

Combined effect of small curvature and variable friction on temporal instability of shallow mixing layers

Ilmars Iltins, Marija Iltina, and Andrei Kolyshkin

Abstract— Temporal stability analysis of shallow mixing layers is performed in the present paper. It is assumed that flow is slightly curved along the longitudinal coordinate. In addition, friction force is assumed to be varied in the transverse direction. The corresponding linear stability problem is solved by a collocation method. Marginal stability curves are obtained for different values of the parameters of the problem. Combined influence of curvature and variable friction on the stability boundary is investigated. It is shown that both curvature and variable friction stabilize the flow.

Keywords—Linear stability analysis, shallow mixing layers, friction coefficient, curvature

I. INTRODUCTION

SHALLOW mixing layers are often observed in nature and engineering. The understanding of mass and momentum exchange between two layers moving with different velocities is quite important for design and maintenance of compound and composite channels. It is recognized nowadays that there are three basic methods of analysis of shallow mixing layers: experimental investigation, numerical modeling and stability analysis [1]. Linear stability of shallow mixing layers under the rigid-lid assumption is investigated in [2], [3] where it is shown that the bed-friction number (proportional to the friction coefficient) plays an important role in preventing the development of instability. Linear stability is also analyzed in [4] for the case where perturbations are also allowed to develop on a free surface. The validity of the rigid-lid assumption is also assessed in [4]. It is shown that errors in stability characteristics of shallow flows are small provided that the Froude number of the flow is much smaller than unity (this is quite reasonable assumption in many environmental flows). The effect of the Froude number of linear stability is analyzed in [5].

Experimental studies of shallow mixing layers are conducted in [6]-[9]. The major conclusions from [6]-[9] are

I. Iltins is with the Department of Engineering Mathematics, Riga Technical University, Riga, LV 1007, Latvia (e-mail: iltins@inbox.lv).

M. Iltina is with the Department of Engineering Mathematics, Riga Technical University, Riga, LV 1007, Latvia (e-mail: marijai@inbox.lv).

A. Kolyshkin is with the Department of Engineering Mathematics, Riga Technical University, Riga, LV 1007, Latvia (e-mail: andrejs.kolisikins@rtu.lv).

as follows: (a) shallowness of mixing layer prevents the development of three-dimensional instabilities, (b) growth of the width of a shallow mixing layer downstream is reduced in comparison with free shear layer width, (c) growth rate of small perturbations is smaller for larger values of the bed-friction number.

The influence of small curvature on the growth rates of small perturbations developed in a free shear layer is investigated in [10]. It is shown in [10] that the effect of curvature is two-fold: (a) it stabilizes the flow if the mixing layer is stably curved and (b) it has a destabilizing influence for the case of unstably curved mixing layer.

In a typical linear stability problem for shallow mixing layers the friction coefficient is assumed to be constant in the region of the flow. However, the presence of vegetation can considerably increase the resistance in shallow mixing layers. Such a case can be observed in nature during floods. Several experimental papers [11]-[15] are devoted to the analysis of such flows. In addition, linear stability of a flow in partially vegetated areas is analyzed in [15] where the friction coefficient is modeled by a step function.

In the present paper temporal stability of shallow mixing layers is analyzed under the following assumptions: (a) rigid-lid assumption (perturbations do not develop on a free surface of the fluid); (b) the flow is slightly curved in the longitudinal direction; and (c) the friction coefficient varies continuously in the transverse direction of the flow. Marginal stability curves are presented for different values of the parameters of the problem. The combined effect of small curvature and variable friction on the stability characteristics of the flow is also investigated.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

Consider the following form of shallow water equations under the rigid-lid assumption

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} + \frac{c_f(y)}{2h} u \sqrt{u^2 + v^2} = 0, \quad (2)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} - \frac{1}{R} u^2 + \frac{\partial p}{\partial y} + \frac{c_f(y)}{2h} v \sqrt{u^2 + v^2} = 0, \quad (3)$$

where p is the pressure, h is water depth, u and v are the depth-averaged velocity components, $c_f(y)$ is the friction coefficient which is assumed to be dependent on the transverse coordinate y , and R is the dimensionless radius of curvature ($R \gg 1$).

