# On Attempt of Stability Investigation of a Cylindrical Couette Flow with Suction/Injection in Radial and Azymuthal Magnetic Fields

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*Abstract*— **The present paper is dedicated to an attempt to find conditions of stability of a flow between two rotating infinite porous cylinders in crossed magnetic fields. A system of differential equation for small perturbations is obtained. An attempt to find combinations of parameters leading to stable flow is made in axisymmetric case.**

***Keywords—stability, Couette flow, magnetic field, magnetohydrodynamics.***

## I. Introduction

The present paper is dedicated to an investigation of stability of the flow of conducting fluid between two infinite rotating cylinders. The same fluid is pushed through the walls, and also radial and azymuthal magnetic fields are applied to the system. In this case the flow could become unstable and turbulent, so it is necessary to know conditions of instability and possibilities to control the flow. The great analysis of a class of similar problems is made in [4], but in this case there is no axial flow and axial magnetic field, and both cylinders are porous.

## II. Formulation of the Problem

The scheme of the flow is shown on the Fig. 1.

An incompressible conducting fluid flows between two co-axial infinite porous rotating cylinders. The flow has two components – the radial and the azymuthal (circular). This flow is under interaction of two magnetic fields – the radial and the azymutal.

The problem can be solved analytically in self-similar formulation as it was made in [3]. Shortly the equations and conditions are shown below.

Let $\omega_1$, $\omega_2$ are angular velocities and $r_1$, $r_2$ – radii of the inner and outer cylinders respectively, $\lambda$ corresponds to intensity of suction/injection, $B_r$ and $B_\varphi$ are respectively the intensity of radial and azymuthal magnetic fields.

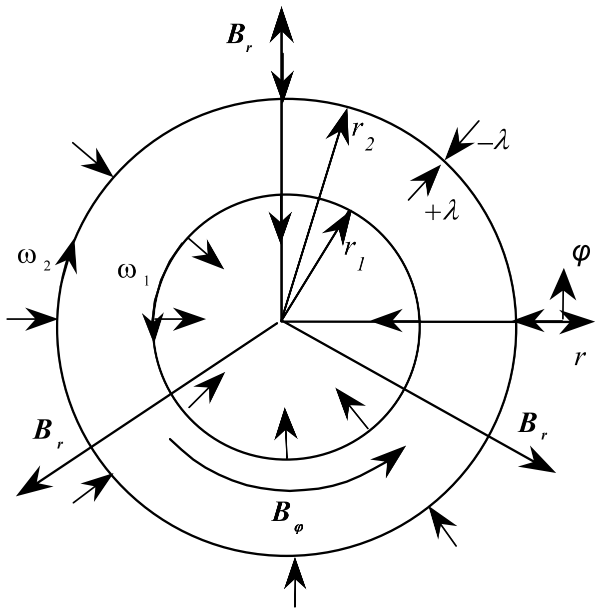


Fig. 1. The scheme of a cross-section of flow.

The stream functions for velocity and magnetic fields in polar co-ordinates are:

$$\psi = K\,R(r')+C\,\varphi,\ \psi_{20} = A\,\varphi+B\,\ln r', \tag{1}$$

which are connected with velocity and magnetic field components:

$$V_r = 1/r \cdot \partial\psi/\partial\varphi = C/(r_2\, r'),\ V_\varphi = -\,\partial\psi/\partial r = -\,K\,R'(r')/r_2, \tag{2}$$

$$B_r = 1/r \cdot \partial\psi_{20}/\partial\varphi = A/\,r',\ B_\varphi = -\,\partial\psi_{20}/\partial r = B/r', \tag{3}$$

$$A = r_2\,B_0,\ B = r_2\,B_1,\ C = (\pm)\lambda,\ K = \nu,\ r'=r/r_2. \tag{4}$$

Here $\nu$ is kinematic viscosity of a fluid, $B_0$ and $B_1$ are intensities of radial and azymuthal magnetic field respectively, $r'$ is self-similar variable (scaled polar radius).

