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MinMaxDM Distribution Family for a Series of Parallel Systems with Defects and the Tensile Strength of a Composite Material

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Abstract: This paper is a review of the authors' previous work devoted to the analysis of tensile strength of unidirectionally fiber-reinforced composite material, considered as a series of parallel systems with defects. Additionally, a specific model is studied based on an assumption that only failure of the damaged parts of a specimen can take place. A version of the Poisson distribution is used for probability mass function of the number of defects. The proposed models allow estimating the probability of the presence of defects in order to clarify improvements of the production technology needed to increase reliability, and to predict the scale effect of tensile strength of the composite. Strength test data of different composite materials are processed and the results analyzed. A numerical comparison of different models is provided.

Keywords: MinMaxDM distribution, series system, reliability, order statistics, fiber, composite

1. Introduction

In order to reduce the mass of airframe and increase the reliability of modern aircraft, composite materials are widely used in primary structural components. It should be noted that the actual strength of a composite material may be considerably smaller than its theoretical strength because, usually, there are defects present in the material. Therefore, it is of practical importance to elaborate a method of estimation of the probability of existence of defects in a specific material in order to understand the necessary improvements in production technology and also to enable prediction of the variation of strength with the size of a specific item. This paper is a review integrating, amending, and developing the approaches developed in authors' previous papers [1-7] devoted to analysis of the tensile strength of a unidirectional (UD) composite material considered as a series of parallel systems with defects. Additionally, a specific model is studied based on an assumption that only the failure of the parts of a specimen containing defects can take place. A version of the Poisson distribution is used for description of the probability mass function of the number of defects.

When modeling the strength of a UD composite under tension in the fiber direction in [1-7], we considered the composite material as a series system, every link of which is a parallel system, or more specifically, a bundle of n_C longitudinal items (LIs). It is a simplified model of a UD composite material in which the longitudinal items (*e.g.*, bundles of fibers) are impregnated by the matrix.

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We make a simplifying assumption that only the longitudinal items carry the applied longitudinal load while the matrix (and, possibly, all the layers with stacking different from the longitudinal one) only redistributes the loads after the failure of some longitudinal items. The total length of the specimen is divided into n_L equal parts ("links"): $L = n_L l_1$. It is supposed that the development of the process of fracture of a specimen takes place in one or in several of these parts which we refer to also as cross sections (CSs). Let the process of monotonic tensile loading (*i.e.,* the process of increase of the nominal stress in the specimen CS) be described by an ascending (up to infinity) sequence $\{x_1, x_2, ..., x_t, ...\}$ of stress levels. We denote the initial number of LIs in every CS by n_C. Let $K_{Ci}(t)$, $0 \le K_{Ci} \le n_C$, be the number of failures of LIs in the *i*-th link at the load x_t. Then the strength of the *i*-th CS is determined as follows

$$X_i^* = \max_t (x_t : n_C - K_{Ci}(t) \ge 0), \tag{1}$$

and the ultimate strength of the specimen (which is a sequence of n_L CSs) is

$$X = \min_{1 \le i \le n_L} X_i^* = \min_{1 \le i \le n_L} \max_t (x_t : n_C - K_{Ci}(t) \ge 0). \tag{2}$$

Different assumptions about the distribution of strength of a link and a priori distribution of initial (technological) defects (failures of some LIs in some CSs) compose a family of the distributions of ultimate tensile strength of the composite material. For shorthand notation of this family, an abbreviation MinMaxDM was introduced in [1] based on the form of Eq. (2) and a specific description of the process of failure of the specimen (see Section 4).

The paper, besides an Introduction, has three subsequent sections. In Section 2 a series system, in Section 3 a parallel system, and in Section 4 a series of parallel systems with damaged LIs are considered. Numerical examples in each case are provided.

