Analytical solution for some MHD problems on a flow of conducting liquid in the initial part of a channel in the case of rotational symmetry

Elena Ligere, Ilona Dzenite

Abstract—This paper presents the analytical solution of magnetohydrodynamical (MHD) problems on a developing flow of conducting liquid in the initial part of a channel for the case, when conducting fluid flows into the channel through its wall in the presence of the rotational symmetry in the geometry of the flow. The problems are solved in Stokes and inductionless approximation, and on using integral transforms. The velocity field of the flow is analyzed numerically by means of the obtained solutions.

Keywords— Magnetohydrodynamics, Navier-Stokes equations, analytical solution, channel flow.

I. Introduction

MAGNETOHYDRODYNAMICS (MHD) is a separate discipline combining the classical fluid mechanics and electrodynamics. The flow of a conducting fluid in the external magnetic field produces new effects, absent in the ordinary hydrodynamics, and which arise due to the electromagnetic Lorenz force generated by the interaction of the moving fluid with electromagnetic field. MHD analyzes these phenomena and it also studies a flow of a conducting fluid caused by the current passing through the fluid.

Nowadays MHD effects are widely exploited both in technical devices (e.g., in pumps, flow meters, generators) and industrial processes in metallurgy, material processing, chemical industry, industrial power engineering and nuclear engineering. Channels, in particular narrow and circular channels, are common parts of many MHD devices. Therefore, investigation of MHD phenomena in channels with conducting fluids is quite important.

The motion of conducting fluid in external magnetic field is described by the system of MHD equations, containing Navier-Stokes equation for the motion of incompressible viscous fluid with the additional term corresponding to the Lorentz force (see [3], [4]).

In magnetohydrodinamics the number of exact solutions, obtained analytically, is limited due to the nonlinearity of the Navier-Stokes equation. The exact solutions have been obtained only for very specific problems. Therefore, numerical methods are widely used for solving these problems.

Analytical solutions are mostly obtained for the simplified flow models and on using some approximations. In the present paper the following two approximations are used. These are the Stokes approximation, when the nonlinear term is neglected in the Navier-Stokes equations, and the inductionless approximation, for which the induced currents are taken into account, but the magnetic field created by these currents is neglected.

In this paper two problems on a flow of conducting liquid in the initial part of a channel are considered for the case, when conducting fluid flows into the channel through its wall in the presence of rotational symmetry in the geometry of the flow. These problems are solved analytically by using integral transforms.

The first problem is the problem on an inflow of a conducting fluid in the plane channel through a round hole of finite radius in its lateral side. In the authors' work [5], this problem was considered for the longitudinal magnetic field, but the case of the strong transverse magnetic field was just briefly mentioned. In the present paper, the case of the strong transverse magnetic field is considered in detail and some new numerical results are also presented.

Additionally, the problem on an inflow of a conducting liquid into a circular channel through a split of finite length in its lateral side is briefly considered. This problem was considered in the author's work [2] and its solution was obtained in the form of convergent improper integrals on using Stokes and inductionless approximation. In [2] on obtaining the solution, the Fourier transform was used together with the assumption that the velocity and pressure gradient are equal to zero in channel at the sufficient distance from the entrance region. But this assumption is not correct, since in longitudinal magnetic field in a round channel the Poiseuille flow appears far away from the entrance region. In the present paper the correct way of obtaining the analytical solution of the problem is considered, although it is shown that the final results obtained in [2] are the same and correct.

II. Problem Formulation

A plane channel with conducting fluid is located in region $D=\{0\le \tilde r\le +\infty,\ 0\le\tilde\varphi\le 2\pi,\ -h\le\tilde z\le h\}$, where $\tilde r,\ \tilde\varphi,\ \tilde z$ are cylindrical coordinates. There is a round hole of finite radius

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$\tilde{R}$ in the channel later side, through which a conducting fluid flows into the channel with the constant velocity $V_0\vec{e}_z$ (see Fig.1).

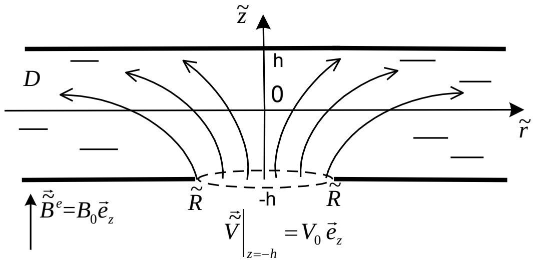

Fig. 1. The geometry of the flow.

