Linear and Weakly Nonlinear Instability of Slightly Curved Shallow Mixing Layers

IRINA EGLITE, ANDREI KOLYSHKIN
Department of Engineering Mathematics
Riga Technical University
Meza str1, block 4, Riga
LATVIA
irina.eglite@gmail.com, akoliskins@rbs.lv

Abstract: - The paper is devoted to linear and weakly nonlinear stability analysis of shallow mixing layers. The radius of curvature is assumed to be large. Linear stability problem is solved numerically using collocation method based on Chebyshev polynomials. It is shown that for stably curved mixing layers curvature has a stabilizing effect on the flow. Weakly nonlinear theory is used to derive an amplitude evolution equation for the most unstable mode. It is shown that the evolution equation in this case is the Ginzburg-Landau equation with complex coefficients. Explicit formulas for the calculation of the coefficients of the Ginzburg-Landau equation are derived. Numerical algorithm for the computation of the coefficients is described in detail.

Key-Words: - Linear stability, weakly nonlinear theory, method of multiple scales, Ginzburg-Landau equation, collocation method

1 Introduction

Shallow mixing layers occur in many engineering applications. Flows in compound and composite channels and flows at river junctions are typical examples of shallow mixing layers. Methods of analysis of shallow mixing layers include experimental investigation, numerical modeling and stability analysis [1]. Experimental investigation of shallow mixing layers is conducted in many papers [2]-[4]. It is shown in [2]-[4] that bottom friction plays an important role in suppressing perturbations. In addition, the rate of growth of the mixing layer is also reduced in comparison with free mixing layers.

Linear stability analysis of shallow flows is performed in [5]-[10]. Rigid-lid assumption is used in [5] to determine the critical values of the bed friction number for wake flows and mixing layers. The applicability of the rigid-lid assumption to the stability analyses of shallow flows is analyzed in [6] where it is shown that for small Froude numbers the error in using the rigid-lid assumption is quite small. The effect of Froude number of the stability of shallow mixing layers in compound and composite channels is studied in [8]. The results presented in [5]-[10] show that the bed friction number stabilizes the flow and reduces the growth of a mixing layer.

Centrifugal instability can also occur in shallow mixing layers. The effect of small curvature of the stability of free mixing layers is investigated in [11]. It is shown in [11] that curvature has a stabilizing effect on a stably curved mixing layer and destabilizing effect on unstably curved mixing layer.

Linear stability analysis can be used to determine how a particular flow becomes unstable. Critical values of the parameters (for example, critical bed friction number, critical wave number and so on) are also estimated from the linear stability theory. Development of instability above the threshold cannot be analyzed by linear theory. Weakly nonlinear theories [12], [13] are used in order to construct an amplitude evolution equation for the most unstable mode. These theories are based on the method of multiple scales [14] and are applicable if the flow is unstable but the value of the parameter (for example, Reynolds number or bed friction number for shallow flows) is close to the critical value. In this case the growth rate of unstable perturbation is small and one can hope to analyze the development of instability by means of relatively simple evolution equations. Such an approach is used in [12] for plane Poiseuille flow, in [15] and [16] in order to analyze instability of waves generated by wind and in [10], [17]-[19] for shallow wake flows. In fact, amplitude equations are used in the literature in two ways. First, a particular form of the evolution equation is selected apriori and the coefficients of the equation are estimated from experimental data. Then the equation with estimated coefficients is used to model the phenomenon of interest. Second, one can actually derive an evolution equation from the equations of motion. This approach is used in [10], [12], [15], [17] and [19]where it is shown that for two-dimensional cases the evolution equation is the complex Ginzbrug-Landau equation.

In the present paper we perform linear stability analysis of slightly curved shallow mixing layers. The critical values of the stability parameters are evaluated. Weakly nonlinear theory is used later to derive an amplitude evolution equation for the most unstable mode. It is shown that the evolution equation is the Ginzburg-Landau equation with complex coefficients. Explicit formulas for the calculation of the coefficients of the Ginzburg-Landau equation are presented. Details of the numerical algorithm are discussed.

2 Linear stability problem

We consider shallow water equations of the following form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{c_f}{2h}u\sqrt{u^2+v^2}=0, \quad (2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}+\frac{c_f}{2h}v\sqrt{u^2+v^2}+\frac{1}{R}u^2=0, \quad (3)$$

where u and v are the depth-averaged velocity components in the x and y -directions, respectively, x is the direction along the streamline, y is perpendicular to x , p is the pressure, h is water depth, c_f is the friction coefficient and R is the radius of curvature. It is assumed here that $1/R \ll 1$.

Introducing the stream function by the relations

$$u=\frac{\partial \psi}{\partial y}, \quad v=-\frac{\partial \psi}{\partial x} \quad (4)$$

and eliminating the pressure we rewrite the system (1)-(3) in the form

$$(\Delta\psi)_t+\psi_y(\Delta\psi)_x-\psi_x(\Delta\psi)_y+\frac{2}{R}\psi_y\psi_{xy}+\frac{c_f}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2}+\frac{c_f}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy}+2\psi_x\psi_y\psi_{xy}+\psi_x^2\psi_{xx})=0, \quad (5)$$

where the subscripts indicate the derivatives with respect to the variables x, y and t and $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

Consider a perturbed solution to (5) in the form

$$\psi(x,y,t)=\psi_0(y)+\varepsilon\psi_1(x,y,t)+\varepsilon^2\psi_2(x,y,t)+\varepsilon^3\psi_3(x,y,t)+..., \quad (6)$$

where the role of the parameter ε will be clarified later. The base flow $u_0(y)$ is related to $\psi_0(y)$ by the formula $u_0=\psi_{0y}$.