2. Reliability of a Series System with Defects

First, we study a special case when there is only one LI in every link, n_C =1, and consider two types of links. Let us denote by K_L the random number of damaged links, $0 \le K_L \le n_L$, with the cumulative distribution function (cdf) of strength designated as $F_Y(x)$ (so there are K_L of Y-type links), and let us denote by $F_Z(x)$ the strength cdf of $n_L - K_L$ links without defects (we call them Z-type links). We suppose that the failure process of the considered system proceeds in two stages. In the first stage the process develops along the specimen and K_L links of Y-type appear. They can appear before loading or during the process of loading, if the stress in a CS exceeds *a defect initiation stress* with cdf $F_K(x)$. Then the second stage takes place: the process of accumulation of elementary damages in crosswise direction up to specimen failure.

Usually, for description of cdf of the strength, *S*, of a LI the Weibull distribution is used: $F(s) = 1 - \exp(-(s/\beta))$. Then, in the logarithmic scale, we have the cdf of the smallest extreme value (sev) $F_{\log S}(x) = 1 - \exp(-\exp((x - \theta_0)/\theta_1))$ for the random variable (r.v.) $\log(S)$, where $\theta_0 = \log(\beta)$ and $\theta_1 = 1/\alpha$.

In this paper we suppose that $F_Y(.)$, $F_Z(.)$ and $F_K(.)$ are sev cdf with specific parameters θ_{0Y}, θ_{1Y} ; θ_{0Z}, θ_{1Z} ; θ_{0K}, θ_{1K} . For the r.v. K_L , the binomial (with the probability mass function (pmf) $b(k,p,m) = p^k (1-p)^{m-k} m!/k!(m-k)!$, $k = 0,1,...,m$) or the Poisson (with pmf $p(k,\lambda) = e^{-\lambda}\lambda^{k-1}/k!$, $k = 0,1,2,...$) distribution or their modifications can be used.

Two levels of differences between LIs with and without defects and three groups (levels) of accuracy of description of the difference of strength inside these groups form six types of corresponding probability structures (p.s.) : A1,…,A3, B1,…, B3 [2]. In the p.s. of type A, it is assumed that the difference between the strength of links of Y and Z types is relatively small and the failure of the specimens can be caused by the failure of a link of either type. For example, the strength of specimen is $X = \min(Y_1,...,Y_{K_L}, Z_1,...,Z_{n_L-K_L})$ in the p.s. A1. In the p.s. of type B, it is assumed that the difference between the strength of links of Y and Z types is very large and the strength of the links of Z type have to be taken into account only if there are no links of Y-type present. For example, in p.s. B3 the strength of specimen $X = Y$ if $K_L > 0$ whereas $X = Z$ if $K_L = 0$. We suppose that the usefulness of considering this set of different p.s. stems from the difference of materials, different requirements to the accuracy of calculations, and different sizes of the test specimen.

In this paper we introduce also a new specific version of p.s. We assume that the failure of a specimen can take place only in a damaged CS (DCS); then the strength of the specimen $X = \min(Y_1, Y_2,...,Y_{K_L})$, $K_L = 1,...,n_L$. In this case, a modified binomial or Poisson distribution can be used for the pmf of the r.v. K_L . The probability of the event “the number of defects is equal to zero” should be equal to zero. For example, if the r.v. K has the binomial or Poisson distribution then the following definition can be used: $K_L = 1 + K$; a conditional binomial or Poisson distribution (under the condition $P(K_L > 0) = 1$) can also be used. In this paper, the following pmf is used: $P(K_L = k) = p_c(k,\lambda) = \lambda^k \exp(-\lambda)/(1-\exp(-\lambda))k!$, $k = 1,2,...$. It is the pmf of a conditional zero-truncated Poisson distribution (ZTPD) meeting the condition stated above. Then cdf of the strength of specimen is $F(x) = 1 - \sum_{k=1}^{\infty} p_c(k,\lambda)(1-F_Y(x))^k / k = (1-e^{-\lambda F_Y(x)})/(1-e^{-\lambda})$. A natural assumption is $\lambda = \lambda_1 * L$, where λ_1 stands for the defect intensity. One can also assume $\lambda_1 = \lambda_{10} F_K(x)$.