Equation (1) is equivalent to the continuity equation in a two-dimensional hydrodynamics so that it is natural to introduce the stream function $\psi(x, y, t)$ by the relations

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (4)$$

Using (1)-(4) we obtain

$$\begin{aligned} & (\Delta \psi)_t + \psi_y (\Delta \psi)_x - \psi_x (\Delta \psi)_y + \frac{2}{R} \psi_y \psi_{xy} + \\ & + \frac{c_f(y)}{2h} \Delta \psi \sqrt{\psi_x^2 + \psi_y^2} + \frac{c_f(y)}{2h \sqrt{\psi_x^2 + \psi_y^2}} (\psi_y^2 \psi_{yy} + \\ & + 2\psi_x \psi_y \psi_{xy} + \psi_x^2 \psi_{xx}) + \frac{c_{fy}(y)}{2h} \psi_y \sqrt{\psi_x^2 + \psi_y^2} = 0, \end{aligned} \quad (5)$$

where $c_{fy}(y)$ is the derivative of the function $c_f(y)$ with respect to y . The friction coefficient $c_f(y)$ is modeled by the formula

$$c_f(y) = c_{f_0} \gamma(y), \quad (6)$$

where $\gamma(y)$ is a differentiable shape function.

A perturbed solution $\psi(x, y, t)$ is assumed to be of the form

$$\psi(x, y, t) = \psi_0(y) + \varepsilon \psi_1(x, y, t) + \dots \quad (7)$$

where $\psi_0(y)$ is the stream function of the base flow $U(y)$, so that $U(y) = \psi_{0y}(y)$. In the present study the function $U(y)$ is given by

$$U(y) = \frac{1}{2} (1 + \tanh y). \quad (8)$$

Substituting (6) and (7) into (5) and linearizing the resulting equation in the neighborhood of the base flow we obtain

$$\begin{aligned} & \psi_{1xxt} + \psi_{1yyt} + \psi_{0y} (\psi_{1xxx} + \psi_{1yyx}) - \psi_{0yy} \psi_{1x} + \\ & + \frac{c_f(y)}{2h} (\psi_{0y} \psi_{1xx} + 2\psi_{0yy} \psi_{1y} + 2\psi_{0y} \psi_{1yy}) \\ & + \frac{c_{fy}(y)}{h} \psi_{0y} \psi_{1x} + \frac{2}{R} \psi_{0y} \psi_{1xy} = 0. \end{aligned} \quad (9)$$

The perturbed solution is represented in the form of a normal mode

$$\psi_1(x, y, t) = \varphi(y) e^{i\alpha(x-ct)}, \quad (10)$$

where α is the wave number and $c = c_r + ic_i$ is a complex eigenvalue. It follows from (9) and (10) that

$$\begin{aligned} & \varphi_{yy} [\alpha(U-c) - iS U \gamma] - iS (\gamma U_y + \gamma_y U) \varphi_y \\ & + \varphi [\alpha^3 (c-U) - \alpha U_{yy} + i\alpha^2 U S \gamma / 2] = 0, \end{aligned} \quad (11)$$

where $S = \frac{c_{f_0} b}{h}$ is the stability parameter and b is a length scale of the problem.

The boundary conditions have the form

$$\varphi(\pm\infty) = 0. \quad (12)$$

Problem (11), (12) is an eigenvalue problem. The sign of the imaginary part of the complex eigenvalue c is used to decide whether flow (8) is linearly stable or unstable: if all eigenvalues satisfy the inequality $c_i < 0$ then base flow (8) is said to be linearly stable. On the other hand, if at least one eigenvalue has a positive imaginary part ($c_i > 0$) then base flow (8) is said to be linearly unstable. Finally, if one eigenvalue satisfies the condition $c_i = 0$ while all other eigenvalues have negative imaginary parts, base flow is said to be marginally stable. The set of all values of the parameters α and S for which flow (8) is marginally stable determines the marginal stability curve.