The self-similar equations and boundary conditions are (the ′ symbol for variable is omitted for simpler notation):

$$r^4R^{IV}+2r^3R'''- r^2R''+rR' - (\pm\lambda)(2r^3R'''+r^2R''- rR') -$$
$$- 2(\pm\lambda)(\pm Ha_1^{\,2})+Ha^2(r^2R''- rR')=0, \quad (5)$$

$$R'(r_1')= -r_2^{\,2}\omega_1 r_1'/\nu = -\mathrm{Re}_1\, r_1',$$
$$R'(1)=-r_2^{\,2}\omega_1 r_1'/\nu = - \mathrm{Re}_2, \quad (6)$$

where $Ha^2 = \sigma B_0^{\,2}r_2^{\,2}/(\rho\nu)$ and $Ha_1^{\,2}=\sigma B_0B_1r_2^{\,2}/(\rho\nu)$ are the Hartmann numbers constructed on radial and mixed magnetic fields respectively, σ is an electrical conductivity and ρ is a density of a fluid, $\mathrm{Re}_1$ and $\mathrm{Re}_2$ are the Reynolds numbers constructed on angular velocities of inner and outer cylinders respectively, $r_1'= r_1/r_2$ is the dimensionless radius of the inner cylinder .

Additionally, the condition of velocity flux conservation in azymutal direction is used:

$$R(1) - R(r_1' )= - (Re_1\, r_1'+Re_2)(1-r_1')/2. \quad (7)$$

To find the azymutal velocity is enough to solve this system with respect to $R'(r)$. The exact self-similar solution is shown in [1], and not shown here because of huge expressions for integration constants.

## III. Stability Problem

To investigate the stability of this flow the linear stability theory was used. In this case all velocity components are considered as a sum of the velocity component of a non-perturbed flow and a small perturbation. Additionally, in spite of the fact that the exact solution has no axial velocity, we need to take it into account. Also, in this case it is impossible to exclude pressure from equations as it was made for non-perturbed problem. The magnetic field remains the same. So, the components of flow velocity are taken in the form:

$$V_r = C/(r_2\, r')+D_1\, \Phi(r')\cdot Exp(-st'+in\varphi+ikz'),$$
$$V_\varphi = -K\, R'(r')/r_2+D_2\, f(r')\cdot Exp(-st'+in\varphi+ikz'),$$
$$V_z = D_3\, N(r')\cdot Exp(-st'+in\varphi+ikz') \quad (8)$$

and pressure

$$p(r',\varphi, z') = p_1(r')+r_2(r')+G\, q(r')\cdot Exp(-st'+in\varphi+ikz'), \quad (9)$$

where in addition to coefficient mentioned in the previous paragraph $D_1 = D_2 = D_3 = \nu/r_2$, $G = \rho\nu^2/r_2^{\,2}$, $z' = z/r_2$, $t' = \tau\, t$ (time), $\tau = \nu/r_2^{\,2}$, $n\in N^0$, $k\in N$, $s\in C$.

Here all variables with ′ represent non-dimensional co-ordinates, wave numbers $n$ and $k$ are natural, but parameter $s$ is complex. Perturbation functions $\Phi$, $f$ and $q$ also are complex.

After using standard linearization procedure the system of ordinary differential equations with respect to variable $r'$ is the following (symbol ′ is omitted as in the previous paragraph):

$$-sr^2\Phi+\lambda(r\,\Phi'+\Phi)-inR'\Phi+2rR'f = -r^2q'+r^2\Phi''+r\Phi'+$$
$$+(ik)^2r^2\Phi+\Phi((in)^2-1-2n)-Ha_2^{\,2}\Phi+Ha_1^{\,2}f, \quad (10)$$

$$-sr^2f^{\,2}+\lambda(rf'+f)+r^2R''\Phi-rR'\Phi - inrR'f =$$
$$= -inrq+r^2f''+rf'+f((in)^2+(ik)^2-1)+2in\Phi+Ha_1^{\,2}\Phi-Ha^2f, \quad (11)$$

$$-sr^2N+\lambda rN'- inrR'N= -ikr^2q+r^2N''+rN'+$$
$$+N((in)^2+(ik)^2r^2 - Ha_2^{\,2}-Ha^2), \quad (12)$$

$$\Phi'+\Phi/r+inf/r+ikN = 0, \quad (13)$$

where $Ha_2^{\,2}=\sigma B_1^{\,2}r_2^{\,2}/(\rho\nu)$ – the Hartmann number constructed on azymuthal magnetic field, and $R$ is the solution of non-perturbed problem.

