant conductivity are used in order to model variability of the electrical conductivity with respect to the vertical coordinate.

The second approach is based on the assumption that for a relatively simple model profiles of electrical conductivity and/or magnetic permeability the solution of the corresponding boundary value problem can be found in terms of known special functions (such as Bessel functions and hypergeometric functions). Examples of analytical solutions of eddy current problems can be found in the literature for the following cases: (a) electrical conductivity of a conducting layer is an exponential function of the vertical coordinate [8] or is modeled by a hyperbolic tangent function [9]; (b) magnetic permeability of a conducting layer is an exponential function of the vertical coordinate [2]. The cases where both electrical conductivity and magnetic permeability are exponential functions of the vertical coordinate are considered in [10]-[12]. In addition, the case where both electrical conductivity and magnetic permeability of each layer in a multilayer conducting tube are power functions of the radial coordinate is investigated in [13].

There are applications where a conducting medium is moving with respect to the coil. One example is steel processing at metallurgical plants. Another important example is the movement of a coin inside a coin validator. Analytical solutions for the case where one conducting layer of a planar multilayer medium with constant properties is moving in a horizontal direction with constant velocity V can be found in [14], [15]. The change in impedance of a coil is found in [14] and [15] by the method of a double Fourier transform with respect to two horizontal coordinates. The case of a moving half-space with varying properties is considered in [16].

In the present paper we construct an analytical solution for an eddy current problem where a coil with alternating current is located above a conducting two-layer medium. The upper layer is moving in a horizontal direction with constant velocity V while the lower half-space is fixed. The electrical conductivity and magnetic permeability of the upper layer are exponential functions of the vertical coordinate. The problem

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is solved by the method of Fourier integral transform in the two horizontal directions.

II. Analytical solution for the case of a two-layer medium

Consider a single-turn coil of radius r_c with alternating current located at a distance h from a two-layer conducting medium (see Fig. 1).

The upper layer is moving in a horizontal direction with constant velocity V while the lower half-space is fixed.

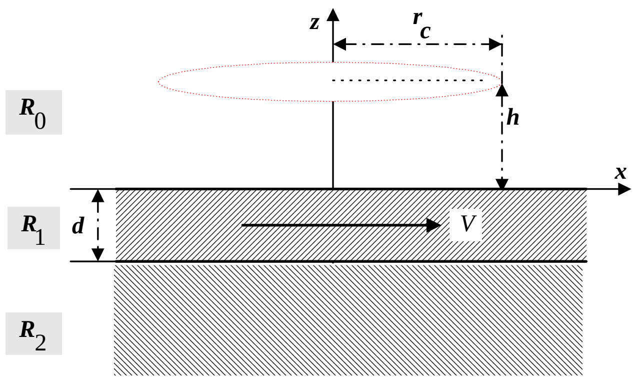

Fig. 1. A single-turn coil above a two-layer medium.

The electrical conductivity and magnetic permeability of region R_1 are exponential functions of the vertical coordinate of the form

$$\sigma_1(z) = \sigma_m e^{\alpha z}, \quad \mu_1(z) = \mu_0 \mu_m e^{\beta z}, \tag{1}$$

where α and β are constants, and μ_0 is the magnetic constant. The electrical conductivity σ_2 and relative magnetic permeability μ_2 of region R_2 are constant.

The problem can be formulated in terms of the vector potential $\vec{A}$ which is defined by the relation

$$\vec{B} = \text{curl}\,\vec{A} \tag{2}$$

where $\vec{B}$ is the magnetic induction vector. The vector potential can be written in the form

$$\vec{A}(x,y,z,t) = A_x(x,y,z)e^{j\omega t}\vec{e}_x + A_y(x,y,z)e^{j\omega t}\vec{e}_y, \tag{3}$$

where ω is the frequency, $\vec{e}_x$ and $\vec{e}_y$ are the unit vectors in the x and y directions, respectively. It is assumed that the external current in the coil has the form

$$\vec{I}^{\,e} = I_\varphi e^{j\omega t}\vec{e}_\varphi, \tag{4}$$

where $\vec{e}_\varphi$ is the unit vector in the $\varphi-$direction of the system of cylindrical polar coordinates (r,φ,z). Using (3) and (4) we rewrite the system of Maxwell's equations in the form (the displacement current is neglected as it is a usual assumption for eddy current testing problems)

$$\Delta\vec{A} - \frac{1}{\mu(z)}\frac{d\mu}{dz}\left(\frac{\partial A_x}{\partial z}\vec{e}_x + \frac{\partial A_y}{\partial z}\vec{e}_y\right) - j\omega\mu_0\mu(z)\sigma(z)\vec{A}$$
$$-\mu_0\mu(z)\sigma(z)V\frac{\partial A_x}{\partial x} = -\mu_0\mu(z)\left(I_x\vec{e}_x + I_y\vec{e}_y\right). \tag{5}$$