The case of transverse magnetic field is considered, i.e., when the external magnetic field $\vec{B}^e = \mathrm{B}_0\vec{e}_z$ is parallel to the $\tilde{z}$ axis. It is also assumed that the channel walls $\tilde{z} = \pm h$ are non-conducting and induced streams do not flow through the hole { $\tilde{z} = -h$, $0 < \tilde{r} < \tilde{R}$ }.

On introducing the dimensionless variables, when the half-width of channel h is used as a length scale, the magnitude of the velocity of fluid in the entrance region V_0 - as a velocity scale, and B_0, V_0B_0 , $\rho\nu V_0/h$ - as scales of magnetic field, electrical field and pressure, respectively, where σ is the conductivity, ρ is the density and ν is the viscosity of the fluid, and on using Stokes and inductionless approximations, the dimensionless MHD equations in cylindrical coordinates take the form

$$\Delta\vec{V} + Ha^2(\vec{E} + \vec{V}\times\vec{e}_B)\times\vec{e}_B = \nabla P, \tag{1}$$

$$div\vec{V} = 0, \tag{2}$$

where $\vec{e}_B$ is the unit vector of external magnetic field,

$\vec{V} = V_r(r,z)\vec{e}_r + V_z(r,z)\vec{e}_z$ is the velocity of the fluid,

$$\Delta\vec{V} = \vec{e}_r(L_0V_r - \frac{V_r}{r^2}) + \vec{e}_z(L_0V_z), \quad L_0 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

P is the pressure, $Ha = B_0h\sqrt{\sigma/\rho\gamma}$ is the Hartmann number, characterizing the ratio of electromagnetic force to viscous one.

Projecting (1) and (2) onto the r and z axes, and taking into account that $\vec{e}_B = \vec{e}_z$ and the intensity of electrical field $\vec{E} = 0$ for this problem (see [3],[4]), the problem takes the form

$$-\frac{\partial P}{\partial r} + L_1V_r - Ha^2V_r = 0, \tag{3}$$

$$-\frac{\partial P}{\partial z} + L_0V_z = 0, \tag{4}$$

$$\frac{\partial V_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\cdot V_r) = 0. \tag{5}$$

with the following boundary conditions

$$z = -1: \quad V_r = 0, \ V_z = \begin{cases} 1, & 0 \le r \le R \\ 0, & r > R \end{cases} \tag{6}$$

$$z = 1: \quad V_r = 0, \quad V_z = 0, \tag{7}$$

$$r \to \pm\infty: \ V_r \to 0, \ \partial P/\partial r \to 0, \tag{8}$$

where $L_1 f = L_0 f - f/r^2$, $R = \tilde{R}/h$.

III. Problem Solving

Due to the rotational symmetry of the problem with respect to r, the Hankel transform (see [1]) is used for the problem solving. The Hankel transform of order 1 with respect to r is applied to the functions V_r and $\partial P/\partial r$, but the Hankel transform of order 0 is applied to the functions V_z and $\partial P/\partial z$:

$$\hat{V}_r(\lambda,z) = \int_0^\infty V_r J_1(\lambda r)r dr, \qquad \hat{V}_z(\lambda,z) = \int_0^\infty V_z J_0(\lambda r)r dr,$$

$$\hat{P}(\lambda,z) = \int_0^\infty P J_0(\lambda r)r dr, \tag{9}$$

where $J_\nu(\lambda r)$ is the Bessel functions of order ν.

On applying the Hankel transform to the system (3)-(5), one gets the system of ordinary differential equations for the Hankel transforms $\hat{V}_r(\lambda,z)$, $\hat{V}_z(\lambda,z)$, $\hat{P}(\lambda,z)$:

$$\lambda\hat{P} - \lambda^2\hat{V}_r + \frac{d^2\hat{V}_r}{dz^2} - Ha^2\hat{V}_r = 0, \tag{10}$$

$$\frac{d\hat{P}}{dz} + \lambda^2\hat{V}_z - \frac{d^2\hat{V}_z}{dz^2} = 0, \tag{11}$$

$$\frac{d\hat{V}_z}{dz} + \lambda\hat{V}_r = 0 \tag{12}$$

with the boundary conditions:

$$z = -1: \quad \hat{V}_r = 0, \quad \hat{V}_z = RJ_1(\lambda R)/\lambda \tag{13}$$

$$z = 1: \quad \hat{V}_r = 0, \ \hat{V}_z = 0. \tag{14}$$

On eliminating $\hat{V}_r$ and $\hat{P}$ from (10)-(12), the following differential equation is obtained for $\hat{V}_z$.

$$\hat{V}_z^{(4)} - \left(2\lambda^2 + Ha^2\right)\cdot\hat{V}_z'' + \lambda^4\cdot\hat{V}_z = 0. \tag{15}$$

The general solution of (15) has the form

$$\hat{V}_z = C_1\sinh k_1z + C_2\sinh k_2z + C_3\cosh k_1z + C_4\cosh k_2z, \tag{16}$$

where $k_1 = \mu + \sqrt{\mu^2 + \lambda^2}$, $k_2 = \mu - \sqrt{\mu^2 + \lambda^2}$, (17)

$\mu = Ha/2$, C_1 - C_4 are arbitrary constants.