The process of gradual accumulation of defects during loading and failure of a series system can be described by a Markov chain (MC). Hereinafter for notation of the p.s. we use an additional letter M: MA1, MA2 and so on. If the process of monotonic tensile loading (*i.e.,* the process of increase of the nominal stress or mean load on one LI) is described by an ascending (up to infinity) sequence $\{x_1, x_2,..., x_t,...\}$ then the number of links of Y-type and the strength of specimens are random functions of time, $K_L(t)$ and $X(t)$. For example, we have $X(t) = \min(Y_1, Y_2,...,Y_{K_L(t)}, Z_1, Z_2,..., Z_{n_L-K_L(t)})$ for the p.s. MA1. Let us consider a MC with $n_L + 2$ states. MC is in state i if there are $i-1$ of Y-type links, i = 1, ... , n_L+1. The state i_{n_L+2} is an absorbing state corresponding to the fracture of specimen. The process of MC state change and the corresponding process $K_L(t)$ are described by the matrix of transition probabilities P

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & \cdots & p_{1(n_L+1)} & p_{1(n_L+2)} \\ 0 & p_{22} & p_{23} & p_{24} & \cdots & p_{2(n_L+1)} & p_{2(n_L+2)} \\ 0 & 0 & p_{33} & p_{34} & \cdots & p_{3(n_L+1)} & p_{3(n_L+3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & p_{(n_L+1)(n_L+1)} & p_{(n_L+1)(n_L+2)} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

At the t-th step of MC the matrix P is a function of t, t=1, 2, ... A priori or initial distribution of K_L is represented by a row vector $\pi_L = (\pi_{L1}, \pi_{L2}, ..., \pi_{L,n+1,} \pi_{L,n+2})$, where $\pi_{L(n+2)}$=0. Now the ultimate strength of a specimen is given by $X = x_{T^*}$, where $T^* = \max(t : X(t) > x_t)$. The cdf of the ultimate strength $F_X(x_t) = \pi_L(\prod_{j=1}^{t} P(j))u$, where the vector-column $u = (0,0,...,1)'$. The examples of specifying of the matrix P for "extreme" p.s. MA1 and MB3, for n_C=1, are given in [2].

2.1 Numerical Examples for a Series System

We processed the following tension test results:

Data_1: glass fibers of length (L_1, L_2, L_3, L_4) = (10, 20, 40, 80 *mm*) at the number of specimens (n_1, n_2, n_3, n_4) = (78, 74, 49, 60) (from [10]).

Data_2: flax fibers of length (L_1, L_2, L_3) = (5, 10, 20 *mm*) at the number of specimens (n_1, n_2, n_3) = (90, 70, 58) (from [11]).

Data_3: carbon fibers of length (L_1, L_2, L_3, L_4) = (1, 10, 20, 50 *mm*) at the number of specimens (n_1, n_2, n_3, n_4) = (57, 64, 70, 66) (from [9]).

Data_4: epoxy-impregnated carbon fiber bundles of length (L_1, L_2, L_3, L_4) = (20, 50, 150, 300 *mm*) at the number of specimens (n_1, n_2, n_3, n_4) = (28, 30, 32, 29) (from [9]).

We used the test results at (L_1, L_2) to estimate the cdf parameters and then predicted the expected values of order statistics for the largest length, L_3 (for flax fibers) or L_4 (for all the other data sets).

A similar processing of these data has already been performed in [4] but for an opposite direction of prediction: fitting of experimental data and estimation of cdf parameters was performed using test results at (L_2, L_3) (for flax fibers) or (L_3, L_4) (for all the other data sets) and prediction of expected values of order statistics - for the smallest length, L_1.