In a classical theory of hydrodynamic stability [20] the base flow is usually a simple solution of the equations of motion. As an example we consider the Navier-Stokes equations where the velocity vector has only one nonzero component which is a function of a radial coordinate only. Solving the Navier-Stokes equations we obtain a parabolic velocity distribution (the Poiseuille flow). This approach does not work for shallow water equations: it is not possible to find a simple analytical solution $u_0(y)$ of (1)-(3). Base flows in the case of shallow water equations are usually chosen in the form of relatively simple model velocity profiles such as hyperbolic tangent profile for shallow mixing layers or hyperbolic secant profile for shallow wake flows. These profiles are chosen on the basis of careful analysis of available experimental data. The following two base flow profiles will be used below:

$$u_0(y)=2+\tanh y, \quad (7)$$

$$u_0(y)=2-\tanh y. \quad (8)$$

Profile (7) corresponds to stably curved mixing layer (the high-speed stream is on the outside of the low-speed stream) while profile (8) represents unstably curved mixing layer (the high-speed stream in on the inside of the low-speed stream).

Substituting (6) into (5) and linearizing the resulting equation in the neighborhood of the base flow $u_0(y)$ we obtain

$$L_1\psi_1 = 0, \qquad (9)$$

where

$$L_1\psi \equiv \psi_{xxt} + \psi_{yyt} + \psi_{0y}\psi_{xxx} + \psi_{oy}\psi_{yyx} - \psi_{0yyy}\psi_x + \frac{c_f}{2h}(\psi_{0y}\psi_{xx} + 2\psi_{oyy}\psi_y + 2\psi_{0y}\psi_{yy}) + \frac{2}{R}\psi_{0y}\psi_{xy}. \qquad (10)$$

Using the method of normal modes we seek the solution to (9) in the form

$$\psi_1(x,y,t) = \varphi_1(y)\exp[ik(x-ct)], \qquad (11)$$

where k is the wave number, c is the phase speed and $\varphi_1(y)$ is the amplitude of the normal perturbation. Substituting (11) into (9) we obtain

$$L\varphi_1 = 0, \qquad (12)$$

where

$$L\varphi \equiv \varphi''[u_0 - c - iSu_0/k) + \varphi'(2u_0/R - iSu_{0y}/k) + \varphi(k^2c - k^2u_0 - u_{0yy} + ikSu_0/2). \qquad (13)$$

and $S = c_f b/h$ is the stability parameter (b is the characteristic length scale).
The boundary conditions are

$$\varphi_1(\pm\infty) = 0. \qquad (14)$$

Problem (12)-(14) is an eigenvalue problem. The complex eigenvalues $c = c_r + ic_i$ determine the linear stability of the base flow $u_0(y)$ which is said to be stable if all $c_i < 0$ and unstable if at least one $c_i > 0$. The condition

$$c_i = 0 \qquad (15)$$

corresponds to neutrally stable perturbations.

2.1 Numerical method

The pseudospectral collocation method based on Chebyshev polynomials is used to solve eigenvalue problem (12)-(14) numerically. The interval $-\infty < y < +\infty$ is transformed into the interval $(-1,1)$ by means of the transformation $r = \frac{2}{\pi}\arctan y$. The solution to (12) is then sought in the form

$$\varphi_1(r) = \sum_{j=0}^{N-1} a_j(1-r^2)T_j(r), \qquad (16)$$

where $T_j(r) = \cos j \arccos r$ is the Chebyshev polynomial of the first kind of degree j and a_j are unknown coefficients. The following set of collocation points is used to solve (12), (14):

$$r_m = \cos\frac{\pi m}{N}, \quad m = 1,1,2,...,N-1. \qquad (17)$$

Substituting (16) into (12) and evaluating the function $\varphi_1(r)$ and its derivatives up to order two inclusive at the collocation points (17) we obtain the generalized eigenvalue problem of the form

$$(B - cD)a = 0, \qquad (18)$$

where B and D are complex-values matrices and $a = (a_0\, a_1 ... a_{N-1})^T$. More detailed description of the numerical algorithm is given in Subsection 3.1 in the context of weakly nonlinear calculations.

The factor $1 - r^2$ guarantees that the boundary conditions (14) in terms of the new variable r are satisfied automatically at $r = \pm 1$. There are at least two reasons why solutions of the form (16) are more convenient than those obtained by “classical” collocation methods [21]: (a) the use of the base functions that satisfy the given zero boundary conditions considerably reduces the condition number [22] and (b) the matrix D in (18) is not singular. Problem (18) is solved numerically by means of the IMSL routine DGVCCG.

The results of numerical calculations are presented in Figs. 1 and 2. Three neutral stability curves for the case of stably curved mixing layer (base flow profile (7)) are shown in Fig. 1 for the following values of the parameter $1/R$: 0, 0.02 and 0.04 (from top to bottom). The flow becomes more stable as curvature increases.