In order to reduce the number of the constants C_1-C_4 in (16) and simplify the problem solving, the problem is divided into two sub-problems: an odd and even problem with respect to z, on considering a plane channel with two holes in its lateral sides $\tilde{z} = \pm h$ in the region $0 < \tilde{r} < \tilde{R}$.

In the **odd problem** with respect to z the fluid with velocities $\mp(V_0\vec{e}_z)/2$ flows into the channel through the both holes at $\tilde{z} = \pm h$. The geometry of the flow for this problem is presented in Fig. 2.

In the **even problem** with respect to z the fluid with velocity $(V_0\vec{e}_z)/2$ flows into the channel through the hole at $\tilde{z} = -h$ and flows out with the same velocity through the hole at $\tilde{z} = h$. The geometry of the flow is presented in Fig. 3.

Then the solution of the **general problem** is equal to the sum of solutions of the odd end even problems.

A. Solution of the Odd Problem

The odd problem with respect to z is the problem on an inflow of fluid into the channel through both holes at $\tilde{z} = \pm h$.

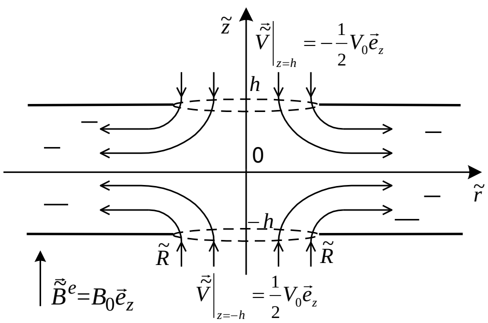

Fig. 2. The odd problem with respect to z.

The dimensionless boundary conditions for the problem are

$$z = \pm 1: \ V_r = 0, \quad V_z = \begin{cases} \mp 1/2, & 0 \le r \le R \\ 0, & r > R \end{cases} \tag{18}$$

$$r \to \pm\infty: \quad V_x \to 0, \quad \partial P/\partial x \to 0. \tag{19}$$

On applying the Hankel transforms (9) to the boundary conditions (18)-(19), one gets

$$z = \pm 1: \quad \hat{V}_r = 0, \quad \hat{V}_z = \mp R \cdot J_1(\lambda R)/(2\lambda) \tag{20}$$

For the odd problem, $\hat{V}_z$ is the odd function with respect to z and, therefore, $C_3 = C_4 = 0$ in (16), i.e.

$$\hat{V}_z(\lambda, z) = C_1 \sinh k_1 z + C_2 \sinh k_2 z \tag{21}$$

In order to determine C_1 and C_2, the boundary condition (20) are used together with the additional boundary condition obtained from (12), i.e.,

$$z = 1: \quad \hat{V}_z = -R \cdot J_1(\lambda R)/(2\lambda) \quad \text{and} \quad d\hat{V}_z/dz = 0. \tag{22}$$

As a result, one obtains

$$\hat{V}_z = \frac{k_1 \cosh k_1 \sinh k_2 z - k_2 \cosh k_2 \sinh k_1 z}{\Delta_1} \cdot \frac{\psi}{\lambda}, \tag{23}$$

where k_1, k_2 are given by (17),

$$\Delta_1 = k_2 \cosh k_2 \cdot \sinh k_1 - k_1 \cosh k_1 \cdot \sinh k_2. \tag{24}$$

$$\psi = \psi(\lambda) = RJ_1(\lambda R)/2 \tag{25}$$

The function $\hat{V}_r$ is determined from (12):

$$\hat{V}_r = \frac{\cosh k_1 \cosh k_2 z - \cosh k_2 \cosh k_1 z}{\Delta_1} \cdot \psi \tag{26}$$

Functions $\lambda\hat{P}$ and $d\hat{P}/dz$ are determined from (10)-(11):

$$\lambda\hat{P} = Ha^2 \cdot \frac{k_2 \cosh k_2 \cosh k_1 z - k_1 \cosh k_1 \cosh k_2 z}{\Delta_1} \cdot \psi \tag{27}$$

$$\frac{d\hat{P}}{dz} = Ha \cdot \frac{\cosh k_2 \sinh k_1 z - \cosh k_1 \sinh k_2 z}{\Delta_1} \cdot \lambda\psi \tag{28}$$