The following relationships hold for the components of the external current:

$$I_x\vec{e}_x = -I\sin\varphi\vec{e}_\varphi, \tag{6}$$

$$I_y\vec{e}_y = I\cos\varphi\vec{e}_\varphi. \tag{7}$$

It is seen from (5)-(7) that equations for A_x and A_y can be separated and solved independently. First, we consider the solution for the x component of the vector potential. Equations for A_x in regions R_0, R_1 and R_2 have the form (see Fig. 1):

$$\Delta A_{x0} = \mu_0 I\delta(z-h)\delta(r-r_c)\sin\varphi, \tag{8}$$

$$\Delta A_{x1} - \frac{1}{\mu_1}\frac{d\mu_1}{dz}\frac{\partial A_{x1}}{\partial z} + k_1^2 A_{x1} - \tilde{V}\frac{\partial A_{x1}}{\partial x} = 0, \tag{9}$$

$$\Delta A_{x2} + k_2^2 A_{x2} = 0, \tag{10}$$

where the subscripts 0, 1 and 2 in (8)-(10) correspond to regions R_0, R_1 and R_2, respectively, $\delta(x)$ is the Dirac delta-function,
$k_1^2 = -j\omega\mu_1(z)\sigma_1(z),\ \tilde{V} = \mu_1(z)\sigma_1(z)V,\ k_2^2 = -j\omega\mu_2\sigma_2.$
The boundary conditions are

$$A_{x0}\,|_{z=0} = A_{x1}\,|_{z=0}, \quad \frac{\partial A_{x0}}{\partial z}|_{z=0} = \frac{1}{\mu_m}\frac{\partial A_{x1}}{\partial z}|_{z=0}, \tag{11}$$

$$A_{x1}\,|_{z=-d} = A_{x1}\,|_{z=-d}, \quad \frac{1}{\tilde{\mu}_1}\frac{\partial A_{x1}}{\partial z}|_{z=-d} = \frac{1}{\mu_2}\frac{\partial A_{x2}}{\partial z}|_{z=-d},. \tag{12}$$

where $\tilde{\mu}_1 = \mu_m e^{-\beta d}$.

The following conditions hold at infinity:

$$A_{xi}, \ \frac{\partial A_{xi}}{\partial x}, \ \frac{\partial A_{xi}}{\partial y} \to 0 \text{ as } x \to \pm\infty, \ i = 0,1,2. \tag{13}$$

Problem (8)-(13) is solved by the successive application of the Fourier integral transforms as follows:

$$\tilde{A}_{xi}(\lambda_x, y, z) = \int_{-\infty}^{\infty} A_{xi}(x, y, z) e^{-j\lambda_x x} dx, \ i = 0,1,2, \tag{14}$$

$$\tilde{\tilde{A}}_{xi}(\lambda_x, \lambda_y, z) = \int_{-\infty}^{\infty} \tilde{A}_{xi}(\lambda_x, y, z) e^{-j\lambda_y y} dy, \ i = 0,1,2. \tag{15}$$

In order to apply transforms (14) and (15) to the right-hand side of (8) we multiply it by $\exp(-j\lambda_x x)\exp(-j\lambda_y y)$ and integrate the resulting expression with respect to x and y. To perform integration we transform the double integral to polar coordinates (r, φ). Thus,

$$\begin{aligned} RHS &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu_0 I \delta(z-h)\delta(r-r_c)\sin\varphi_r e^{-j(\lambda_x x + \lambda_y y)} dx dy \\ &= \mu_0 I \delta(z-h)\int_0^{\infty} \delta(r-r_c) r dr \int_0^{2\pi} \sin\varphi_r e^{-j\lambda r\cos(\varphi_\lambda - \varphi_r)} d\varphi_r, \end{aligned} \tag{16}$$

where

$$\lambda = \sqrt{\lambda_x^2 + \lambda_y^2}, \ \cos\varphi_\lambda = \frac{\lambda_x}{\lambda}, \ \sin\varphi_\lambda = \frac{\lambda_y}{\lambda}.$$