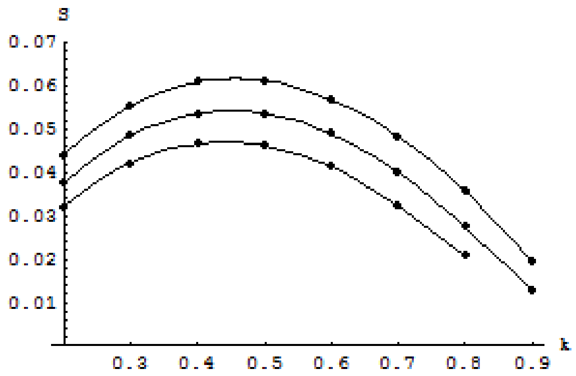

Fig.1. Neutral stability curves for base flow profile (7). The values of the parameter $1/R$ are 0, 0.02 and 0.04 (from top to bottom).

Neutral stability curves for the case of unstably curved mixing layer (base flow profile (8)) are shown in Fig. 2. As can be seen from the figure, curvature has a destabilizing influnece on the flow (the critical values of the stability parameter S increase as $1/R$ increases).

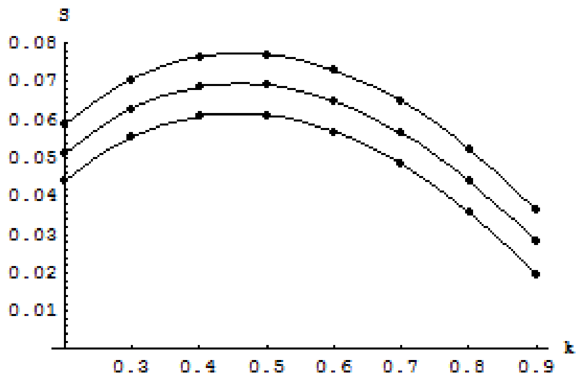

Fig.2. Neutral stability curves for base flow profile (8). The values of the parameter $1/R$ are 0, 0.02 and 0.04 (from bottom to top).

3 Weakly nonlinear analysis

Using linear stability theory one can determine the conditions under which a particular flow becomes unstable. Numerical solution of the corresponding eigenvalue problem allows one to obtain the critical values of the parameters of the problem and determine the structure of the unstable mode. However, linear theory cannot be used to predict the evolution of the most unstable mode above the threshold. In the unstable region perturbation grows exponentially with time (see (11)). If the growth rate is large then nonlinear effects quickly become dominant and there is little hope to analyze the development of instability analytically. However, if the growth rate of the unstable mode is relatively small then weakly nonlinear theories can be used in order to develop an amplitude evolution equation for the most unstable mode. Such equations are obtained in the past for the case of plane Poiseuille flow, shallow water flows, waves on the surface generated by wind and in some other situations (see [10], [12], [15]-[19]).

Suppose that S_c, k_c and c_c are the critical values of the stability parameter, wave number and wave speed, respectively. Then the most unstable mode (in accordance with the linear theory) is given by (11) with $S = S_c, k = k_c$ and $c = c_c$ where the eigenfunction $\varphi_1(y)$ can be replaced by $C\varphi_1(y)$. The constant C cannot be determined from the linear stability theory. In order to analyze the development of instability analytically in the framework of weakly nonlinear theory we consider a small neighborhood of the critical point in the (k,S)-plane where parameter S is assumed to be slightly below the critical value:

$$S = S_c(1-\varepsilon^2).$$

The constant C in this case will be replaced by a slowly varying amplitude function A. Following the paper by Stewartson and Stuart [12] we introduce the "slow" time and longitudinal coordinates τ and ξ by the relations

$$\tau = \varepsilon^2 t, \quad \xi = \varepsilon(x - c_g t),$$

where c_g is the group velocity. Thus, $A = A(\xi,\tau)$ and the function ψ_1 in (11) now has the form

$$\psi_1(x,\xi,y,t,\tau) = A(\xi,\tau)\varphi_1(y)\exp[ik(x-ct)] + c.c., \qquad (19)$$

where the abbreviation c.c. means the complex conjugate.

The stream function in (6) can be represented as follows:

$$\psi = \psi(x,y,t,\xi(x,t),\tau(t)).$$

Using the chain rule we can rewrite the derivatives of ψ with respect to t and x in the form

$$\frac{\partial \psi(x,y,t,\xi(x,t),\tau(t))}{\partial t} = \frac{\partial \psi}{\partial t} - \varepsilon c_g \frac{\partial \psi}{\partial \xi} + \varepsilon^2 \frac{\partial \psi}{\partial \tau},$$

$$\frac{\partial \psi(x,y,t,\xi(x,t),\tau(t))}{\partial x} = \frac{\partial \psi}{\partial x} + \varepsilon c \frac{\partial \psi}{\partial \xi}.$$

In other words, the differential operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ are replaced by

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \varepsilon c_g \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau}, \quad (20)$$