Then on using the inverse complex Hankel transform, the solution of the problem (3)–(5) with boundary conditions (18)-(19) is obtained in the form of the convergent improper integrals:

$$V_r = \int_0^\infty \frac{\cosh k_1 \cosh k_2 z - \cosh k_2 \cosh k_1 z}{\Delta_1} \psi \lambda J_1(\lambda r) d\lambda, \tag{29}$$

$$V_z = \int_0^\infty \frac{k_1 \cosh k_1 \sinh k_2 z - k_2 \cosh k_2 \sinh k_1 z}{\Delta_1} \psi J_0(\lambda r) d\lambda \tag{30}$$

$$\frac{\partial P}{\partial r} = Ha^2 \int_0^\infty \frac{k_2 \cosh k_2 \cosh k_1 z - k_1 \cosh k_1 \cosh k_2 z}{\Delta_1} \psi \lambda J_1(\lambda r) d\lambda \tag{31}$$

$$\frac{\partial P}{\partial z} = Ha \int_0^\infty \frac{\cosh k_2 \sinh k_1 z - \cosh k_1 \sinh k_2 z}{\Delta_1} \psi \lambda^2 J_0(\lambda r) d\lambda \tag{32}$$

B. Solution of the Even Problem

The geometry of the flow for the even problem with respect to z is shown in Fig. 3.

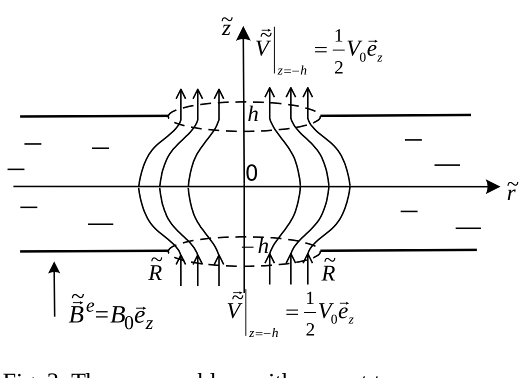

Fig. 3. The even problem with respect to z.

The dimensionless boundary conditions for this problem are

$$z=\pm 1:\ \ V_r=0\,,\qquad V_z=\begin{cases}1/2, & 0\le r\le R\\ 0, & r>R\end{cases} \tag{33}$$

$$r\to\pm\infty:\qquad V_x\to 0\,,\quad \partial P/\partial x\to 0\,. \tag{34}$$

On applying the Hankel transform (9) to boundary conditions (32)-(33), one gets

$$z=\pm 1:\qquad \hat{V}_r=0\,,\quad \hat{V}_z=RJ_1(\lambda R)/(2\lambda) \tag{35}$$

For the even problem, the function $\hat{V}_z$ is even with respect to z, therefore, $C_1=C_2=0$ in (16), i.e.,

$$\hat{V}_z(\lambda,z)=C_3\cosh k_1 z+C_4\cosh k_2 z\,. \tag{36}$$

In order to determine C_3 and C_4, the boundary condition (35) and (12) are used, as a result, one gets

$$\hat{V}_z=\frac{k_2\sinh k_2\cosh k_1 z-k_1\sinh k_1\cosh k_2 z}{\Delta_2}\cdot\psi \tag{37}$$

where k_1, k_2 are given by (17), ψ is given by (25) and

$$\Delta_2=k_2\cosh k_1\cdot\sinh k_2-k_1\cosh k_2\cdot\sinh k_1\,.$$

The function $\hat{V}_r$ is determined from (12):

$$\hat{V}_r=\frac{\sinh k_1\sinh k_2 z-\sinh k_2\sinh k_1 z}{\Delta_2}\cdot\psi \tag{38}$$

The functions $\lambda\hat{P}$ and $d\hat{P}/dz$ are determined from (10)-(11):

$$\lambda\hat{P}=Ha\cdot\frac{k_2\sinh k_2\sinh k_1 z-k_1\sinh k_1\sinh k_2 z}{\Delta_2}\cdot\psi \tag{39}$$

$$\frac{d\hat{P}}{dz}=Ha\cdot\frac{\sinh k_1\cosh k_2 z-\sinh k_2\cosh k_1 z}{\Delta_2}\cdot\lambda\psi \tag{40}$$

