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# Analysis of the Sensitivity of a Vibration-Based Procedure for Structural Identification

**J. P. Auzins and A. F. Boyko**

Riga Technical University (RTU), Riga, Latvia

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# Анализ чувствительности вибрационного метода структурной идентификации

**Я. П. Аузиньш, А. Ф. Бойко**

Рижский технический университет, Рига, Латвия

*Проанализирована точность определения свойств материалов элементов конструкций (модуль упругости, коэффициент Пуассона, плотность и др.) методом измерения частот собственных колебаний. Оценены точность определения частоты методом Фурье и влияние ее на конечный результат. С помощью метода метамоделей выведены зависимости между ошибками измерения частот и ошибками идентифицируемых параметров. Метод проиллюстрирован на примере элемента обшивки с ребром жесткости.*

***Ключевые слова***: анализ чувствительности, структурная идентификация, собственные частоты, метамодель.

**Introduction**. The method of eigenfrequencies of oscillations is widely used for estimation of structural material properties. The methodology of identification is similar to that described for composite plates in [1–3]. According to this method, the eigenfrequencies of structures are measured (by the resonance method and Fourier transformation) and, at the same time, a finite element model (FEM) is developed. The values of the identified parameters are obtained by minimizing the difference between the experimentally obtained frequencies and those calculated using the FEM [1–3]. In this process, crucial errors are made the source of which being measurements of oscillation eigenfrequencies, errors made in the design of structures, as well as the discrepancy between the mathematical (FEM) and physical models. The influence of those errors on the finite model is usually evaluated using the sensitivity matrix [3]. In practice, it is very important to evaluate the accuracy of the final result and also to determine with what accuracy the measurements of eigenfrequencies should be made in order to obtain a satisfactory accuracy of the final result.

1. **Finite Element Model**. The curved plate (Fig. 1) was designed using the ANSYS program. The number of elements with oriented meshes is 1160. A solid-state element (a component with elastic characteristics $E = 210$ GPa, Poisson’s ratio $\mu = 0.3$, and density $\rho = 7800$ kg/m$^3$ corresponding to a carbon steel) was chosen as a finite element [2]. In calculations, damping was not taken

into consideration. The elastic properties of the steel curved stiffened panel were determined employing both eigenfrequencies and mode shapes calculated by ANSYS with fixed boundary conditions (with one fixed end).

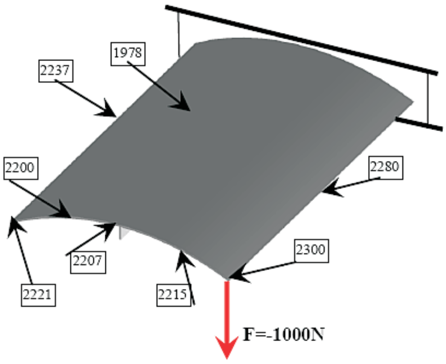


Fig. 1. Shell element with checkpoints ($0.4\times 0.25\times 0.002$ m, $s = 0.015$ s).

2. **Methodology of Identification**. To identify the material properties, i.e., three components of the vector $x=\{E, \mu, \rho\}$, it is necessary to minimize the functional: the total relative discrepancy between the experimentally measured frequencies $f_i^{exp}$ and numerically calculated frequencies $f_i(x)$ ($i=1,2,\ldots, m$) is determined by

$$\Phi(x)=\sum_{i=1}^{m}\left(\frac{f_i^{exp}-f_i(x)}{f_i^{exp}}\right)^2, \tag{1}$$

where $m$ is the number of the frequencies employed.

Instead of the direct minimization of functional (1), it is proposed to use the metamodel technology. This technology employs the so-called numerical experiments for obtaining approximating functions

$$f_i = f_i(x), \quad i=1, \ldots, m. \tag{2}$$

Previous investigations [1] showed that the second order approximations should be used to build the approximations $f$. In this case, the relative error of the approximation is less than 0.03%, which is much smaller than the error of the FEM.