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \xi}. \quad (21)$$

Substituting (6) into (5), using (20), (21) and collecting the terms of order ε^2 we obtain the following equation for the function ψ_2:

$$\begin{aligned} L_1\psi_2 &= c_g(\psi_{1xx\xi} + \psi_{1yy\xi}) - 2\psi_{1x\xi t} - 3u_0\psi_{1xx\xi} \\ &- \psi_{1y}\psi_{1xxx} - \psi_{1y}\psi_{1yyx} - u_0\psi_{1\xi yy} + \psi_{1x}\psi_{1xxy} \\ &+ \psi_{1x}\psi_{1yyy} + u_{0yy}\psi_{1\xi} - \frac{S}{2}[\psi_{1xx}\psi_{1y} + 2u_0\psi_{1x\xi} \\ &+ 2\psi_{1yy}\psi_{1y} - 2u_0u_{0y} + 2\psi_{1x}\psi_{1xy}] \\ &- \frac{2}{R}[u_0\psi_{1\xi y} + \psi_{1y}\psi_{1xy}]. \end{aligned} \quad (22)$$

Note that the operator L_1 on the left-hand side of (22) is the same as in (9) and it will be the same for all orders in ε.

Similarly, collecting the terms of order ε^3 we obtain

$$\begin{aligned} L_1\psi_3 &= c_g(\psi_{2xx\xi} + \psi_{2yy\xi}) - \psi_{1xx\tau} - 2\psi_{2x\xi t} \\ &+ 2c_g\psi_{1x\xi\xi} - \psi_{1\xi\xi t} - \psi_{1yy\tau} - 3u_0\psi_{2xx\xi} - 3u_0\psi_{1x\xi\xi} \\ &- \psi_{1y}\psi_{2xxx} - 3\psi_{1y}\psi_{1xx\xi} - \psi_{2y}\psi_{1xxx} - \psi_{2y}\psi_{1yyx} \\ &- \psi_{1y}\psi_{2yyx} - \psi_{1y}\psi_{1\xi yy} - u_0\psi_{2\xi yy} + \psi_{2x}\psi_{1xxy} \\ &+ \psi_{1\xi}\psi_{1xxy} + \psi_{1x}\psi_{2xxy} + 2\psi_{1x}\psi_{1xy\xi} + \psi_{1x}\psi_{2yyy} \\ &+ \psi_{2x}\psi_{1yyy} + \psi_{1\xi}\psi_{1yyy} + \psi_{2\xi}u_{0yy} - \frac{S}{2}[\psi_{1xx}\psi_{2y} \\ &+ 1.5\psi_{1xx}\psi_{1x}^2/u_0 + \psi_{2xx}\psi_{1y} + 2\psi_{1x\xi}\psi_{1y} + 2u_0\psi_{2x\xi} \\ &+ u_0\psi_{1\xi\xi} + \psi_{1yy}\psi_{2y} + \psi_{2yy}\psi_{1y} - u_0\psi_{1xx} - 2u_{0y}\psi_{1y} \\ &- 2u_0\psi_{1yy} + \psi_{1yy}\psi_{2y} + \psi_{1y}\psi_{2yy} + 2\psi_{1x}\psi_{2xy} \\ &+ 2\psi_{1x}\psi_{1\xi y} + 2\psi_{2x}\psi_{1xy} + 2\psi_{1\xi}\psi_{1xy}] \\ &- \frac{2}{R}[u_0\psi_{2\xi y} + \psi_{1y}\psi_{2xy} + \psi_{1y}\psi_{1\xi y} + \psi_{2y}\psi_{1xy}]. \end{aligned} \quad (23)$$

Next, we consider the solution of (22). It can be shown that substituting (19) into the right-hand side of (22) the following three groups of terms will emerge: (a) the terms that are independent on time, (b) the terms proportional to the first harmonic $\exp[ik(x-ct)]$, and (c) the terms proportional to the second harmonic $\exp[2ik(x-ct)]$ (here and in sequel we drop the subscripts and use the notation $k = k_c$ and $c = c_c$ for convenience). Thus, the function ψ_2 should also contain the same three groups of terms. More precisely, we seek the solution to (22) in the form

$$\begin{aligned} \psi_2 &= AA^*\varphi_2^{(0)}(y) + A_\xi\varphi_2^{(1)}(y)\exp[ik(x-ct)] \\ &+ A^2\varphi_2^{(2)}(y)\exp[2ik(x-ct)], \end{aligned} \quad (24)$$

where $\varphi_2^{(0)}(y)$, $\varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$ are unknown functions of y, A^* denotes the complex conjugate of A, the superscript reflects the index of the harmonic component and the subscript represents the order of approximation.

Substituting (19) and (24) into (22) and collecting the terms proportional to AA^* yields

$$2S[u_{0y}(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)})+u_0(\varphi_{2yy}^{(0)}+\varphi_{2yy}^{*(0)})]$$
$$=ik(\varphi_{1y}\varphi_{1yy}^{*}-\varphi_{1y}^{*}\varphi_{1yy}+\varphi_1\varphi_{1yyy}^{*}-\varphi_1^{*}\varphi_{1yyy})$$
$$-\frac{S}{2}[k^2(\varphi_1\varphi_{1y}^{*}+\varphi_1^{*}\varphi_{1y}++2(\varphi_{1y}^{*}\varphi_{1yy}+\varphi_{1yy}^{*}\varphi_{1y})]$$
$$=0. \tag{25}$$