In order to compute the integral in (16) we use the formula (see [17]):

$$e^{j\xi\cos\varphi} = J_0(\xi) + 2\sum_{k=1}^{\infty} j^k J_k(\xi)\cos k\varphi, \tag{17}$$

where $J_k(\xi)$ is the Bessel's function of the first kind of order k. Substituting (17) into (16) we obtain

$$\begin{aligned} RHS &= \mu_0 I \delta(z-h)\int_0^{\infty} \delta(r-r_c) r dr \int_0^{2\pi} \sin\varphi_r [J_0(\xi) \\ &+ 2\sum_{k=1}^{\infty} j^k J_k(\xi)]\cos(\varphi_\lambda - \varphi_r) d\varphi_r \end{aligned} \tag{18}$$

The integral with respect to φ_r in (18) is computed using the formula

$$\int_0^{2\pi} \sin\varphi_r \cos k\varphi_r d\varphi_r = 0, \tag{19}$$

$$\int_0^{2\pi} \sin\varphi_r \sin k\varphi_r d\varphi_r = \begin{cases} 2\pi, & \text{if } k = 1 \\ 0, & \text{if } k \neq 1 \end{cases} \tag{20}$$

Hence,

$$\begin{aligned} RHS &= -2j\mu_0 \pi I \sin\varphi_\lambda \delta(z-h)\int_0^{\infty} J_1(\lambda r)\delta(r-r_c) r dr \\ &= -2j\mu_0 \pi I \sin\varphi_\lambda J_1(\lambda r_c) r_c \delta(z-h). \end{aligned} \tag{21}$$

Applying Fourier transforms (14) and (15) to (8) and using (21) we obtain

$$\frac{d^2\tilde{\tilde{A}}_{x0}}{dz^2} - \lambda^2 \tilde{\tilde{A}}_{x0} = \gamma\delta(z-h), \tag{22}$$

where $\gamma = -2j\mu_0 \pi I \sin\varphi_\lambda r_c J_1(\lambda r_c)$.

Similarly, applying transforms (14) and (15) to (9) and (10) we obtain

$$\frac{d^2\tilde{\tilde{A}}_{x1}}{dz^2} - \beta\frac{d\tilde{\tilde{A}}_{x1}}{dz} - q_1^2 \tilde{\tilde{A}}_{x1} = 0, \tag{23}$$

$$\frac{d^2\tilde{\tilde{A}}_{x2}}{dz^2} - q_2^2 \tilde{\tilde{A}}_{x2} = 0, \tag{24}$$

where

$$\begin{aligned} q_1^2 &= \lambda^2 - k_1^2 + j\lambda\cos\varphi_\lambda \tilde{V}, \\ q_2^2 &= \lambda^2 - k_2^2, \\ k_1^2 &= -j\omega\mu_0\mu_m\sigma_m e^{(\alpha+\beta)z}, \\ k_2^2 &= -j\omega\mu_0\mu_2\sigma_2, \\ \tilde{V} &= \mu_0\mu_m\sigma_m e^{(\alpha+\beta)z} V. \end{aligned} \tag{25}$$

The boundary conditions are

$$\tilde{\tilde{A}}_{x0}|_{z=0} = \tilde{\tilde{A}}_{x1}|_{z=0}, \quad \frac{d\tilde{\tilde{A}}_{x0}}{dz}|_{z=0} = \frac{1}{\mu_m}\frac{d\tilde{\tilde{A}}_{x1}}{dz}|_{z=0}, \tag{26}$$

$$\tilde{\tilde{A}}_{x1}|_{z=-d} = \tilde{\tilde{A}}_{x1}|_{z=-d}, \quad \frac{1}{\tilde{\mu}_1}\frac{d\tilde{\tilde{A}}_{x1}}{dz}|_{z=-d} = \frac{1}{\mu_2}\frac{d\tilde{\tilde{A}}_{x2}}{dz}|_{z=-d}.. \tag{27}$$

In order to solve equation (22) we consider the following two sub-regions of R_0 : $0 < z < h$ and $z > h$. The solutions in these regions are denoted by $\tilde{\tilde{A}}_{x00}$ and $\tilde{\tilde{A}}_{x01}$, respectively. Hence,

$$\frac{d^2 \tilde{\tilde{A}}_{x00}}{dz^2} - \lambda^2 \tilde{\tilde{A}}_{x00} = 0, \qquad 0 < z < h, \tag{28}$$

$$\frac{d^2 \tilde{\tilde{A}}_{x01}}{dz^2} - \lambda^2 \tilde{\tilde{A}}_{x01} = 0, \qquad z > h, \tag{29}$$