For computations, a numerical experiment was planned using the criteria of $D$-optimality. Unlike the experimental designs of $D$-optimal natural experiments, where repeated experiments were used, in this method Latin hypercube-type plans are employed. $D$-optimal plans for the second order approximations using three factors, have to consist of $4\cdot 5/2=10$ experiments (experimental runs). Usually a twice as large number of runs is used. In order to compare the results obtained with cubic approximation, it is necessary to have $4\cdot 5\cdot 6/6=20$ experimental runs. Since in this case the FEM calculations are not very time-consuming, the $D$-optimal Latin hypercube-type design [2] with $N = 70$ runs and $K = 3$ variables has been selected (Fig. 2).

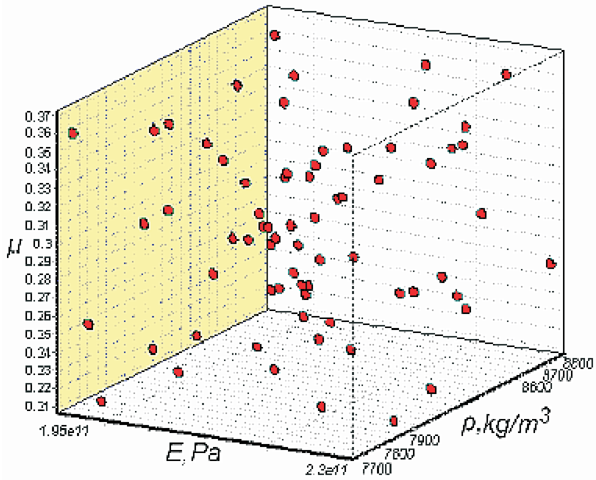


Fig. 2. Points of experimental design in 3D space.

The domain of interest is chosen as follows: $E \in [195, 230]$, $\mu \in [0.21, 0.37]$, and $\rho \in [7700, 8800]$.

3. **Short Analysis of the Source of Inaccuracy in Determining Frequencies by the Fourier Method**. The first thirty eigenfrequencies were determined by the ANSYS at the first stage of calculation. This is a fragment of the range: 71.1; 141.2; 242.7; 295.2; 321.3; 559.8; 560.1; 572.1; 738.8; 855.3; 864.2; 869.5; … .

To disclose the frequencies existing in the plate, the time integration was carried out with a small step, namely, 0.0001 s. During modeling it became clear that it was necessary to apply an asymmetric impulse load to the structure for excitation of the greatest number of frequencies.

3.1. ***Spectral Analysis***. To complete the analysis of the facts established during spectral analysis based on the data calculated using the ANSYS program, we will carry out the synthesis of frequency-amplitude processes using the Polyharmonic program created in the RTU MMD Laboratory. This program was created for the simulation of the eigenfrequency measurement process using the Fourier transformation. The program can generate transitional frequency-amplitude processes by the first 30 frequencies and add the effects of noise and energy dissipation. The process synthesized by the Polyharmonic program and that calculated using the ANSYS agree well with each other when the amplitudes of the processes are inversely proportional to the squared eigenfrequency.

Verification of the frequency spectra obtained from calculations by the ANSYS and from the synthesizing by the Polyharmonic program shows that the lowest eigenfrequencies are obtained using the Polyharmonic with three-decimal digit accuracy $f_1 = 71.2$ and $f_2 = 141.27$ Hz. Nevertheless, the higher frequencies (higher than the 15th frequency) attenuate quickly and it is impossible to identify them in the both programs. Perhaps, this is associated with the numerical integration algorithm and the influence of the boundary conditions on the results of identification of the eigenfrequency.

The frequency $f_3 = 242.71$ Hz is not visible in the process given by the ANSYS. However, in the synthesized process this frequency is clearly seen. Hence, the above-mentioned frequency has not been excited in the current experiment, but it can be displayed if another loading mode or load combination is added.