The boundary conditions have the form

$$\varphi_2^{(0)}(\pm\infty)=0. \tag{26}$$

Similarly, substituting (19) and (24) into (22) and collecting the terms proportional to $\exp[ik(x-ct)]$ we obtain the following equation for the function $\varphi_2^{(1)}(y)$

$$(u_0-c-Su_0\frac{i}{k})\varphi_{2yy}^{(1)}+(2\frac{u_0}{R}-Su_{0y}\frac{i}{k})\varphi_{2y}^{(1)}$$
$$+(k^2c-k^2u_0-u_{0yy}+ku_0S\frac{i}{2})$$
$$=-\frac{i}{k}(c_g-u_0)\varphi_{1yy}+2\frac{iu_0}{kR}\varphi_{1y} \tag{27}$$
$$+(2ikc-3iku_0-\frac{i}{k}u_{0yy}+ikc_g-u_0S)\varphi_1$$

with the boundary conditions

$$\varphi_2^{(1)}(\pm\infty)=0. \tag{28}$$

Comparing (27) and (12) one can see that the right-hand side of (27) is exactly the same as the right-hand side of (12) if $\varphi_2^{(1)}(y)$ is replaced by $\varphi_1(y)$. Thus, (27) is resonantly forced and solvability condition should be applied at this stage to guarantee the existence of the solution. Using the Fredholm's alternative [23] we conclude that equation (27) has a solution if and only if the right-hand side of (27) is orthogonal to all eigenfunctions of the corresponding homogeneous adjoint problem.

The adjoint operator L^a and adjoint eigenfunction φ_1^a are defined by the relation

$$\int_{-\infty}^{+\infty}\varphi_1^a L\varphi_1 dy=\int_{-\infty}^{+\infty}\varphi_1 L^a\varphi_1^a dy. \tag{29}$$

The left-hand side of (29) is equal to zero since φ_1 is the solution to (12). Thus, the adjoint equation is defined by the formula

$$L^a\varphi_1^a=0. \tag{30}$$

Integrating the left-hand side of (29) by parts and using the boundary conditions (14) we obtain the adjoint operator in the form

$$L^a\varphi_1^a\equiv\varphi_{1yy}^a(u_0-c-Su_0\frac{i}{k})$$
$$+\varphi_{1y}^a(2u_{0y}-Su_{0y}\frac{i}{k}-2\frac{u_0}{R})$$
$$+\varphi_1^a(k^2c-k^2u_0+\frac{ik}{2}Su_0-2\frac{u_{0y}}{R}).$$

The boundary conditions are

$$\varphi_1^a(\pm\infty)=0. \tag{31}$$

The adjoint eigenfunction φ_1^a is the solution of the problem (30), (31).

Applying the solvability condition to (27) we obtain

$$\int_{-\infty}^{+\infty}\varphi_1^a[(c_g-u_0)\varphi_{1yy}-2\frac{u_0}{R}\varphi_{1y}$$
$$+(-2k^2c+3k^2u_0+u_{0yy} \tag{32}$$
$$-k^2c_g+iku_0S)\varphi_1]dy=0.$$

Equation (32) defines the group velocity

$$c_g=\frac{\eta_1}{\eta}, \tag{33}$$

where

$$\eta=\int_{-\infty}^{+\infty}\varphi_1^a(\varphi_{1yy}-k^2\varphi_1)dy \tag{34}$$

and

$$\eta_1 = \int_{-\infty}^{+\infty} \varphi_1^a [u_0 \varphi_{1yy} + 2\frac{u_0}{R}\varphi_{1y} + (2k^2 c - 3k^2 u_0 - u_{0yy} - iku_0 S)\varphi_1]dy. \quad (35)$$

Finally, substituting (19) and (24) into (22) and collecting the terms proportional to $\exp[2ik(x-ct)]$ we obtain

$$8ik^3 c\varphi_2^{(2)} - 2ikc\varphi_{2yy}^{(2)} - 8ik^3 u_0 + 2iku_0\varphi_{2yy}^{(2)} - 2iku_{0yy}\varphi_2^{(2)} + S[-4k^2 u_0 \varphi_2^{(2)} + 2u_{0y}\varphi_{2y}^{(2)} + 2u_0\varphi_{2yy}^{(2)}] + 4iku_0\varphi_{2y}^{(2)}/R = ik(\varphi_1\varphi_{1yyy} - \varphi_{1y}\varphi_{1yy}) - S(2\varphi_{1y}\varphi_{1yy} - 3k^2\varphi_1\varphi_{1y}) - 2ik\varphi_{1y}^2/R \quad (36)$$

with the boundary conditions

$$\varphi_2^{(2)}(\pm\infty) = 0. \quad (37)$$

Solving three boundary value problems (25)-(28), (36), (37) numerically we obtain the functions $\varphi_2^{(0)}(y)$, $\varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$. The function ψ_2 (the second order correction) is then given by (24).