The general solution to (28) can be written in the form

$$\tilde{\tilde{A}}_{x00} = C_1 e^{\lambda z} + C_2 e^{-\lambda z}. \tag{30}$$

The bounded solution to (29) is

$$\tilde{\tilde{A}}_{x01} = C_3 e^{-\lambda z}. \tag{31}$$

The functions $\tilde{\tilde{A}}_{x00}(\lambda_x, \lambda_y, z)$ and $\tilde{\tilde{A}}_{x01}(\lambda_x, \lambda_y, z)$ satisfy the following conditions at $z = h$:

$$\tilde{\tilde{A}}_{x00}\,|_{z=h} = \tilde{\tilde{A}}_{x01}\,|_{z=h}, \qquad \frac{d\tilde{\tilde{A}}_{x01}}{dz}\Big|_{z=h} - \frac{d\tilde{\tilde{A}}_{x00}}{dz}\Big|_{z=h} = \gamma. \tag{32}$$

The first condition in (32) represents continuity of the function $\tilde{\tilde{A}}_{x0}(\lambda_x, \lambda_y, z)$ at $z = h$. The second condition in (32) is obtained by integrating (22) with respect to z from $z = h - \varepsilon$ to $z = h + \varepsilon$ and considering the limit in the resulting expression as $\varepsilon \to +0$.

It follows from (30)-(32) that

$$C_1 = -\frac{\gamma}{2\lambda} e^{-\lambda h}, \quad C_3 = C_2 - \frac{\gamma}{2\lambda} e^{\lambda h}. \tag{33}$$

Using (30), (31) (33) we obtain

$$\begin{cases} \tilde{\tilde{A}}_{x00} = C_2 e^{-\lambda z} - \dfrac{\gamma}{2\lambda} e^{-\lambda(h-z)} \\ \tilde{\tilde{A}}_{x01} = C_2 e^{-\lambda z} - \dfrac{\gamma}{2\lambda} e^{-\lambda(z-h)} \end{cases} \Rightarrow \tilde{\tilde{A}}_{x0} = C_2 e^{-\lambda z} - \frac{\gamma}{2\lambda} e^{-\lambda|h-z|} \tag{34}$$

Solution to (23) can be expressed in terms of Bessel functions (see [18]):

$$\tilde{\tilde{A}}_{x1} = e^{\frac{\beta z}{2}} \left[C_4 J_\nu\left(\tfrac{2\sqrt{b}}{\alpha+\beta} e^{\frac{(\alpha+\beta)}{2} z} \right) + C_5 Y_\nu\left(\tfrac{2\sqrt{b}}{\alpha+\beta} e^{\frac{(\alpha+\beta)}{2} z} \right) \right], \tag{35}$$

where

$$\nu = \frac{\sqrt{\beta^2 + 4\lambda^2}}{\alpha + \beta}, \; b = -j\mu_0 \mu_m \sigma_m (\omega + \lambda \cos\varphi_\lambda V).$$

The bounded solution to (24) is

$$\tilde{\tilde{A}}_{x2} = C_6 e^{q_2 z}. \tag{36}$$

Using boundary conditions (26) and (27) we obtain the system of equations for the unknowns C_2, C_4, C_5 and C_6:

$$C_2 - \frac{\gamma}{2\lambda} e^{-\lambda h} = C_4 J_\nu(z_0) + C_5 Y_\nu(z_0) \tag{37}$$

$$-\lambda C_2 - \frac{\gamma}{2} e^{-\lambda h} = \frac{C_4}{\mu_m}\left(\frac{\beta}{2} J_\nu(z_0) + \sqrt{b} J_\nu^{'}(z_0) \right) + \frac{C_5}{\mu_m}\left(\frac{\beta}{2} Y_\nu(z_0) + \sqrt{b} Y_\nu^{'}(z_0) \right) \tag{38}$$

$$C_6 e^{-q_2 d} = C_4 e^{-\frac{\beta d}{2}} J_\nu\left(z_0 e^{-\frac{(\alpha+\beta)d}{2}} \right) + C_5 e^{-\frac{\beta d}{2}} Y_\nu\left(z_0 e^{-\frac{(\alpha+\beta)d}{2}} \right) \tag{39}$$

$$\frac{C_6}{\mu_2} q_2 e^{-q_2 d} = \frac{e^{\frac{\beta d}{2}}}{\mu_m} \left[C_4 \left(\frac{\beta}{2} J_\nu\left(z_0 e^{-\frac{(\alpha+\beta)d}{2}} \right) + \sqrt{b} e^{-\frac{(\alpha+\beta)d}{2}} J_\nu^{'}\left(z_0 e^{-\frac{(\alpha+\beta)d}{2}} \right) \right) + C_5 \left(\frac{\beta}{2} Y_\nu\left(z_0 e^{-\frac{(\alpha+\beta)d}{2}} \right) + \sqrt{b} e^{-\frac{(\alpha+\beta)d}{2}} Y_\nu^{'}\left(z_0 e^{-\frac{(\alpha+\beta)d}{2}} \right) \right) \right] \tag{40}$$

where $z_0 = \dfrac{2\sqrt{b}}{\alpha + \beta}$.