III. NUMERICAL METHOD

Numerical solution of (11), (12) is obtained by a collocation method. It is convenient to map the interval $-\infty < y < +\infty$ onto the interval $-1 \leq \xi \leq 1$ using the new

variable $\xi = \frac{2}{\pi} \arctan y$. In terms of the transformed variable the solution to (11) is sought in the form

$$\varphi(\xi) = \sum_{k=0}^{N-1} a_k (1 - \xi^2) T_k(\xi), \quad (13)$$

where $T_k(\xi) = \cos k \arccos \xi$ is the Chebyshev polynomial of the first kind of order k and a_k are unknown coefficients.

The factor $(1 - \xi^2)$ in (13) guarantees that zero boundary conditions at $\xi = \pm 1$ will be satisfied automatically:

$$\varphi(\pm 1) = 0. \quad (14)$$

The collocation points are chosen in the form

$$\xi_m = \cos \frac{\pi m}{N}, \quad m = 1, 2, \dots, N-1. \quad (15)$$

Using (11), (13) and (15) we obtain the following generalized eigenvalue problem

$$(A + \alpha B) a = 0, \quad (16)$$

where A and B are complex-valued matrices and $a = (a_0 a_1 \dots a_{N-1})^T$. Note that both A and B are non-singular (this fact simplifies the solution of the generalized eigenvalue problem (16)).

IV. RESULTS OF COMPUTATIONS

Numerical results are presented in the paper for the case where variability of the friction coefficient in the transverse direction is described by (6), where

$$\gamma(y) = \frac{\beta + 1}{2} + \frac{(\beta - 1)}{2} \tanh y, \tag{17}$$

with $\beta = \frac{c_{f1}}{c_{f0}} \geq 1$. Here c_{f1} and c_{f0} are the friction coefficients in the vegetated area and main channel, respectively. Note that in [16] the friction coefficient varied in such a way that $c_{f0} = 0$ as $y \rightarrow -\infty$.

Marginal stability curves for the case $R = \infty$ and three values of the parameter β , namely, $\beta = 1, 1.5$ and 2 are shown in Fig. 1. Note that $\beta = 1$ corresponds to a constant friction coefficient in the whole region of the flow while values $\beta > 1$ represent the degree of non-uniformity of the friction force in the transverse direction. It is seen from Fig. 1 that the case with non-constant friction is more stable than the case of a uniform friction. In addition, the critical value of S decreases as the parameter β increases.

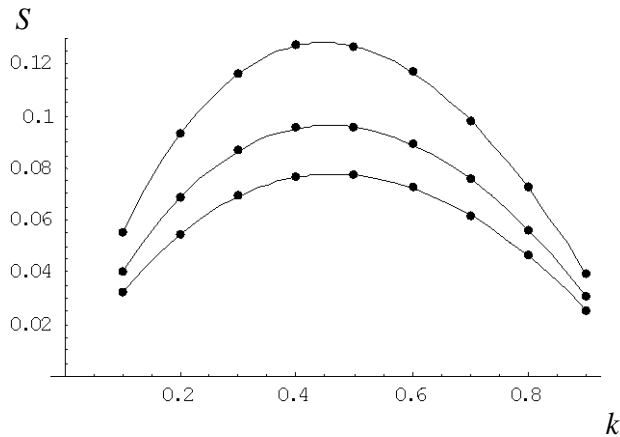


Fig. 1. Marginal stability curves for the case $R = \infty$ and three values of β : $\beta = 1, \beta = 1.5$ and $\beta = 2$ (from top to bottom).

Fig. 2 plots the marginal stability curves for the case $1/R = 0.03$ and three values of β : $1, 1.5$ and 2 . It is also seen from the graph that non-uniformity of the friction coefficient stabilizes the flow in the presence of a small curvature.