Let us consider the solution at the third order in ε. Equation (23) also has a solution if and only if the right-hand side of (23) is orthogonal to all eigenfunctions φ_1^a of the corresponding homogeneous adjoint problem (30), (31). Substituting (19) and (24) into the right-hand side of (23) and applying the solvability condition we obtain the amplitude evolution equation for slowly varying amplitude function $A(\xi,\tau)$ of the form

$$\frac{\partial A}{\partial \tau} = \sigma A + \delta \frac{\partial^2 A}{\partial \xi^2} - \mu |A|^2 A.. \quad (38)$$

Equation (38) is the complex Ginzburg-Landau equation with complex coefficients σ, δ and μ where

$$\sigma = \frac{\sigma_1}{\eta}, \ \delta = \frac{\delta_1}{\eta}, \ \mu = \frac{\mu_1}{\eta}. \quad (39)$$

The coefficients σ_1, δ_1 and μ_1 are given by

$$\sigma_1 = \frac{S}{2}\int_{-\infty}^{+\infty} \varphi_1^a (-k^2 u_0 \varphi_1 + 2u_{0y}\varphi_{1y} + 2u_0\varphi_{1yy})dy,$$

$$\delta_1 = \int_{-\infty}^{+\infty} \varphi_1^a [(c_g - u_0)\varphi_{2yy}^{(1)} - 2\frac{u_0}{R}\varphi_{2y}^{(1)} + \varphi_2^{(1)}(-k^2 c_g - 2k^2 c + 3k^2 u_0 + u_{0yy} - ikSu_0) + \varphi_1(2ikc_g + ikc - 3iku_0 - u_0\frac{S}{2})]dy,$$

$$\mu_1 = \int_{-\infty}^{+\infty} \varphi_1^a \{6ik^3\varphi_2^{(2)}\varphi_{1y}^* - 2ik\varphi_{1y}^*\varphi_{2yy}^{(2)} + 3ik^3\varphi_1^*\varphi_{2y}^{(2)} + ik^3\varphi_1(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) - ik\varphi_{1yy}(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) + ik\varphi_{2y}^{(2)}\varphi_{1yy}^* - ik\varphi_1^*\varphi_{2yyy}^{(2)} + ik\varphi_1(\varphi_{2yyy}^{(0)} + \varphi_{2yyy}^{*(0)}) + 2ik\varphi_{1yyy}^*\varphi_2^{(2)} - \frac{S}{2}[-k^2\varphi_1(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) + 3k^2\varphi_1^*\varphi_{2y}^{(2)} - \frac{3k^4}{2u_0}\varphi_1^2\varphi_1^* + 2\varphi_{1yy}(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) + 2\varphi_{1yy}^*\varphi_{2y}^{(2)} + 2\varphi_{1y}(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{*(0)}) + 2\varphi_{2yy}^{(2)}\varphi_{1y}^*] - 2\frac{ik}{R}(\varphi_{2y}^{(2)}\varphi_{1y}^* + \varphi_{2y}^{(0)}\varphi_{1y})\}dy.$$

The coefficient η in (39) is given by (34).

Formulas (39) represent the coefficients of equation (38) in terms of the characteristics of the linear stability of the flow. More precisely, in order to obtain σ, δ and μ we need to perform the following calculations: (i) solve the linear stability problem (12), (14) and determine the critical values of the parameters k, S, c and the corresponding eigenfunction $\varphi_1(y)$; (ii) solve the homogeneous adjoint problem (30), (31) and determine the adjoint eigenfunction φ_1^a; (iii) solve three boundary value problems (25)-(28), (36), (37) and determine the functions $\varphi_2^{(0)}(y)$, $\varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$; (iiii) evaluate the integrals in (39).

3.1 Numerical method

In this subsection we propose numerical method for the calculation of the coefficients of the Ginzburg-Landau equation.

The solutions of linear stability problem (12)-(14), adjoint problem (30), (31), boundary value problems (25)-(28), (36), (37) are sought in the form

$$\varphi(r) = \sum_{j=0}^{N-1} a_j (1-r^2) T_j(r), \quad (40)$$

where $\varphi(r)$ represents any of the functions $\varphi_1(r),\ \varphi_1^a(r),\ \varphi_2^{(0)}(r),\ \varphi_2^{(1)}(r),\ \varphi_2^{(2)}(r)$ (recall that $r=\frac{2}{\pi}\arctan y$). Using the chain rule we compute the derivatives of the first, second and third order of φ with respect to y:

$$\frac{d\varphi}{dy}=\frac{2}{\pi}\cos^2\frac{\pi r}{2}\frac{d\varphi}{dr},$$
$$\frac{d^2\varphi}{dy^2}=\frac{4}{\pi^2}\cos^4\frac{\pi r}{2}\frac{d^2\varphi}{dr^2}-\frac{4}{\pi}\sin\frac{\pi r}{2}\cos^3\frac{\pi r}{2}\frac{d\varphi}{dr},$$
$$\frac{d^3\varphi}{dy^3}=\frac{(-4+12\tan^2\frac{\pi r}{2})}{\pi}\cos^6\frac{\pi r}{2}\frac{d\varphi}{dr} \quad (41)$$
$$-\frac{24}{\pi^2}\sin\frac{\pi r}{2}\cos^5\frac{\pi r}{2}\frac{d^2\varphi}{dr^2}+\frac{8}{\pi^2}\cos^6\frac{\pi r}{2}\frac{d^3\varphi}{dr^3}.$$

The derivatives of φ with respect to r are evaluated using (40):

$$\frac{d\varphi}{dr}=\sum_{j=0}^{N-1}a_j[-2rT_j(r)+(1-r^2)T_j'(r)],$$
$$\frac{d^2\varphi}{dr^2}=\sum_{j=0}^{N-1}a_j[-2T_j(r)-4rT_j'(r)+(1-r^2)T_j''(r)],$$
$$\frac{d^3\varphi}{dr^3}=\sum_{j=0}^{N-1}a_j[-6T_j'(r)-6rT_j''(r) \quad (42)$$
$$+(1-r^2)T_j'''(r)].$$