Solving (37)-(40) we obtain

$$C_5 = \gamma \mu_m e^{-\lambda h} \frac{S_1}{D}, \tag{41}$$

$$C_4 = -\gamma \mu_m e^{-\lambda h} \frac{S_2}{D}, \tag{42}$$

$$C_6 = C_4 e^{-\frac{\beta d}{2} + q_2 d} J_\nu(z_1) + C_5 e^{-\frac{\beta d}{2} + q_2 d} Y_\nu(z_1), \tag{43}$$

$$C_2 = \frac{\gamma}{2\lambda} e^{-\lambda h} \left(1 - \frac{2\lambda \mu_m \left(D_5 D_1 - \mu_2 \sqrt{b}\, e^{\frac{(\beta-\alpha)d}{2}} D_3 \right)}{D_5\left(D_6 D_1 + \sqrt{b} D_2 \right) + \mu_2 \sqrt{b}\, e^{\frac{(\beta-\alpha)d}{2}} \left(D_6 D_3 + \sqrt{b} D_4 \right)} \right) \tag{44}$$

where

$$S_1 = \left(\frac{\beta}{2}\mu_2 e^{\beta d} - \mu_m q_2\right) J_\nu(z_1) + \mu_2 \sqrt{b}\, e^{\frac{(\beta-\alpha)d}{2}} J'_\nu(z_1),$$

$$D = D_5\left(D_6 D_1 + \sqrt{b} D_2\right) + \mu_2 \sqrt{b}\, e^{\frac{(\beta-\alpha)d}{2}} \left(D_6 D_3 + \sqrt{b} D_4\right),$$

$$D_1 = J_\nu(z_0) Y_\nu(z_1) - J_\nu(z_1) Y_\nu(z_0),\ D_5 = \frac{\beta}{2}\mu_2 e^{\beta d} - \mu_m q_2,$$
$$D_2 = J'_\nu(z_0) Y_\nu(z_1) - J_\nu(z_1) Y'_\nu(z_0),\ D_6 = \lambda \mu_m + \frac{\beta}{2}.$$
$$D_3 = J_\nu(z_0) Y'_\nu(z_1) - J'_\nu(z_1) Y_\nu(z_0),$$
$$D_4 = J'_\nu(z_0) Y'_\nu(z_1) - J'_\nu(z_1) Y'_\nu(z_1),$$

$$S_2 = \left(\frac{\beta}{2}\mu_2 e^{\beta d} - \mu_m q_2\right) Y_\nu(z_1) + \mu_2 \sqrt{b}\, e^{\frac{(\beta-\alpha)d}{2}} Y'_\nu(z_1),$$

$$z_1 = z_0 e^{-\frac{(\alpha+\beta)d}{2}}.$$

The induced vector potential in region R_0 is

$$\tilde{\tilde{A}}_{x0}^{ind}(\lambda_x, \lambda_y, z) = C_2 e^{-\lambda z}, \tag{45}$$

where C_2 is given by (44).

Applying the inverse Fourier transform of the form

$$A_{xi}(x,y,z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\tilde{A}}_x(\lambda_x,\lambda_y,z)\exp\left(j(\lambda_x x + \lambda_y y)\right) d\lambda_x\, d\lambda_y \tag{46}$$

to (45) we obtain

$$A_{x0}^{ind}(x,y,z) = -\frac{1}{4\pi} jIr_c \mu_0 \int_0^\infty J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \times \int_0^{2\pi} C \sin\varphi_\lambda [J_0(\lambda r) + 2\sum_{k=1}^\infty j^k J_k(\lambda r)\cos k(\varphi_\lambda - \varphi_r)] d\varphi_\lambda, \tag{47}$$

where

$$C = 1 - \frac{2\lambda\mu_m\left(D_5 D_1 - \mu_2\sqrt{b}\, e^{\frac{(\beta-\alpha)d}{2}} D_3\right)}{D_5\left(D_6 D_1 + \sqrt{b} D_2\right) + \mu_2\sqrt{b}\, e^{\frac{(\beta-\alpha)d}{2}}\left(D_6 D_3 + \sqrt{b} D_4\right)}.$$