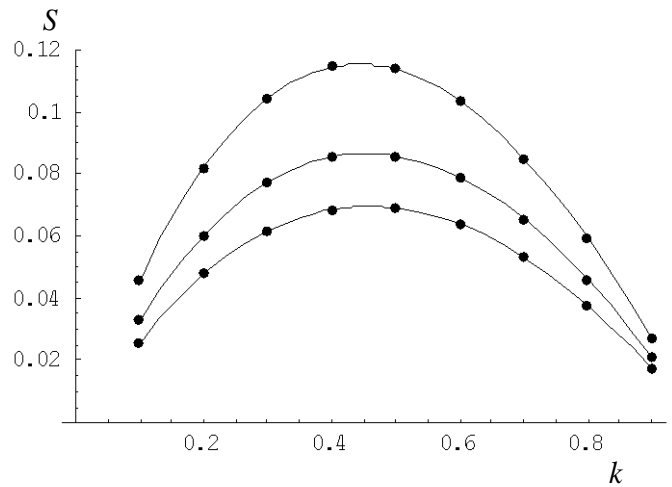


Fig. 2. Marginal stability curves for the case $1/R = 0.03$ and three values of β : $\beta = 1, \beta = 1.5$ and $\beta = 2$ (from top to bottom).

The effect of small curvature on the stability boundary is shown in Fig. 3 for the case of a non-constant friction. The marginal stability curves in Fig. 3 correspond to one value $\beta = 2$. It is seen from the graph that the increase in curvature leads to a more stable flow.

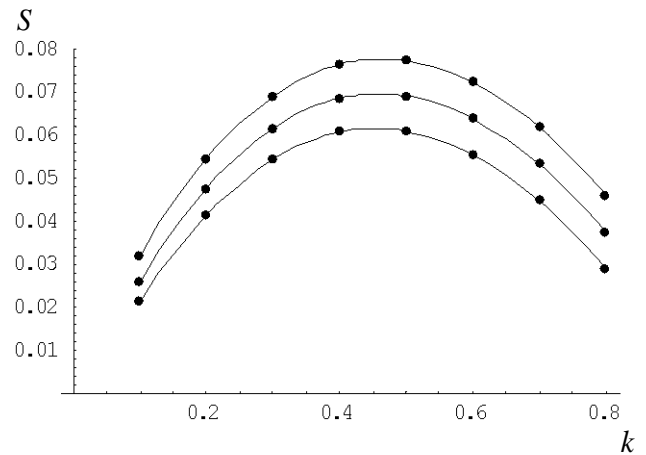


Fig. 3. Marginal stability curves for the case $\beta = 2$ and three values of $1/R$: $0, 0.03, 0.06$ (from top to bottom).

The combined effect of the variable friction and small curvature on the stability boundary is shown in Fig. 4 where the critical value of the parameter S (defined as $S_{cr} = \max_k S(k)$) is plotted versus β for three different values of $1/R$: $0, 0.03, 0.06$ (from top to bottom). Both parameters (small curvature $1/R$ and non-uniformity of the friction coefficient β) have a stabilizing influence on the flow.

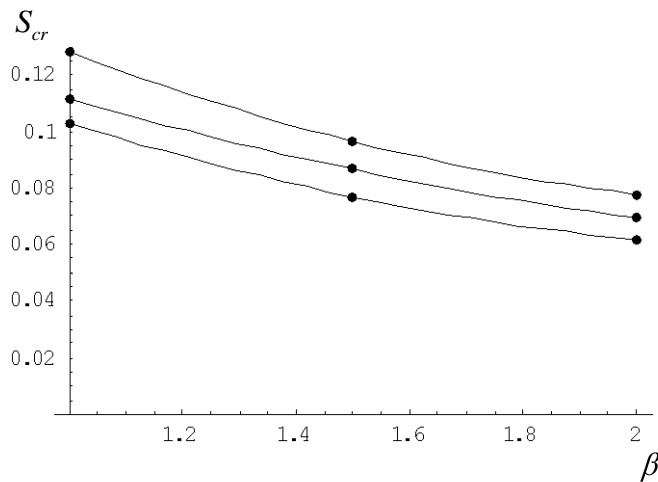


Fig. 4. Critical values S_{cr} versus β for three values of $1/R$: 1, 1.5 and 2 (from bottom to top).