On applying the inverse complex Hankel transform to (37)-(40), the solution to the problem (3)–(5) with boundary conditions (33), (34) is obtained in the form of convergent improper integrals:

$$V_r=\int_0^\infty\frac{\sinh k_2\sinh k_1 z-\sinh k_1\sinh k_2 z}{\Delta_2}\psi\lambda_1(\lambda r)d\lambda\,, \tag{41}$$

$$V_z=\int_0^\infty\frac{k_2\sinh k_2\cosh k_1 z-k_1\sinh k_1\cosh k_2 z}{\Delta_2}\psi J_0(\lambda r)d\lambda \tag{42}$$

$$\frac{\partial P}{\partial r}=Ha\int_0^\infty\frac{k_1\sinh k_1\cdot\sinh k_2 z-k_2\sinh k_2\cdot\sinh k_1 z}{\Delta_2}\psi\lambda J_1(\lambda r)d\lambda \tag{43}$$

$$\frac{\partial P}{\partial z}=Ha\int_0^\infty\frac{\sinh k_1\cdot\cosh k_2 z-\sinh k_2\cdot\cosh k_1 z}{\Delta_2}\psi\lambda^2 J_0(\lambda r)d\lambda \tag{44}$$

where k_1, k_2 are given by (17) and ψ by (25) .

IV. Numerical Results

A. *Numerical Results for the Odd Problem*

On the base of obtained solution, the velocity field is studied numerically. Results of calculations of the velocity radial component V_r for the odd problem at the Hartmann numbers Ha=10 and Ha=50 are presented graphically in Fig. 4. The component V_r is an odd function with respect to z. It can be seen from Fig.4 that V_r has the M-shaped profiles only near the entrance hole ($1\le r\le 1.1$ at Ha=10 and $1\le r<1.1$ Ha=50). Even at a small distance from the entrance, the profiles of V_r take the shape peculiar to the Hartmann flow in a plane channel in transverse magnetic field. The magnitude of the velocity is inversely proportional to the distance from the hole.

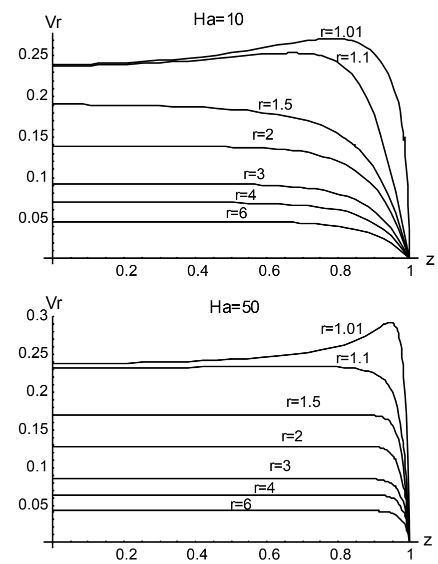

Fig. 4. Profiles of the velocity radial component V_r for the odd problem at R=1.

B. *Numerical Results for the Even Problem*

Fig. 5 presents the results of calculation of V_r by means of formula (41). One can see that V_r differs from zero only near the entrance region. Additionally, in Fig. 5 V_r is positive for some values of r, e.g., for $0<r\le 1$ and Ha=10. However, since the fluid flows out through the hole at $z=1$, the r-component of the velocity must be negative for $0<r<1$ at *Ha=0*. It means that in the transverse magnetic field in the region $0<z<1$ there exists an opposite flow, which occurs due to vortices generated in the channel (see Fig. 6). The vector field of the velocity for Ha=10 is shown in Fig. 6.

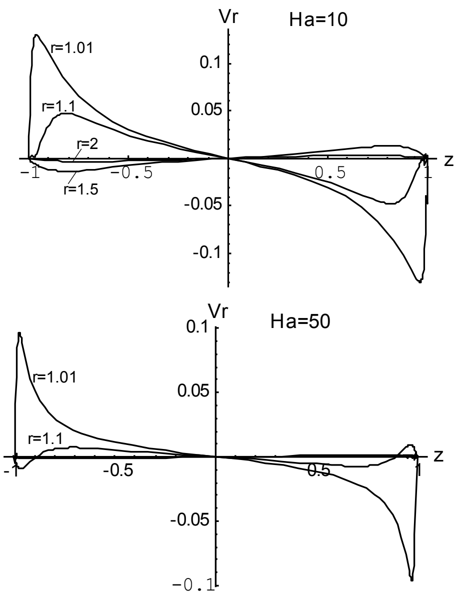

Fig. 5. Profiles of V_r for the even problem at R=1.