Similarly, the $y-$component of the induced vector potential is given by

$$A_{y0}^{ind}(x,y,z) = \frac{1}{4\pi} jIr_c \mu_0 \int_0^\infty J_1(\lambda r_c) e^{-\lambda(z+h)} d\lambda \times \int_0^{2\pi} C \cos\varphi_\lambda [J_0(\lambda r) + 2\sum_{k=1}^\infty j^k J_k(\lambda r)\cos k(\varphi_\lambda - \varphi_r)] d\varphi_\lambda. \tag{48}$$

The induced change in impedance of the coil is given by the formula

$$Z^{ind} = \frac{j\omega}{I}\oint_L \vec{A}_0^{ind}(x,y,z) d\vec{l}, \tag{49}$$

where L is the contour of the coil, and

$$\vec{A}_0^{ind}(x,y,z) = A_{x0}^{ind}(x,y,z)\vec{e}_x + A_{y0}^{ind}(x,y,z)\vec{e}_y. \tag{50}$$

Substituting (47), (48) and (50) into (49) we obtain

$$Z^{ind} = -\frac{j}{2}\omega r_c^2 \mu_0 \int_0^\infty J_1^2(\lambda r_c) e^{-2\lambda h} d\lambda \int_0^{2\pi} C d\varphi_\lambda. \tag{51}$$

Formula (51) can be rewritten in the form

$$Z^{ind} = \omega r_c \mu_0 Z,$$

where

$$Z = -\frac{j}{2}\int_0^\infty J_1^2(u) e^{-2u\hat{h}} du \int_0^{2\pi} \hat{C} d\varphi_\lambda \tag{52}$$

and

$$\hat{C} = 1 - \frac{2u\mu_m(\hat{D}_5\hat{D}_1 - \hat{D}_7\hat{D}_3)}{\hat{D}_5(\hat{D}_6\hat{D}_1 + \sqrt{-j\hat{b}}\hat{D}_2) + \hat{D}_7(\hat{D}_6\hat{D}_3 + \sqrt{-j\hat{b}}\hat{D}_4)},$$

$$\hat{D}_1 = J_{\hat{\nu}}(\hat{z}_0) Y_{\hat{\nu}}(\hat{z}_1) - J_{\hat{\nu}}(\hat{z}_1) Y_{\hat{\nu}}(\hat{z}_0),$$

$$\hat{D}_2 = J'_{\hat{\nu}}(\hat{z}_0) Y_{\hat{\nu}}(\hat{z}_1) - J_{\hat{\nu}}(\hat{z}_1) Y'_{\hat{\nu}}(\hat{z}_0),$$

$$\hat{D}_3 = J_{\hat{\nu}}(\hat{z}_0) Y'_{\hat{\nu}}(\hat{z}_1) - J'_{\hat{\nu}}(\hat{z}_1) Y_{\hat{\nu}}(\hat{z}_0),$$

$$\hat{D}_4 = J'_{\hat{\nu}}(\hat{z}_0) Y'_{\hat{\nu}}(\hat{z}_1) - J'_{\hat{\nu}}(\hat{z}_1) Y'_{\hat{\nu}}(\hat{z}_0),$$

$$\hat{D}_5 = \frac{\hat{\beta}}{2}\mu_2 e^{\hat{\beta}\hat{d}} - \mu_m \hat{q}_2, \quad \hat{D}_6 = u\mu_m + \frac{\hat{\beta}}{2},$$

$$\hat{D}_7 = \mu_2 \sqrt{-j\hat{b}} e^{\frac{(\hat{\beta}-\hat{\alpha})\hat{d}}{2}},$$

$$\hat{b} = \sqrt{\hat{\delta}_1^2 + \hat{\rho} u \cos\varphi_\lambda}, \quad \hat{\delta}_1 = r_c \sqrt{\omega\mu_0\mu_m\sigma_m},$$

$$\hat{\rho} = r_c^3 \mu_0 \mu_m \sigma_m V, \quad \hat{q}_2 = \sqrt{u^2 + j\hat{\delta}_2^2}, \quad \hat{\delta}_2 = \hat{\delta}_1 \sqrt{\frac{\mu_2\sigma_2}{\mu_m\sigma_m}},$$

$$\hat{\nu} = \frac{\sqrt{\hat{\beta}^2 + 4u^2}}{\hat{\alpha} + \hat{\beta}}, \quad \hat{\alpha} = \alpha r_c, \quad \hat{\beta} = \beta r_c.$$

Formula (52) is used to compute the change in impedance for the following values of the parameters of the problem:
$\hat{\alpha} = 1, \hat{\beta} = 1, \hat{h} = 0.05, \hat{d} = 0.05, \mu_m = 5, \mu_2 = 1,$

$$\kappa = \sqrt{\frac{\mu_2\sigma_2}{\mu_m\sigma_m}} = 1.5.$$

The results are shown in Fig. 2.