In order to evaluate the function $\varphi(r)$ and its derivatives up to the third order we need to compute the values of the Chebyshev polynomial $T_j(r)$ and its derivatives at the collocation points (17):

$$T_j(r_m)=\cos\frac{m\pi j}{N},$$
$$T_j'(r_m)=\frac{j\sin\frac{m\pi j}{N}}{\sin\frac{\pi m}{N}},$$
$$T_j''(r_m)=\frac{j\cos\frac{\pi m}{N}\sin\frac{m\pi j}{N}}{\sin^3\frac{\pi m}{N}}-\frac{j^2\cos\frac{m\pi j}{N}}{\sin^2\frac{\pi m}{N}},$$

$$T_j'''(r_m)=\left[\frac{j-j^3}{\sin^3\frac{\pi m}{N}}+\frac{3j\cos^2\frac{\pi m}{N}}{\sin^5\frac{\pi m}{N}}\right]\sin\frac{m\pi j}{N} \quad (43)$$
$$-\frac{3j^2\cos\frac{\pi m}{N}}{\sin^4\frac{\pi m}{N}}\cos\frac{\pi mj}{N}.$$

The values of $\varphi_1(r)$, its derivatives up to order two inclusive and the coefficients of equation (12) at the collocation points (17) can be evaluated using formulas (41)-(43) so that the elements of the matrices B and D (see (18)) can be computed and the generalized eigenvalue problem (18) can be solved numerically.

Similar approach can be used in order to solve boundary value problems (25), (26) and (36), (37). System of linear algebraic equations of the form

$$Fa=G \quad (44)$$

is obtained in each case after discretization where $a=(a_0\,a_1...a_{N-1})^T$. The matrix F is not singular for problems (25), (26) and (36), (37). Therefore, any linear equation solver can be used in order to find a. Thus, the functions $\varphi_2^{(0)}(y)$ and $\varphi_2^{(2)}(y)$ can be evaluated by means of the expansions of the form (40).

The same form of the expansion (40) is used to solve boundary value problem (27), (28). Equation of the form (44) is also obtained after discretization in this case, but the matrix F is singular since the corresponding homogeneous part of (27) has a nontrivial solution at $S=S_c, k=k_c$ and $c=c_c$. Equation (44) is solved in this case by means of the singular value decomposition method [24]. It is known that if F is a complex $N\times N$ matrix, then there exist orthogonal $N\times N$ matrices U and V such that

$$U^H FV=\Sigma, \quad (45)$$

where $\Sigma=diag(\gamma_1,\gamma_2,...,\gamma_N)$.

Equation (45) is called the singular value decomposition of the matrix F and $\gamma_1,\gamma_2,...,\gamma_N$ are the singular values of F. In our case only the last of the singular values will be equal to zero

($\gamma_1 > \gamma_2 > ... > \gamma_{N-1} > \gamma_N = 0$). Hence, the solution to (44) in this case can be written in the form

$$a = V\Sigma^{-1}U^H G, \quad (46)$$

where the last column of V, the last row of U^H, the last column and the last row of Σ^{-1} are deleted. In component form the solution to (46) is

$$a = \sum_{i=1}^{N-1} \frac{U_i^H D V_i}{\gamma_i}, \quad (47)$$

where U_i^H and V_i are vectors (columns of the matrices U^H and V, respectively). Hence, the values of the function $\varphi_2^{(1)}(y)$ can be computed using formula (40) where the coefficients a_j are the components of the vector a in (47).

The final step of the computational procedure involves the calculation of integrals in (39). Adaptive quadrature formula described in [30] can be used to compute the integrals in (39).

3.2 Ginzburg-Landau model

Spatio-temporal dynamics of complex flows is often analyzed by relatively simple evolution equations such as Landau or Ginzburg-Landau equations [25]-[26]. The models are relatively simple so that physicists try to use them in order to describe complex phenomena. In many cases the form of the equation is assumed but the coefficients of the equation are calculated from experimental data (in other words, a phenomenological model is used). Examples can be found in [27]-[29] where it is shown that the Landau and Ginzburg-Landau equations can be successfully used in order to describe experimental observations of flows behind bluff bodies in a wide range of Reynolds numbers.

In the present paper we show that the Ginzburg-Landau equation does not need to be assumed for slightly curved shallow mixing layers, it is actually derived from the shallow water equations under the rigid-lid assumption.

4 Conclusion

In the present paper stability analysis of slightly curved shallow mixing layers is performed. Linear stability characteristics of a hyperbolic tangent velocity profile are calculated. It is shown that a stably curved mixing layer is even more stabilized by the increasing curvature while for unstably curved mixing layers curvature has a destabilizing effect on the flow.

Method of multiple scales is used in the paper to derive an amplitude evolution equation for the most unstable mode in the framework of weakly nonlinear theory. It is shown that the evolution equation is the complex Ginzburg-Landau equation. Explicit formulas for the calculation of the coefficients of the Ginzburg-Landau equation are presented. Numerical algorithm for the calculation of the coefficients is proposed and analyzed in detail.

References:

[1] G.H. Jirka, Large scale flow structures and mixing processes in shallow flows, *Journal of Hydraulic Research*, Vol.39, 2001, pp. 567-573.