It has to be admitted that if some eigenfrequencies are closely situated during processing by the ANSYS, the peaks converge and we shall see only one frequency (Fig. 3). This phenomenon is intensified with an increase in the frequency. However, in the synthesizing spectrum by the Polyharmonic program, the nearly situated eigenfrequencies are presented and clearly seen (Fig. 4).

| No. | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|
| Frequency | 559.80 | 560.19 | 572.17 | 738.88 | 855.30 | 864.29 | 869.55 |

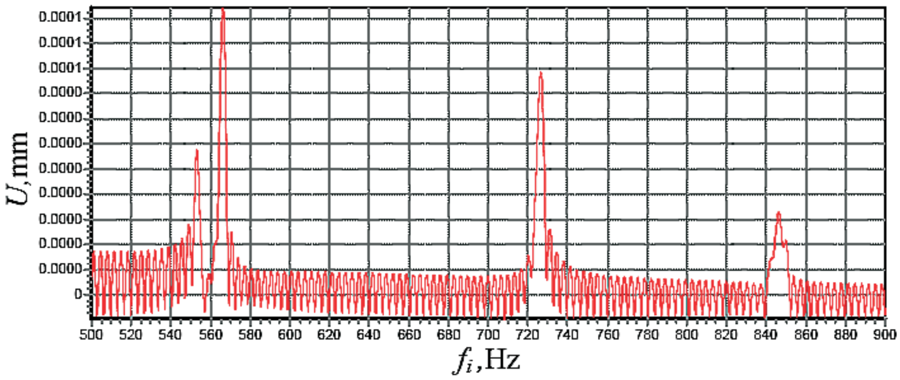


Fig. 3. Frequency spectrum in the range from 500 to 900 Hz obtained by ANSYS.

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|---|---|---|---|---|---|---|---|
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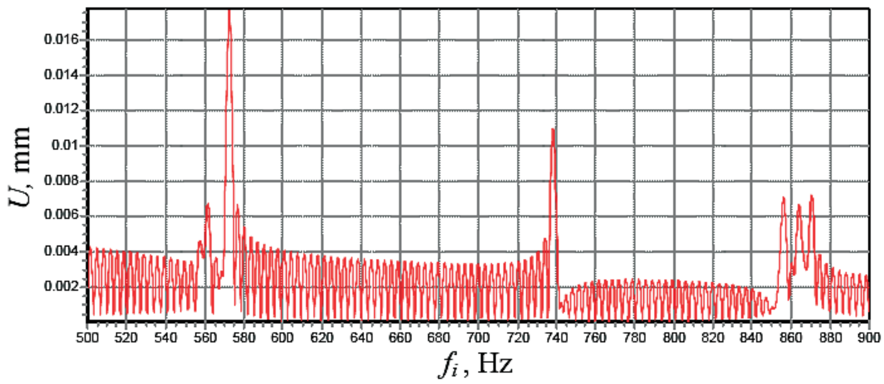


Fig. 4. Frequency spectrum in the range from 500 to 900 Hz obtained by Polyharmonic program.

In addition, the influence of the noise level, attenuation, the number and quantity of the discretization steps of the Fourier transformation on the frequency calculation accuracy was analyzed. It was found that at a noise level of 20%, eigenfrequencies up to the 10th number are calculated with exactness in second number. It is possible to precisely determine the eigenfrequencies up to the 6th–8th frequencies with a simultaneous application of the attenuation in the amount of about $(1-6)\cdot 10^{-6}$.

Analysis of the number and quantity of discretization steps of the Fourier transformation reveals that the decrease in the step number from 3000 to 2000 leads to a significant loss in the accuracy of the determination of eigenfrequencies above the 10th frequency.

4. **Sensitivity Analysis of Identification**. When the identification parameters are obtained, it is important to estimate how the precision of the natural eigenfrequency measurement influences the accuracy of the identification results and which sample of the measured eigenfrequencies gives the highest precision of the identification results. Traditionally, the sensitivity coefficients are used as partial derivatives of a model response with respect to a model parameter or in the case of material identification [3]. But in practice it is also important to clarify the relative standard deviation of the frequency measurement, so that the standard deviation of the identified $E$ value does not exceed, for example, 1%.