First, processing of the data was performed for the same models as considered in [4]: for (a) p.s. MB3 with a priori distribution $\pi = (p, 1-p, 0)$, $p = 0.15$; (b) p.s. B3F (it is a

B3 type p.s. with $P(K_L = 0) = 1 - F_K(x)$); (c) p.s. MA with $p_k = p_k^P$, $k = 0,1,...,n_L - 1$, $p_{n_L} = \sum_{k=n_L}^{\infty} p_k^P$, $p_k^P = \exp(-\lambda)\lambda^k / k!$).

These models were compared with the so called Linear-Weibull (LW) model, with distribution function $F(x) = 1 - \exp(-(L/l_1)(x/\beta)^\alpha))$, and the Power-Weibull (PW) model with $F(x) = 1 - \exp(-(L/l_1)^\gamma (x/\beta)^\alpha)$.

The maximum likelihood method can be used for parameter estimation, but it is excessively labor-consuming. The estimates of parameters θ_0 and θ_1 (at fixed other parameters) can be found easily, using the regression analysis of order statistics. Our purpose here is only investigation of the possibility of using the considered models for prediction of the changes of fiber strength distribution when fiber length is varied, and comparison of the models. So we have limited ourselves by the use of regression analysis. Let r.v. X (the logarithm of strength) have cdf with a location and a scale parameter and let x_{ij} be the *j*-th order statistic of tensile strength for *i*-th fiber length. Here $j = 1,2,...,n_i$, n_i is the number of specimens with $L = L_i$, $i = 1,2,...,k_L$, k_L is the number of different L_i. Let us denote by $E(X_{ij})$ the expected value of the random order statistic X_{ij}, and by $E(\overset{0}{X}_{ij})$ - the same for the particular case of θ_0=0 and θ_1=1. Then we have $E(X_{ij}) = \theta_0 + \theta_1 E(\overset{0}{X}_{ij})$, where $E(\overset{0}{X}_{ij})$ is a function of n_i, *j* and L_i (and λ_1 for a ZTPD). For the LW model, $E(X_{ij}) = \theta_0 + \theta_1(-\log(L_i / l_1) + E(\overset{0}{X}_{ij}))$. For the PW model $E(X_{ij}) = \theta_{00} - \theta_{01}\log(L_i / l_1) + \theta_1 E(\overset{0}{X}_{ij}))$, where there are three unknown parameters θ_{00}=$\log(\beta)$, $\theta_{01} = \gamma / \alpha$ and θ_1. These equations can be used for a linear regression (LR) estimation of $\theta_0, \theta_{00}, \theta_{01}$ and θ_1.

For numerical evaluation of the quality of fitting and comparison of different models, the following three additional statistics were calculated: $\overline{R}_{LR}$, OSPPt, and Q_1 (a detailed definition of these statistics is provided in [4]).

In the first block of Table 1, the results (quality statistics and model parameters) corresponding to the best from the mentioned models (for the specific test data) are given. Let us note also that in order to reduce the number of unknown parameters, the following simplification was introduced: for every data set with specific L values considered, it was assumed that $l_1 = L_1$. Selecting an optimal value of l_1 would, of course, improve the fitting of experimental data by the models (but it would not necessarily lead to a better prediction). As the estimates of $\lambda_1 = \lambda / L_1$ and p, the values of these parameters that provided the best fit (*i.e.*, minimum of $\overline{R}_{LR}$) were used. In the following blocks, the results (quality statistics and model parameters) for ZTPD, PW and LW models are presented.

In the two last blocks, linear regression parameter estimates, obtained either using only data at L_1, L_2 (_LRA_2) or the whole data set (_LRA_k_L), and the parameters obtained by the maximum likelihood method (_ML) are given (if there were ML-estimates available in the papers from which the test data have been taken).