[2] V.H. Chu, and S. Babarutsi, Confinement and bed-friction effects in shallow turbulent mixing layers, *Journal of Hydraulic Engineering, ASCE*, Vol.114, 1988, pp. 1257-1274.

[3] W.S.J Uijttewaal, and R. Booij, Effect of shallowness on the development of free-surface mixing layers, *Physics of Fluids*, Vol.12, 2000, pp. 392-402.

[4] W.S.J. Uijttewaal, and J. Tukker, Development of quasi two-dimensional structures in a shallow free-surface mixing layer, *Experiments in Fluids*, Vol.24, 1998, pp. 192-200.

[5] V.H. Chu, J.H. Wu, and R.E. Khayat, Stability of transverse shear flows in shallow open channels, *Journal of Hydraulic Engineering, ASCE*, Vol.117, 1991, pp. 1370-1388.

[6] D. Chen, and G.H. Jirka, Linear stability analysis of turbulent mixing layers and jets in shallow water layers, *Journal of Hydraulic Research*, Vol.36, 1998, pp. 815-830.

[7] M.S. Ghidaoui, and A.A. Kolyshkin, Linear stability analysis of lateral motions in compound open channels, *Journal of Hydraulic Engineering, ASCE*, Vol.125, 1999, pp. 871-880.

[8] A.A. Kolyshkin, and M.S. Ghidaoui, Gravitational and shear instabilities in compound and composite channels, *Journal of Hydraulic Engineering, ASCE*, Vol.128, 2002, pp. 1076-1086.

[9] B.C. Prooijen, and W.S.J. Uijttewaal, A linear approach for the evolution of coherent structures in shallow mixing layers, *Physics of Fluids*, Vol.14, 2002, pp. 41105-4114.

[10] A.A. Kolyshkin, and M.S. Ghidaoui, Stability analysis of shallow wake flows, *Journal of Fluid Mechanics*, Vol.494, 2003, pp. 355-377.
[11] W.W. Liou, Linear instability of free curved shear layers, *Physics of Fluids*, Vol.6, 1993, pp. 541-549.
[12] K. Stewartson, and J.T. Stuart, A non-linear instability theory for a wave system in plane Poiseuille flow, *Journal of Fluid Mechanics*, Vol.48, 1971, pp. 529-545.
[13] C. Godréche, and P. Manneville, *Hydrodynamics and nonlinear instabilities*, Cambridge University Press, 1998.
[14] J. Kevorkian, and J.D. Cole, *Multiple scale and singular perturbation methods*, Springer, 1996.
[15] P.J. Blennerhassett, On the generation of waves by wind, *Philosophical Transaction of the Royal Society of London, Ser. A.*, Vol.298, 1980, pp. 451-494.
[16] S. Leblanc, Amplification of nonlinear surface waves by wind, *Physics of Fluids*, Vol.19, 2007, pp. 101705-1-101705-4.
[17] F. Feddersen, Weakly nonlinear shear waves, *Journal of Fluid Mechanics*, Vol.372, 1998, pp. 71-91.
[18] M.S. Ghidaoui, A.A. Kolyshkin, J.H. Liang, F.C. Chan, Q. Li, and K. Xu, Linear and nonlinear analysis of shallow wakes, *Journal of Fluid Mechanics*, Vol.548, 2006, pp. 309-340.
[19] A.A. Kolyshkin and S. Nazarovs, Linear and weakly nonlinear analysis of two-phase shallow wake flows, *WSEAS Transactions on Mathematics*, Vol.6, No. 1, 2007, pp. 1-8.
[20] P.G. Drazin, and W.H. Reid, *Hydrodynamic stability*, Cambridge University Press, 1981.
[21] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang, *Spectral methods: fundamentals in single domain*, Springer, 2006.
[22] W. Heinrichs, Improved condition number for spectral methods, *Mathematics of Computation*, Vol.53, No. 187, 1989, pp. 103-119.
[23] D. Zwillinger, *Handbook of differential equations*, Academic Press, 1998.
[24] G.H. Golub, and C.F. van Loan, *Matrix computations*, The Johns Hopkins University Press, 1983.
[25] L.S. Aranson ,and L. Kramer, The world of the complex Ginzburg-Landau equation, *Review of Modern Physics*, Vol.74, 2002, pp. 99-143.
[26] M.C. Cross, and P.C. Honenberg, Pattern formation outside of equilibrium, *Review of Modern Physics*, Vol.65, 1993, pp. 851-1112.
[27] M. Provansal, C. Mathis, and L. Boyer, The Bénard-von Kármán instability: transient and forced regimes, *Journal of Fluid Mechanics*, Vol.182, 1987, pp. 1-22.
[28] T. Leweke, and M. Provansal, The flow behind rings: bluff body wakes without end effects, *Journal of Fluid Mechanics*, Vol.288, 1995, pp. 265-310.
[29] M. Shumm, E. Berger, and P.A. Monkewitz, Self-excited oscillations in the wake of two-dimensional bluff bodies, *Journal of Fluid Mechanics*, Vol.271, 1994, pp. 17-53.
[30] G.E. Forsythe, M.A. Malcolm, and C.B. Moler, *Computer methods for mathematical computations*, Prentice Hall, 1977.