Since the measurement precision for the highest frequencies is lower, scaled values of the frequencies are used for identification, dividing them by the respective value of the measured frequencies. Normalized frequencies are denoted as $g_i$, $i = 1, 2, \ldots, m$.

Sensitivity analysis is made using linear models, and, accordingly, linear approximations:

$$g_i = A_i x + B_i, \quad i = 1, 2, \ldots, m, \tag{3}$$

or, the same in the matrix form:

$$G = Ax + B, \tag{4}$$

where $A$ is the matrix $m \times 3$ and $B$ is the column vector of length $m$, which are calculated using linear approximations.

To identify the material properties $x$, functional (1) should be minimized:

$$\Phi = \sum_{i=1}^{m} (A_i x + B_i - 1)^2. \tag{5}$$

The functional is quadratic, the minimum can be found equating partial derivatives to zero

$$\begin{aligned} \frac{\partial \Phi}{\partial x_1} &= 2\sum_{i=1}^{m} (A_i x + B_i - 1) A_{i1} = 0, \\ \frac{\partial \Phi}{\partial x_2} &= 2\sum_{i=1}^{m} (A_i x + B_i - 1) A_{i2} = 0, \\ \frac{\partial \Phi}{\partial x_3} &= 2\sum_{i=1}^{m} (A_i x + B_i - 1) A_{i3} = 0. \end{aligned} \tag{6}$$

This gives linear equations in the matrix form:

$$Sx + B^* = 0, \tag{7}$$

where

$$S = A^T A, \tag{8}$$

$$B^* = A^T (B - I), \quad I = [1, 1, 1, 1, ..., 1]^T. \tag{9}$$

Let's presume that relative errors of the frequency measurements $\varepsilon_i^g$ are cross-uncorrelated, normally distributed random values with zero means and standard deviations $\sigma_i$ $(i = 1, 2, ..., m)$.

Then the column vector $\varepsilon^x$ error of identification errors $\varepsilon_i^x$ $(i = 1, 2, 3)$ is

$$\varepsilon^x = C\varepsilon, \tag{10}$$

where $C = (A^T A)^{-1} A^T$ is the matrix $3 \times m$.

Thus, the errors of the identified parameters $\varepsilon_i^x$, $i = 1, 2, 3$ are normally distributed random values with zero means and standard deviations $\sigma_i^x$ [4]

$$\sigma_i^x = \sqrt{\sum_{j=1}^{m} (C_{ij} \sigma_j)^2}, \quad i = 1, 2, 3. \tag{11}$$

If all measured frequencies have equal relative standard deviations $\sigma_i = \sigma^f$ $(i = 1, 2, ..., m)$, then

$$\sigma_i^x = \sqrt{\sum_{j=1}^{m} (C_{ij})^2}\, \sigma^f, \quad i = 1, 2, 3. \tag{12}$$

The relative standard deviation of the identified parameters can be calculated dividing absolute deviations by identified values of $x_i$

$$\sigma_i^{xrel} = \frac{\sigma_i^x}{x_i}, \quad i = 1, 2, 3. \tag{13}$$

The relations between the standard deviation of the frequency measurement error and the parameter identification errors are linear, so we can divide $\sigma_i^{xrel}$ by $\sigma^f$ to obtain the sensitivity coefficient $k_i$:

$$\sigma_i^{xrel} = \frac{\sigma_i^{xrel}}{x_i}. \tag{14}$$

This coefficient shows that the percentage error of the $i$th parameter identification is $k_i$ times larger than the percentage error of the frequency measurement.

5. **Results of Identification and Sensitivity Analysis**. For the purpose of verification, we identify parameters of the materials with finite “measured” frequencies specified in Section 3. It is necessary to identify the following parameters: $E = 210$ GPa, $\mu = 0.3$, and $\rho = 7800$ kg/m$^3$.

The identified model coincides precisely with the results of calculation in the frequency range to the 20th frequency ($f_{20}$ = 1584.7 Hz) that was not used in identification of parameters, with an accuracy of 0.01% (in Table 1 – a fragment of the range). Here the result of parameters identification is not as accurate: $E = 210$ GPa with an error of 3.9%, $\mu = 0.3$ with an error of 0.04% and $\rho = 7800$ kg/m$^3$ with an error of 4% that differs significantly from the “experimental” result in the ANSYS program.