Table 1: Criteria Statistics. Estimates of Parameters

№	Criteria statistics. Estimates of Parameters	Glass Fibers [12]	Flax Fibers [13]	Carbon Fibers [11]	Bundles 1000 of Epoxy-impreg. Carbon Fibers [11]
1	($\overline{R}_{LR}$, OSPPt, Q_1)_MMDM	0.156 **0.174** **0.124** MB	**0.119** **0.197** **0.316**	0.182 0.284 0.230	0.417 0.578 0.433
	Structure θ_0 , θ_1 p λ_1	MB3 7.63 0.246 0.15	B3F 6.76 0.479	MA 8.436 0.152 0.99	MA 7.97 0.044 0.025
2	($\overline{R}_{LR}$, OSPPt, Q_1)_ZTPD	0.1567 0.3800 0.4184	0.1394 0.6483 1.2933	**0.1614** 0.3246 0.2499	**0.2107** 0.7054 0.5855
	$\hat{\theta}_0$, $\hat{\theta}_1$ λ_1	7.85 , 0.17 0.142	7.1, 0.29 0.27	8.5, 0.15 1.66	7.98 , 0.045 0.0198
3	($\overline{R}_{LR}$, OSPPt, Q_1)_PW	**0.1525** 0.2155 0.1644	0.1534 0.3332 0.5121	0.1705 0.4026 0.2809	0.2109 1.1647 1.1228
	(β , α γ)_LRA_2	2 381 5.440 0.601	1 068 3.15 0.580	4 543 6.23 0.887	2 896 20.7 0.149
	(β , α γ)_LRA_ k_L	2 394 5.159 0. 608	1 076 3.09 0.623	4 582 6.15 1.022	2 965 16.6 0.732
	(β , α γ)_ML	3 030 5.43 0. 580	1 400 2.80 0.460	4 630 5.31 0.9	3 250 16.8 0.580
4	($\overline{R}_{LR}$, OSPPt, Q_1)_LW	0.1855 0.4760 0.6702	0.1890 0.7937 1.6501	0.1803 **0.2778** **0.2268**	0.3680 **0.5386** **0.3793**
	(β , α)_LRA_2	3 663 5.605	1 810 3.25	4 575 6.58	3 340 23.3
	(β , α)_LRA_ k_L	3 691 5.827	1 836 3.31	4 571 6.07	3 510 18.8
	(β , α)_ML	3010 8.99	1 836 3.31	-	-

The results indicate that the difference between the _LRA_ k_L -estimates and maximum likelihood _ML-estimates is not significant, particularly taking into account that the likelihood function actually has no distinct maximum (see also [4]). So we are not too far from ML-estimates if we use LR-estimates of the parameters.

In Table 1 the criterion values for the best results are shown in bold script. It is seen that for different data sets and different criteria, the best models also differ. For the sequence {glass fibers, flax fibers, carbon fibers, bundles 1000 of epoxy-impregnated carbon fibers} the best models for fitting the data sets are PW, BF, ZTPD and ZTPD. But for criteria OSPPt and Q_1, the best models are MB3, B3F, LW and LW, respectively. Although this is only a preliminary conclusion, it should be useful to take it into account in processing different test data sets.

3. Reliability of a Series of Parallel Systems with Defects

The most significant contribution to the solution of the problem of reliability of parallel systems was made by Peirce [12] and Daniels [13, 14]. A thorough analysis of Daniels results is provided in [8]. It was shown that this model yields accurate results only in specific cases. In this paper some development of the Daniels model is considered.

3.1 Randomized Daniels Model

Let $(X_1,...,X_n)$ be order statistics which are ordered random strengths of n LIs in a link. If there is a uniform distribution of load between intact LIs and the applied load increases monotonically, then the ultimate strength of this link is given by $X^* = \max_{1\le j\le n} X_j(n-j+1)/n_C$, where $n = n_C - K_C$, $K_C < n_C$, and the r.v. K_C denotes the number of (technological) failures. Daniels studied the case $K_C = 0$.