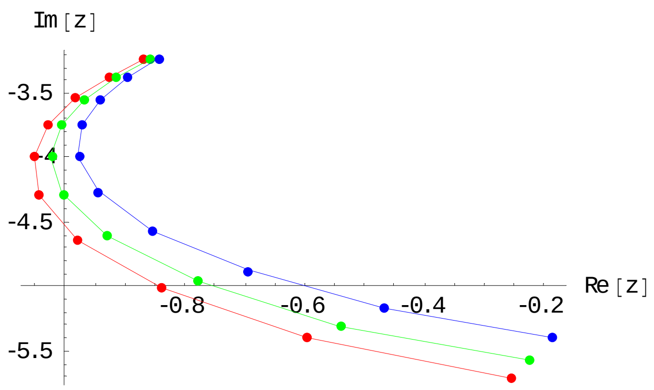

Fig. 2. The change in impedance computed by formula (52) for three values of $\hat{\rho} = 5,10$ and 15 (from bottom to top). The points on each graph correspond to different values of $\hat{\delta}_1 = 1,2,...,10$ (from right to left).

Note that the solution for the problem of a moving plate can be obtained from (51) in the case $\sigma_2 = 0$.

III. Analytical Solution for the Case of a Moving Half-Space

Consider a particular case of the problem solved in the previous section. Suppose that a single-turn coil with radius r_c is located at distance h above a conducting half-space moving in the $x-$ direction with constant velocity V (see Fig. 3).

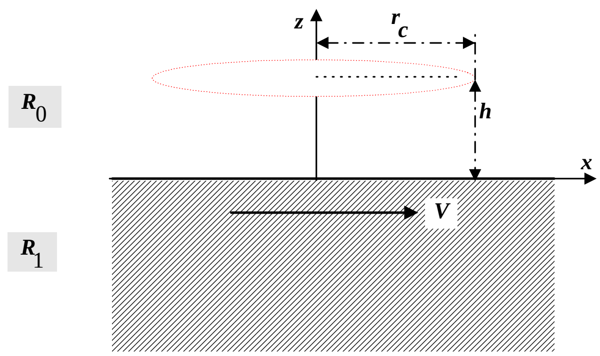

Fig. 3. A single-turn coil above a moving half-space.

The electrical conductivity σ_1 and magnetic permeability μ_1 of region R_1 are given by (1). The $x-$component of the vector potential satisfies the system of equations (8) and (9) in regions R_0 and R_1, respectively, with boundary conditions (11). In addition, conditions (13) for $i = 0,1$ are imposed at infinity. The problem is solved by the method of Fourier transforms (14) and (15). Solution in region R_0 in the transformed space is given by (34). Bounded solution to (23) in the transformed space is

$$\tilde{\tilde{A}}_{x1} = e^{\frac{\beta z}{2}} C_4 J_\nu \left(\frac{2\sqrt{b}}{\alpha+\beta} e^{\frac{(\alpha+\beta)}{2}z} \right), \tag{52}$$

where $\nu = \dfrac{\sqrt{\beta^2 + 4\lambda^2}}{\alpha + \beta}$.

Arbitrary constants C_2 and C_4 in (34) and (52) are obtained from (26). In particular, C_2 has the form

$$C_2 = \frac{\gamma}{2\lambda} e^{-\lambda h} \frac{\left(\frac{\beta}{2} - \lambda\mu_m\right) J_\nu(z_0) + \sqrt{b} J_\nu'(z_0)}{\left(\frac{\beta}{2} + \lambda\mu_m\right) J_\nu(z_0) + \sqrt{b} J_\nu'(z_0)}, \tag{53}$$

where $z_0 = \dfrac{2\sqrt{b}}{\alpha + \beta}$.

The $x-$component of the induced vector potential has the form (45), where C_2 is given by (53). Applying the inverse Fourier transform of the form (46) to (45) we obtain

$$A_{x0}^{ind}(x,y,z) = -\frac{1}{4\pi} jIr_c\mu_0 \int_0^\infty J_1(\lambda r_c)e^{-\lambda(z+h)}d\lambda \times \int_0^{2\pi} D\sin\varphi_\lambda \begin{pmatrix} J_0(\lambda r) \\ +2\sum_{k=1}^\infty j^k J_k(\lambda r)\cos k(\varphi_\lambda - \varphi_r) \end{pmatrix} d\varphi_\lambda, \tag{54}$$

where

$$D = \frac{\left(\frac{\beta}{2} - \lambda\mu_m\right)J_\nu(z_0) + \sqrt{b}J_\nu^{'}(z_0)}{\left(\frac{\beta}{2} + \lambda\mu_m\right)J_\nu(z_0) + \sqrt{b}J_\nu^{'}(z_0)}.$$