All boundary conditions for perturbations are trivial.

A flow is stable if the real part of parameter $s$ is positive: $\mathrm{Re}\, s \ge 0$, and is convectively unstable if $\mathrm{Re}\, s < 0$. So, the problem is to find such combination of flow parameters that makes the flow stable, and also to find the border between these two regimes.

To solve this problem the method of a pseudo spectral collocation is used. In this case all perturbation functions are represented as finite sums of Chebyshev polynomials $T_i(x)$

$$\begin{Bmatrix} \Phi \\ f \\ N \\ q \end{Bmatrix} = \sum_{i=0}^{m-2} \begin{Bmatrix} a_i \\ b_i \\ c_i \\ d_i \end{Bmatrix} (1-x^2)T_i(x),\ x\in(-1;\ 1) \quad (14)$$

at Chebyshev-Gauss-Lobatto points $x_j$:

$$x_j=cos(j\ \pi/m),\ j=0,1,\dots,m. \quad (15)$$

The relation between dimensionless variable $r'$ and additional variable $x$ is:

$$r' = (1-r_1)x/2+(1+r_1)/2. \quad (16)$$

Using this discretization method, all boundary conditions are fulfilled automatically.

Putting these expressions into the system of differential equations, we obtain system of algebraic equations that could be represented in matrix form and forms a general eigenvalue problem:

$$A\boldsymbol{v} = s\, B\boldsymbol{v} \quad (17)$$

with respect to $s$, where vector $\boldsymbol{v}$ is vector of coefficients in (14). The right-hand matrix has a two-block structure, but the left-hand matrix is of general form:

$$\left(\boxed{\mathbf{A}}\right) \mathbf{V} = s \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \mathbf{V}$$

Solving this eigenvalue problem, we need to find a combination of flow parameters such as suction/injection intensity, magnetic field intensity and angular velocities of cylinders for which real parts of all eigenvalues are non-negative. So, the flow is convectively stable, if Re $s \le 0$, and unstable, if Re $s > 0$. The curve Re $s = 0$ is borderline between these two regions.

This task is very complicated, so in this work we try to find stability only for axisymmetric case: $n = 0$. According to the theory (see [2] and [3]), this case is more unstable.

For the axisymmetric perturbations system of differential equations (8) – (13) takes the form (all functions are used as shown in (8) – (9)):

$$-sr^2\Phi+\lambda(r\Phi'+\Phi)+2rR'f = -r^2q'+r^2\Phi''+r\Phi'+ +(ik)^2r^2\Phi - \Phi - Ha_2^2\Phi+Ha_1^2f, \quad (18)$$

$$-sr^2f^2+\lambda(rf'+f)+r^2R''\Phi - rR'\Phi = =r^2f''+rf'+f((ik)^2-1)+Ha_1^2\Phi - Ha^2f, \quad (19)$$

$$-sr^2N+\lambda rN' = -ikr^2q+r^2N''+rN'+ +N((ik)^2r^2 - Ha_2^2 - Ha^2), \quad (20)$$

$$\Phi'+\Phi/r+ikN = 0. \quad (21)$$

This system looks simpler than (10) – (13). It is discretized in the same way as shown above. But if just insert expressions (14) into equations (18) – (21), the matrix $B$ in eigenvalue problem contains zero rows, because the equation (21) does not contain parameter $s$. To avoid this, the equations (20) and (21) are used to exclude the pressure and the axial perturbations from the system equations, hoping that all necessary boundary conditions for higher derivatives of $\Phi$ and $f$ is fulfilled, because it is not possible to get them analytically.

The eigenvalue problem (14) for axisymmetric case was solved using Wolfram *Mathematica* package, version 8.0. For the present moment some results are obtained, but solving still is in progress, so more concrete results and analysis will be presented on the Conference.

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