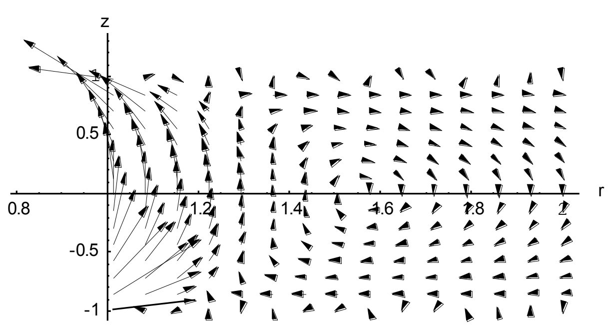

Fig. 6. Velocity field for the even problem at R=1 and Ha=10.

C. Numerical Results for the General Problem

The solution of the general problem is equal to the sum of solutions to the odd and even problems with respect to z. Results of calculation of V_r for the general problem at the Hartmann numbers Ha=10 and Ha=50 are presented in Fig. 7 and Fig. 8.

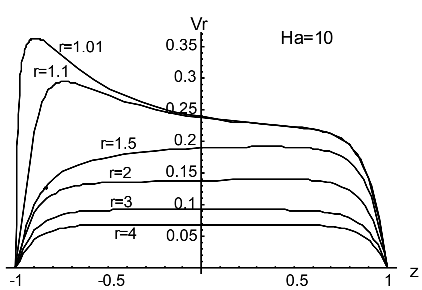

Fig. 7. Profiles of V_r for the general problem at R=1 and Ha=10.

One can see that as in the previous case, the profiles of the velocity component V_r differ from the Hartmann flow profiles only near the entrance region. The magnitude of the velocity is inversely proportional to the distance from the hole.

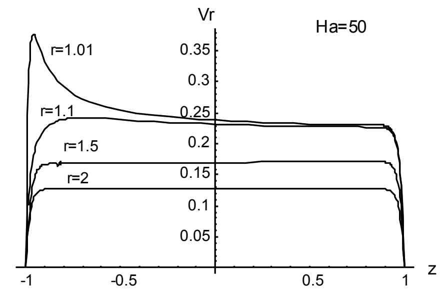

Fig. 8. Profiles of V_r for the general problem at R=1 and Ha=50.

V. Remarks to the Solution of MHD Problem on an Inflow of Conducting Fluid into a Circular Channel through the Channel's Lateral Side

A. Formulation of the Problem

A circular channel is located in the region $D = \{ 0 \le \tilde{r} < R,\ 0 \le \tilde{\varphi} < 2\pi,\ -\infty < \tilde{z} < +\infty \}$. There is a split in the channel lateral surface in the region $\{ \tilde{r} = R,\ -\tilde{d} \le \tilde{z} \le \tilde{d} \}$, through which a conducting fluid flows into the channel with the constant velocity $\tilde{\vec{V}} = -V_0 \vec{e}_r$ (see Fig.9). The case of longitudinal magnetic field is considered, i.e., when the external magnetic field $\vec{B}^e = B_0 \vec{e}_z$ is parallel to the $\tilde{z}$ axis.

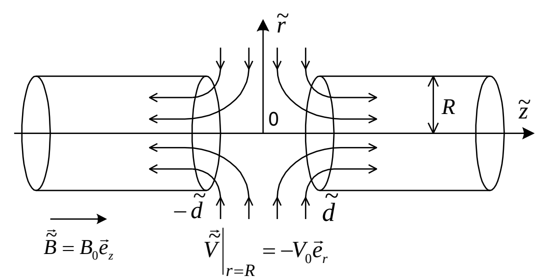

Fig. 9. The geometry of the flow in the circular channel.

It is supposed that walls $\tilde{r} = R$ are nonconducting and induced streams do not flow through the split $\tilde{r} = R$, $-\tilde{d} < \tilde{z} < \tilde{d}$ in the region $R < \tilde{r} < +\infty$. In this problem $\vec{E} = 0$ (see [3]). The dimensionless variables are introduced similarly as it was done for the first problem, but the chanel radius R is used as the length scale.

The problem is described by the system of equations (3)-(5) with the boundary conditions:

$$r = 1:\ V_z = 0, \quad V_r = \begin{cases} -1, & z \in (-d, d) \\ 0, & z \notin (-d, d) \end{cases} \qquad (45)$$

$z \to \pm\infty$: $V_z \to V_\infty(r)\cdot \mathrm{sign}(z)$,

$$\frac{\partial P}{\partial z} \to \frac{\partial P_\infty}{\partial z}\cdot sign(z) \equiv A \cdot sign(z) \ , \tag{46}$$

where $d = \tilde{d}/R$, $A = const.$ The functions $V_\infty(r)$ and dP_∞/dz are the velocity of the flow and the pressure gradient in the channel at the sufficient distance from the entrance region, and which satisfy the following equation (see [3], [4]):

$$-\frac{dP_\infty}{dz} + \frac{1}{r}\frac{d}{dr}(r\cdot\frac{dV_\infty}{dr}) = 0 \tag{47}$$

with the boundary condition: $r = 1$: $V_\infty(r) = 0$.