If the number n is sufficiently large, then for r.v. X^* there is a convergence in probability to a constant $\mu = \max_x x(1-F_X(x))$. For the Weibull distribution of single LI strength (without defects), $F(x) = 1-\exp(-\exp((\log(x)-\theta_0)/\theta_1))$, one arrives at $\mu = \theta_1^{\theta_1}\exp(\theta_0-\theta_1)$. Daniels has shown that, for sufficiently large *n*, the r.v. X^* has an approximately normal distribution. For the case of K_C=0, the parameters of this distribution are μ and $\sigma = \mu(\exp(\theta_1-1)/n_C)^{1/2}$. But if the number of damaged LIs $K_C = k_C$ (*i.e.,* there are only $n_C - k_C$ intact LIs) then we should use $\mu_n = \mu(n_C - k_C)/n_C$ and $\sigma_n = \sigma(n_C - k_C)/n_C$ (the denominator is equal to n_C instead of $n_C - k_C$ because the specimen strength is calculated taking into account the initial number of LIs, n_C).

We suppose that n_C is large enough and the r.v. K_C has a "truncated at n_C" conditional binomial a priori distribution with parameters (n_C, p_C), ("truncated at n_C" because a priori we should eliminate the equality $K_C = n_C$). Then the cdf of X^*

$$F_{X^*}(x) = \sum_{n=1}^{n_C} F_{X_n^*}(x) b(n_C - n, p_C, n_C)/(1-b(n_C, p_C, n_C)),$$

where $F_{X_n^*}(x) = \Phi((x-\mu_n)/\sigma_n)$; $\Phi(.)$ is the cdf of the standard normal distribution. Let us call this model a randomized Daniels model and denote it by RDM while NRDM stands for the traditional, non-randomized Daniels model.

3.2 Reliability of a Parallel System with Defects using the Markov Chains Theory

Let us recall that the process of monotonic tensile loading is described by an ascending sequence $\{x_1, x_2,..., x_t,...\}$, with $K_{Ci}(t)$, $0 \le K_{Ci} \le n_C$, designating the number of random failures of LIs under the load x_t in the *i*-th link and n_C - the initial number of LIs. The *i*-th link fails when $K_{Ci} = n_C$. We again consider the process of accumulation of failures as an inhomogeneous finite Markov chain (MC) with a finite state-space $I = \{1, 2, ..., n_C + 1\}$. MC is in the state i if $(i-1)$ of LIs have failed, $i = 1, ..., n_C + 1$; the state ($n_C + 1$) is an absorbing state corresponding to the failure of the link. The process of MC state change is described by the matrix of transition probabilities P; at the t-th step of MC the matrix P is a function of t, t=1,2,...

The cdf of strength of a link, X^*, is defined on the sequence $\{x_1, x_2, ..., x_t, ...\}$: $F_{X^*}(x_t) = \pi_C(\prod_{j=1}^{t} P(j))u$, where $P(j)$ is the transition probability matrix for t=j, and the column vector $u = (0, ..., 0, 1)'$.

Four main versions (hypotheses) of the structure of matrix P, denoted as P_a, P_{an_C}, P_b and P_c have been considered in [2]. The matrix P_a corresponds to the assumption that, in a single step of MC, only one LI can fail, and it is the nearest one to the already failed LI; P_{an_C} relates to the case of failure of the weakest item in the considered cross section; P_b - to a binomial distribution of the number of failures at every step of MC; P_c - to the case when the stress concentration function is known.