Similarly, the induced component of the vector potential in the y – direction is

$$A_{y0}^{ind}(x,y,z) = \frac{1}{4\pi} jIr_c\mu_0 \int_0^\infty J_1(\lambda r_c)e^{-\lambda(z+h)}d\lambda \times \int_0^{2\pi} D\cos\varphi_\lambda \begin{pmatrix} J_0(\lambda r) \\ +2\sum_{k=1}^\infty j^k J_k(\lambda r)\cos k(\varphi_\lambda - \varphi_r) \end{pmatrix} d\varphi_\lambda \tag{55}$$

Substituting (54), (55) and (50) into (49) we obtain the induced change in impedance

$$Z^{ind} = -\frac{j}{2}\omega r_c^2\mu_0 \int_0^\infty J_1^2(\lambda r_c)e^{-2\lambda h}d\lambda \times \int_0^{2\pi} \frac{\left(\frac{\beta}{2} - \lambda\mu_m\right)J_\nu(z_0) + \sqrt{b}J_\nu^{'}(z_0)}{\left(\frac{\beta}{2} + \lambda\mu_m\right)J_\nu(z_0) + \sqrt{b}J_\nu^{'}(z_0)} d\varphi_\lambda \tag{56}$$

Formula (56) can be rewritten in the form

$$Z^{ind} = \omega r_c \mu_0 Z, \tag{57}$$

where

$$Z = -\frac{j}{2}\int_0^\infty J_1^2(u)e^{-2u\hat{h}}du \times \int_0^{2\pi} \frac{\left(\frac{\hat{\beta}}{2} - u\mu_m\right)J_{\hat{\nu}}(\hat{z}_0) + \sqrt{-j\hat{b}}J_{\hat{\nu}}^{'}(\hat{z}_0)}{\left(\frac{\hat{\beta}}{2} + u\mu_m\right)J_{\hat{\nu}}(\hat{z}_0) + \sqrt{-j\hat{b}}J_{\hat{\nu}}^{'}(\hat{z}_0)} d\varphi_\lambda \tag{58}$$

and the dimensionless parameters are defined as follows

$$\hat{\nu} = \frac{\sqrt{\hat{\beta}^2 + 4u^2}}{\hat{\alpha} + \hat{\beta}}, \ \hat{b} = \hat{\delta}_1^2 + \hat{\rho}u\cos\varphi_\lambda,$$

$$\hat{\delta}_1 = r_c\sqrt{\omega\mu_0\mu_m\sigma_m}, \ \hat{\rho} = r_c\mu_0\mu_m\sigma_m V.$$

Formula (58) is used to compute the change in impedance of a single-turn coil due to the presence of a conducting half-space for the following values of the parameters of the problem: $\hat{\alpha} = 0$, $\hat{\beta} = 2$, $\hat{h} = 0.05$, $\mu_m = 5$. The results of the calculations are shown in Fig. 4.

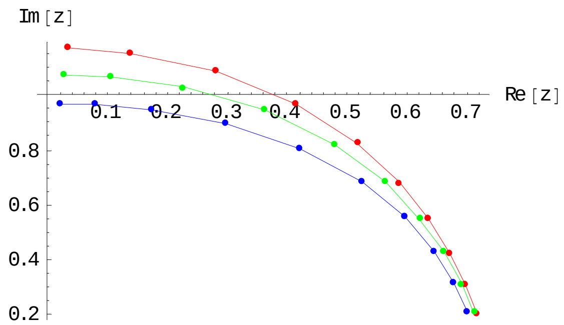

Fig. 4. The change in impedance computed by formula (58) for three values of $\hat{\rho} = 5, 10$ and 15 (from top to bottom). The points on each graph correspond to different values of $\hat{\delta}_1 = 1, 2, ..., 10$ (from left to right).

IV. Conclusions

Method of Fourier integral transform in the x and y directions is used in the present paper in order to construct analytical solution of eddy current problem where a coil with alternating current is located above a conducting two-layer medium. The upper layer is moving in the x direction with constant velocity V. The electrical conductivity and magnetic permeability of the upper layer are exponential functions of the vertical coordinate. The lower half-space has constant properties and is fixed. The change in impedance of the coil is obtained in the form of a double integral containing Bessel functions. In addition, the change in impedance of a coil located above a moving half-space with varying properties is also computed in closed form. Results of numerical computations are presented.

The solution can be generalized for the case where an upper layer of a multilayer conducting medium is moving in the horizontal direction.

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