B. Solution of the Problem.

On solving the problem, the symmetry of the problem with respect to z is used, i.e., the velocity component $V_r(r,z)$ and pressure $P(r,z)$ are even functions with respect to z, but the component $V_z(r,z)$ is the odd function with respect to z. It means that the functions $V_r(r,z)$ and $V_z(r,z)$ satisfy additional boundary conditions:

$$z = 0,\ 0 < r < 1:\quad V_z = 0,\ \partial V_r/\partial z = 0. \tag{48}$$

The problem can be solved by the Fourier cosine and Fourier sine transforms, but since V_z and $\partial P/\partial z$ do not tends to zero at $z \to \pm\infty$, the new functions for the velocity and pressure gradient are to be introduced before using these transforms:

$$\vec{V}^{new} = \vec{V} - \frac{2}{\pi}\arctan(z)\cdot V_\infty(r)\cdot \vec{e}_z \tag{49}$$

$$\frac{\partial P^{new}}{\partial z} = \frac{\partial P}{\partial z} - \frac{2}{\pi}\arctan(z)\cdot A \tag{50}$$

As a result, the problem has the form:

$$-\frac{\partial P^{new}}{\partial r} + (L_1 - Ha^2)V_r = 0 \ , \tag{51}$$

$$-\frac{\partial P^{new}}{\partial z} + L_0 V_z^{new} - \frac{2}{\pi}V_\infty(r)\cdot\frac{2z}{\left(1+z^2\right)^2} = 0 \ , \tag{52}$$

$$\frac{\partial V_z^{new}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{2}{\pi}V_\infty(r)\cdot\frac{1}{1+z^2} = 0 . \tag{53}$$

Boundary conditions are

$$r = 1:\ V_z^{new} = 0 \ ,\quad V_r = \begin{cases} -1, & z \in (-d, d) \\ 0, & z \notin (-d, d) \end{cases} \tag{54}$$

$$z \to \pm\infty:\quad V_z^{new} \to 0 \ ,\quad \partial P^{new}/\partial z \to 0 . \tag{55}$$

The Fourier cosine transform with respect to z is applied to (51), (53) and to V_r in boundary conditions (54), but the Fourier sine transform is applied to (52) and to V_z in boundary conditions (54):

$$V_r^c(r,\lambda) = F^c[V_r(r,z)] = \sqrt{\frac{2}{\pi}}\int_0^\infty V_r(r,z)\cos\lambda z\, dz \ ,$$

$$V_z^s(r,\lambda) = F^s\left[V_z^{new}(r,z)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty V_z^{new}(r,z)\sin\lambda z\, dz \ ,$$

$$P^c(r,\lambda) = F^c\left[P^{new}(r,z)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty P^{new}(r,z)\cos\lambda z\, dz \ .$$

It is also used that

$$F^s\left[\frac{2z}{\left(1+z^2\right)^2}\right] = \lambda F^c\left[\frac{1}{1+z^2}\right]. \tag{56}$$

As a result, the following system of ordinary differential equations for the unknown functions V_r^c, V_z^s, P^c is obtained:

$$-\frac{dP^c}{dr} + (L_{1r} - Ha^2)V_r^c = 0 \ , \tag{57}$$

$$\lambda P^c + L_{0r}V_z^s - \frac{2}{\pi}V_\infty(r)\cdot\lambda\cdot F^c(\lambda) = 0 \ , \tag{58}$$

$$\lambda V_z^s + \frac{1}{r}\frac{d}{dr}\left(rV_r^c\right) + \frac{2}{\pi}V_\infty(r)\cdot F^c(\lambda) = 0 \ , \tag{59}$$

were $L_{0r} = \dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} - \lambda^2$, $L_{1r} = L_{0r} - \dfrac{1}{r^2}$,

un $F^c(\lambda) = F^c\left[\dfrac{1}{1+z^2}\right]$.