3.3 Modeling of Reliability of a Parallel System using the Monte Carlo Method

Let $E_1, E_2, \ldots E_n$ be the elastic moduli and $f_1, ..., f_n$ - the cross-sectional areas of n LIs; $\bar{\varepsilon}_i = \varepsilon_i / \varepsilon$, where ε_i is the strain in the i-th LI at a mean strain ε in the cross section and the distribution of r.v. $\bar{\varepsilon}_i$ does not depend on ε. Then we have a random stress-strain function: $\sigma(\varepsilon) = \varepsilon \sum_{\varepsilon\bar{\varepsilon}_i E_i^p < X_i} E_i^p \bar{\varepsilon}_i f_i / \sum_{i=1}^{n_C} f_i$, where $i = 1, ..., n_C$, r.v. E_i^p is equal to zero with probability p_C and equal to E_i with probability 1- p_C. The ultimate (nominal) random strength $\sigma = \max_{\varepsilon} \sigma(\varepsilon)$.

Using the Monte Carlo method, the relations above allow easy evaluation of the cdf of CS strength if the combined distribution of random variables $E_i, \bar{\varepsilon}_i$ and f_i, $i = 1, ..., n$, is known. Examples of predicting $\sigma(\varepsilon)$ for n_C=5, normal distribution of $\log(X)$ and $\log(E)$ with correlation coefficient r, independent normal distribution of $\log(\bar{\varepsilon})$ with the expected value equal to zero and with $f_i = f$, $i = 1, ..., n$ can be found in [2].

3.4. Numerical Example for a Parallel System

The results of tests of 64 bundles of carbon fibers separated from a monolayer (Data_A1) and the same number of strips consisting of 10 similar bundles of the same, 20-mm length (Data_A2) are reported in [15]. Using Data_A1, estimates of the parameters of sev distribution were obtained. These estimates were used to predict the order statistics of Data_A2 by the different models considered in previous sections. The following values of OSPPt statistics were obtained: 0.695, 0.3 and 0.27 for NRDM, RDM models (with p_C=0.2) and using MC model with the matrix of type P_b, respectively.

The average strength is reasonably accurately predicted using Monte Carlo method with the two-dimensional lognormal distribution of strength and elastic modulus, but the standard deviation is underestimated and the corresponding value of OSPPt is rather great: 0.58. Note that, using "appropriate estimates" $\hat{n}_C$=4 and $\hat{p}_C = 0.05$, the value of OSPPt can be reduced to 0.25.

4. MinMaxDM Distribution Family

For the case n_C=1, considered in Section 2, the types of cdf $F_Y(x)$ and $F_Z(x)$ should be chosen "a priori". But clearly, all the other ideas considered in Section 2 can be used also for the series system in which the links are parallel systems with n_C>1. Cdfs $F_Y(x)$ and $F_Z(x)$ define then cdfs of the strength of parallel systems of Y-type or Z-type, respectively, with n_C>1. Note that the strength of a specimen is described by Equation (2).

When obtaining the cdf of X in the numerical examples we suppose that the logarithm of strength of a single LI without defects has a sev distribution. Of course, it is not the only possible assumption. Different assumptions about the distribution of the strength of one link and a priori distribution of initial (technological) defects compose a family of the distributions of ultimate tensile strength of composite material. Taking into account Equation (2), the corresponding family of distributions of X can be denoted by abbreviation MinMaxD (in honor of Daniels). If for calculation of cdf the MC theory is used then the abbreviation MinMaxM is more appropriate. The abbreviation MinMaxDM can be used for the whole family. It should be mentioned that, if the strength of a damaged LI is equal to zero, it may be assumed that the cdfs $F_Y(x)$ and $F_Z(x)$ differ only in the a priori distribution of the number of damaged LIs in a cross section. This number is equal to zero for Z-type links.

The models of MinMaxDM distribution family contain two groups of parameters. The first group includes the strength and stiffness parameters of single LIs (fibers, fiber bundles, *etc*.): for example, θ_0 and θ_1 are parameters of the Weibull cdf of strength of a single LI: $F(x) = 1 - \exp(-\exp((\log(x) - \theta_0)/\theta_1))$. The second group contains the structural parameters: $n_C, p_C, l_1, p_L, \sigma_{L\varepsilon}$, where L is the specimen length (it is worth noting that, in a general case, for the processing of test data of specimens with different lengths, we can use the following definition of the "effective" number of links: $n_L = 1 + [(L - L_1)/l_1]$, where $[x]$ is the integer part of x, and l_1 is a parameter). The structural parameters can be used for a numerical estimate of the quality of the technology used to manufacture the tested specimens.