T a b l e 1

**Calculated Errors**

| No. | Experimental value (Hz) | Calculated value (Hz) | Error (%) |
|---|---|---|---|
| 1 | 71.179 | 71.1749 | 0.0057 |
| 2 | 141.280 | 141.2800 | 0 |
| 3 | 242.710 | 242.7100 | 0 |
| 4 | 295.240 | 295.2400 | 0 |
| 5 | 321.390 | 321.3900 | 0 |
| 6 | 559.800 | 560.0280 | −0.0410 |
| 7 | 560.190 | 560.0270 | 0.0291 |
| 8 | 572.170 | 572.1090 | 0.0107 |
| 9 | 738.880 | 738.8800 | 0 |
| 10 | 855.300 | 855.2900 | 0.0012 |
| 11 | 864.290 | 864.2900 | 0 |
| 12 | 869.550 | 869.5700 | −0.0020 |

The analysis of sensitivity performed according to the aforementioned methodology proves that even in determination of the frequencies with an accuracy of 0.1% the error of parameters determination would be: $E = 210 \pm 2612.41$ (1244%), $\mu = 0.3 \pm 0.004029$ (1.34%), and $\rho = 7800 \pm 10141.1$ (1300%), which attests to insensibility of $E$ and $\rho$ determination. The result presented should have been expected, providing proportional dependence of the mass and stiffness matrix ratio on $\rho$ and $E$, respectively. In this case, the method is unable to exactly identify the moduli of elasticity $E$ and density $\rho$ simultaneously as they compensate each other with respect to frequencies. That is, in simultaneous proportional change the values of $E$ and $\rho$ values of frequencies do not change. Hence it can be seen that it is possible to identify Poisson’s ratio with an error of ±0.004 that would have been a rather good result.

Considering the aforementioned proportionality, the assignment of identification might be reduced to identification of two parameters $E$ and $\mu$ by assigning a constant value of material density $\rho = 7800$ kg/m$^3$. Hence, by using the first 6 frequencies with determination accuracy of 1% the result would have been $E = 210.12$ GPa $\pm 0.8\%$ and $\mu = 0.304 \pm 12.9\%$, but with the first 12 frequencies used – $E = 209.8$ GPa $\pm 0.56\%$ and $\mu = 0.302 \pm 10.8\%$. Then, in identification according to the first two frequencies, the error consequently will be 1.5% and 22.3%, but with the use of the 11th and 12th frequencies, the error will be 3.7% and 108%. Since there is a linear correlation between the measurement and identification errors, then in frequency measurements with an error of 0.1%, the identification error shall be 10 times lower, respectively.

## Conclusions

1. The noise level and attenuation influence the accuracy of frequency determination insignificantly.

2. The number and quantity of the discretization steps of the Fourier transformation have a great influence on the accuracy of the determination of frequencies higher than the 10th one.

3. The developed method of sensitivity analysis allows estimation of the identification error and, in principle, gives the possibility of identifying several parameters (e.g., Poisson's ratio or shear modulus).

4. The errors of frequency determinations caused by manufacturing errors (dimensions, density etc. differ from their nominal values) and measurement errors (caused, e.g., by discretization of the frequency band using fast Fourier transformation) can produce an unacceptable error of identification, therefore all parameters, not only frequencies, must be measured with split-hair accuracy.

## Резюме

Проаналізовано точність визначення властивостей матеріалів елементів конструкцій (модуль пружності, коефіцієнт Пуассона, щільність і ін.) методом вимірювання частот власних коливань. Оцінено точність визначення частоти методом Фур'є і вплив її на кінцевий результат. За допомогою методу метамоделей виведено залежності між помилками вимірювання частот та помилками ідентифікованих параметрів. Метод проілюстровано на прикладі елемента обшивки з ребром жорсткості.

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