The boundary conditions are

$$\text{r=1:}\quad V_r^c(r,\lambda) = -\sqrt{\frac{2}{\pi}}\frac{\sin\lambda L}{\lambda} \ ,\qquad V_z^s = 0 \tag{60}$$

On eliminating the functions V_z^s, P^c from system (57)-(59), the equation for V_r^c is obtained in the form

$$\frac{d}{dr}(L_r\tilde{L}_{0r})V_r^c - 2\lambda^2\tilde{L}_{1r}V_r^c + \lambda^2(\lambda^2 + Ha^2)V_r^c = 0. \tag{61}$$

where $\tilde{L}_{0r} = \dfrac{1}{r} + \dfrac{d}{dr}$, $L_r = \dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr}$, $\tilde{L}_{1r} = L_r - \dfrac{1}{r^2}$.

Differential equation (61) completely coincides with differential equation for V_r^c obtained in [2], therefore, the solution of this equation is the same as in [2], i.e.,

$$V_r^c(r\lambda) = c_1(\lambda)I_1(k_1 r) + c_2(\lambda)I_1(k_2 r) \tag{62}$$

where

$$c_1(\lambda) = A(\lambda)k_2 I_0(k_2)/\Delta,\quad c_2(\lambda) = -A(\lambda)k_1 I_0(k_1)/\Delta \ , \tag{63}$$

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$$k_1 = \sqrt{\lambda^2 + iHa\lambda}, \quad k_2 = \sqrt{\lambda^2 - iHa\lambda}\,, \tag{64}$$

$$A(\lambda) = \sqrt{\frac{2}{\pi}}\frac{\sin\lambda L}{\lambda}, \qquad \Delta = k_1 I_0(k_1) I_1(k_2) - k_2 I_0(k_2) I_1(k_1)\,.$$

The function $\hat{V}_z^{\,s}$ is determined from (59):

$$V_z^s(\mathrm{r},\lambda) = -\frac{1}{\lambda}\left(C_1 k_1 I_0(k_1 r) + C_2 k_2 I_0(k_2 r)\right) - \frac{2}{\pi}\cdot\frac{F^c(\lambda) V_\infty(r)}{\lambda} \tag{65}$$

The functions λP^c and dP^c/dr are determined from (57) and (58) on taking into account the following formulas

$$\widetilde{L}_{1r} I_1(k\cdot r) = k^2 I_1(k\cdot r) \quad \text{un} \quad L_r I_0(k\cdot r) = k^2 I_0(k\cdot r)\,.$$

Then

$$\frac{dP^c}{dr} = \widetilde{c}_1(\lambda) I_1(k_1 r) - \widetilde{c}_2(\lambda) I_1(k_2 r)\,, \tag{66}$$

$$-\lambda P^c = iHa\cdot\left(c_2 k_2 I_0(k_2 r) - c_1 k_1 I_0(k_1 r)\right) - \frac{2}{\pi}\cdot\frac{F^c(\lambda) A}{\lambda}\,. \tag{67}$$

Note, that $V_z^{\,s}$ and $-\lambda P^c$ differ from result obtained in [2] by only last terms. On applying the inverse cosine and sine Fourier transforms to the functions $V_r^{\,c}$, $V_z^{\,s}$, dP^c/dr and $-\lambda P^c$, the solution of the problem is obtained and it has the form of convergent improper integrals, which coincide with the solution of the problem obtained in [2]:

$$V_r = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[c_1(\lambda) I_1(k_1 r) + c_2(\lambda) I_1(k_2 r)\right]\cos\lambda z\, d\lambda\,, \tag{68}$$

$$V_z = -\sqrt{\frac{2}{\pi}}\int_0^\infty \left[c_1(\lambda) k_1 I_0(k_1 r) + c_2(\lambda) k_2 I_0(k_2 r)\right]\frac{\sin\lambda z}{\lambda}\, d\lambda\,, \tag{69}$$

$$\frac{\partial P}{\partial r} = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[\widetilde{c}_1(\lambda) I_1(k_1 r) - \widetilde{c}_2(\lambda) I_1(k_2 r)\right]\cos\lambda z\, d\lambda\,, \tag{70}$$

$$\frac{\partial P}{\partial z} = iHa\sqrt{\frac{2}{\pi}}\int_0^\infty \left[c_2(\lambda) k_2 I_0(k_2 r) - c_1(\lambda) k_1 I_0(k_1 r)\right]\sin\lambda z\, d\lambda\,, \tag{71}$$

where $\widetilde{c}_1(\lambda) = Ha\cdot(i\lambda - Ha)\cdot c_1(\lambda),$

$\widetilde{c}_2(\lambda) = Ha\cdot(i\lambda + Ha)\cdot c_2(\lambda)$

and $c_1(\lambda),\ c_2(\lambda)$ are given by (63).

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