4.1 Numerical Example for a Series of Parallel System

In [11] carbon fiber strength data are presented for $(L_1,...,L_4)$ = (1, 10, 20, 50 mm) (Data_B1); in [10] the mean values, μ_X, and standard deviations, σ_X, of dry bundle strength (of the same fibers, $n_C = 1000$) with $(L_1,...,L_4)$ = (5, 20, 100, 200 mm) are given (Data_B2). Just as in [10], we perform fitting and parameter estimation using Data_B1 for L=20 mm and attempt to predict the strength of bundles (Data_B2) at different lengths. Naturally, we cannot consider all the versions of models in the framework of MinMaxDM distribution family. In Table 2 the results of application of the randomized Daniels's model (RDM) are presented and compared with the results of similar processing of the same data using the NRDM model, provided in [10].

A good agreement of NRDM prediction of mean strength of dry bundles with the same length, 20 mm, is observed in [10], but this agreement does not extend to other values of L and there is a significant mismatch of σ_S for all four lengths.

Using p.s. B3 with $F(x) = (1 - p_0)F_Y(x) + p_0F_Z(x)$, $p_0 = (1 - p_L)^{n_L}$, $l_1 = 20\, mm$ and the corresponding RDM with $F_Y(x) = F_{X^*}(.)$ with a specific p_C, and $F_Z(x) = \Phi((x - \mu)/\sigma)$, we obtained the results shown in Table 2. For the model considered, closer estimates of σ_S and reasonably good estimates of μ_S, at least for L≥20 mm, are obtained using the strength parameters of single fibers and the specific structural parameters mentioned above.

Table 2: Summary of Parameter Estimation using Data_B1 and Prediction of Data_B2

L (mm) / Number of tests		5 / 28	20 / 25	100 / 29	200 / 7	$\hat{\theta}_0$	$\hat{\theta}_1$	p_C	p_L
Mean (GPa)	Experimental	1.92	1.68	1.58	1.38	7.88	0.18	0	0
	NRDM [8]	2.19	1.71	1.28	1.14	7.88	0.18	0	0
	_RDM	1.61	1.57	1.48	1.4	7.88	0.18	0.2	0.1
Std (GPa)	Experimental	0.07	0.1	0.13	0.11	7.88	0.18	0	0
	NRDM [8]	0.031	0.024	0.018	0.016	7.88	0.18	0	0
	RDM	0.057	0.114	0.161	0.156	7.88	0.18	0.2	0.1

5. Conclusions

The considered models, as a part of the MinMaxDM distribution family, provide a good fitting of the results of tensile strength tests of fibers and composites, and can explain some specific features of the strength of LIs in the framework of a more complex structure. A good fitting is not surprising in this case, of course, because the considered models may have a rather large number of parameters. But, unlike some other models, the parameters of this distribution family allow a natural interpretation and provide additional numerical information about the quality of fibers (strands) and the structure of a composite. In fact, the best results in size effect prediction are obtained using models with the smallest number of unknown parameters: models MB3, B3F and LW have only two parameters to be determined (see Table 1).

Apparently, the MinMaxDM distribution family opens a broad field for study: one can consider different versions of the fiber strength distribution, the a priori distribution of defect number, the matrix of transition probability for MC models *etc*. The results obtained so far should be regarded only as preliminary.

The models of MinMaxDM distribution family allow estimation of the intensity of defects responsible for the difference between the expected and actual strength of fibers and bundles. Knowledge of this information could provide the basis for improving the production technology and, hence, increasing the reliability of structures in which the composite material is used.

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