V. CONCLUSION

The combined influence of two factors (non-uniformity of the friction coefficient in the transverse direction and small curvature) on temporal stability of shallow mixing layers is investigated in the present paper. The linear stability problem is solved by a pseudospectral collocation method based on Chebyshev polynomials. Marginal stability curves are presented for different values of the parameters of the problem. It is shown that both parameters stabilize the flow.

Linear stability analysis can be used only to determine the critical values of the parameters (in this particular case, the stability parameter S) for which the base flow becomes unstable. In order to study the evolution of the most unstable mode for the value of S which is slightly smaller than the critical value one can use weakly nonlinear theory. It can be shown that in this case the amplitude evolution equation for the most unstable mode in accordance with linear stability theory is the complex Ginzburg-Landau equation. The authors are currently working on the derivation of the Ginzburg-Landau equation and the analysis of the stability properties of the equation.

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REFERENCES

- [1] G.H. Jirka, "Large scale flow structures and mixing processes in shallow flows," *J. Hydr. Res.*, vol. 39, pp. 567–573, 2001.
- [2] V.H. Chu, J.H. Wu, and R.E. Khayat, "Stability of transverse shear flows in shallow open channels," *J. Hydr. Engng.*, vol. 117, pp. 1370–1388, 1991.
- [3] D. Chen, and G.H. Jirka, "Linear stability analysis of turbulent mixing layers and jets in shallow water", *J. Hydr. Res.*, vol. 36, pp. 815–830, 1998.

- [4] M.S. Ghidaoui, and A.A. Kolyshkin, "Linear stability analysis of motions in compound open channels", *J. Hydr. Engng.*, vol. 125, pp. 871–880, 1999.
- [5] A.A. Kolyshkin, and M.S. Ghidaoui, "Gravitational and shear instabilities in compound and composite channels, *J. Hydr. Engng.*, vol. 128, pp. 1076–1086, 2002.
- [6] V.H. Chu, and S. Babarutsi, "Confinement and bed-friction effects in shallow turbulent mixing layers", *J. Hydr. Engng.*, vol. 114, pp. 1257–1274, 1988.
- [7] W.S.J. Uijttewaala, and J. Tukker, "Development of quasi two-dimensional structures in shallow free-surface mixing layer," *Exp. in Fluids*, vol. 24, pp. 192–200, 1998.
- [8] W.S.J. Uijttewaala, and R. Booij, "Effect of shallowness on the development of free-surface mixing layers," *Phys. Fluids*, vol. 12, pp. 392–402, 2000.
- [9] B.C. Prooijen, and W.S.J. Uijttewaala, "A linear approach for the evolution of coherent structures in shallow mixing layers", *Phys. Fluids*, vol. 14, pp. 4105–4114, 2002.
- [10] W. Liou, "Linear instability of free curved shear layers", *Phys. Fluids*, vol. 6, pp. 541–549, 1993.
- [11] H. Nepf, "Hydrodynamics of vegetated channels," *J. Hydr. Res.*, vol. 50, pp. 262–279, 2012.
- [12] H. Nepf, "Flow and transport in regions with aquatic vegetation," *Ann. Rev. of Fluid Mech.*, vol. 44, pp. 123–142, 2012.
- [13] X. Zhang, and H. Nepf, "Exchange flow between open water and floating vegetation," *Environ. Fluid Mech.*, vol. 11, pp. 531–546, 2011.
- [14] Jamali, M., X. Zhang, and H. Nepf, "Exchange flow between canopy and open water," *J. Fluid Mech.*, vol. 611, pp. 237–254, 2008.
- [15] B. White, and H. Nepf, "Shear instability and coherent structures in shallow flow adjacent to a porous layer," *J. Fluid Mech.*, vol. 593, pp. 1–32, 2007.
- [16] I. Eglite, and A. Kolyshkin, "Spatial stability analysis of curved shallow mixing layers", in *Mathematical models and methods in modern science*, WSEAS Press, Athens, pp. 104–108, 2013.